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Course

On

Introductory Course in Real Analysis

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Lecture 59: Rolle's Mean Value Theorem and Its Applications

Okay now from here we can also drag the results which is known as the Rolle's theorem, the statement of the Rolle's theorem is suppose that F is continuous, if F is a real continuous function on that closed and bounded interval A, B which is differentiable in the open interval A, B, which is differentiable in the interval A, B and that the value of the function at the endpoint coincides F(a) = F(b) and say equal to 0, we can even take it without 0 also it will work 0, then there exists a point X in the interval A, B such that the derivative of the function at this point is 0.

The proof follows again from the Lagrange's Mean Value theorem, in the Lagrange's case, if we take F(a) = F(b) then implies the derivative of the function must be 0 for some X belonging to the interval A, B, that is again the meaning A, suppose we have a curve, say suppose we have a curve of this type say like this, and here is the point A, this is the point say B, function is continuous and differentiable over the open interval, and at the end point both are attending the same value N equal to say 0,
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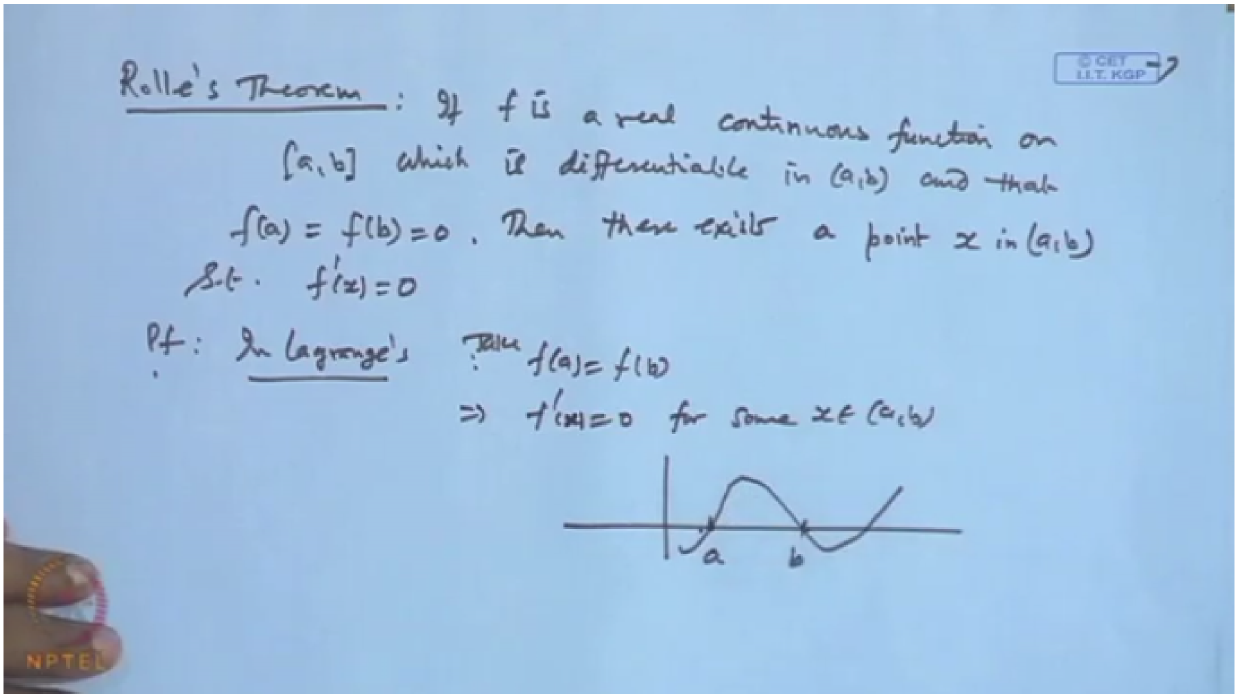

then according to this there will exist one point X where the derivative vanishes F dash(x) will be 0 and this point means the line is parallel to axis of X, slope will be 0, slope will be 0, so this shows the result for them, okay.

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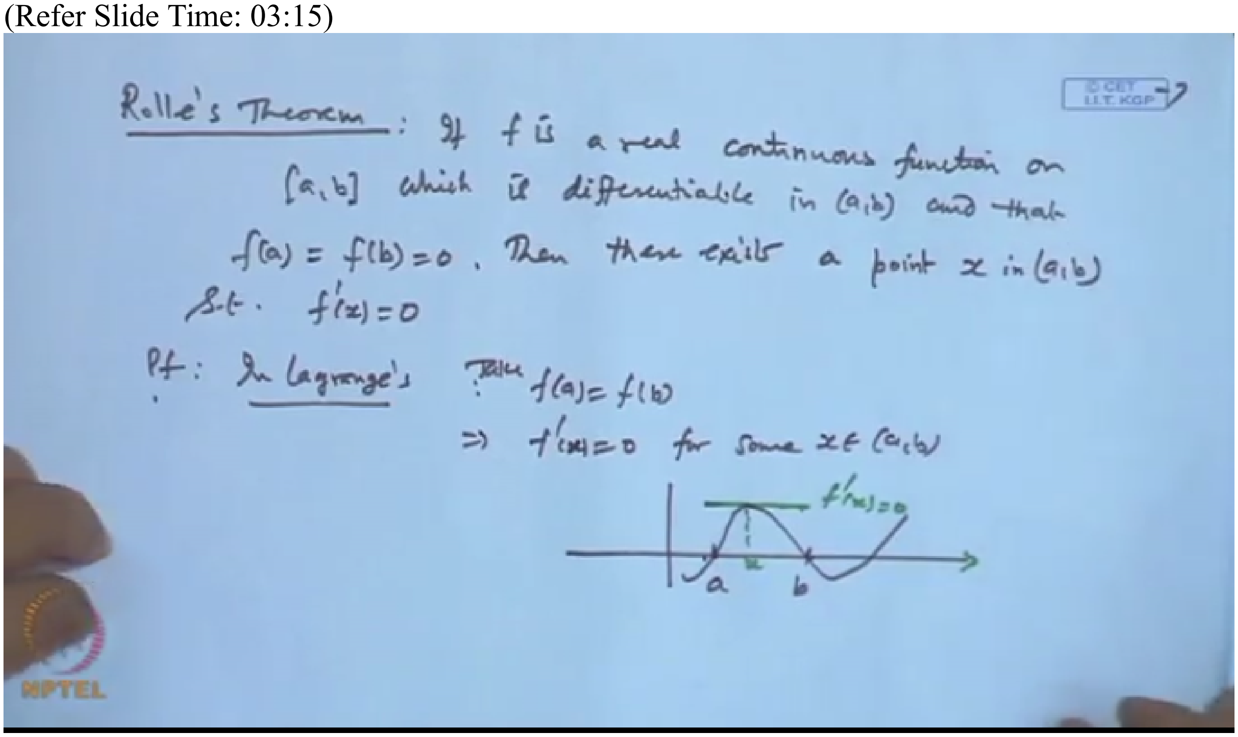

Now using this we can come to know, consequence of this here means we can get one more result, what this result says is suppose F is continuous on the closed interval I, and F is differentiable in the open interval I at each point in the open interval, and the derivative F prime X equal to 0 for every X belonging to the open interval A, B then F is a constant function, and obviously the proof is very simple because if we apply this theorem Lagrange's Mean Value theorem then over the interval we get AX, so proof is apply Lagrange's Mean Value theorem over the interval say AX, there exists some point, say A, X is only real so let us take the point A by or T, let us take the point A, T where the T is lying between A less than B, okay, T is this over this interval, so apply this then what you get is F(t) – F(a) = T –A into the derivative so there exists a point X belongs to the interval A, T which is of course subset of A, B such that this result hold, okay.

Now if at this point it is 0, but this is given that the derivative vanishes for every X belongs to the interval A, B so this implies that F(t) = F(a), and for every T, but T is arbitrary, so this implies the function F is constant function. And as a consequence of this result we can say as a corollary that suppose F and G are continuous on the closed interval A, B and differentiable, and they are differentiable on the open interval A, B and they satisfy the condition that F prime X = G prime X for all X belonging to the interval A, B, then there exists a constant C, then there exists a constant capital C such that the difference of this is equal to C, means that differ by a constant, so proof follows immediately, so I will not go for this proof further, okay.
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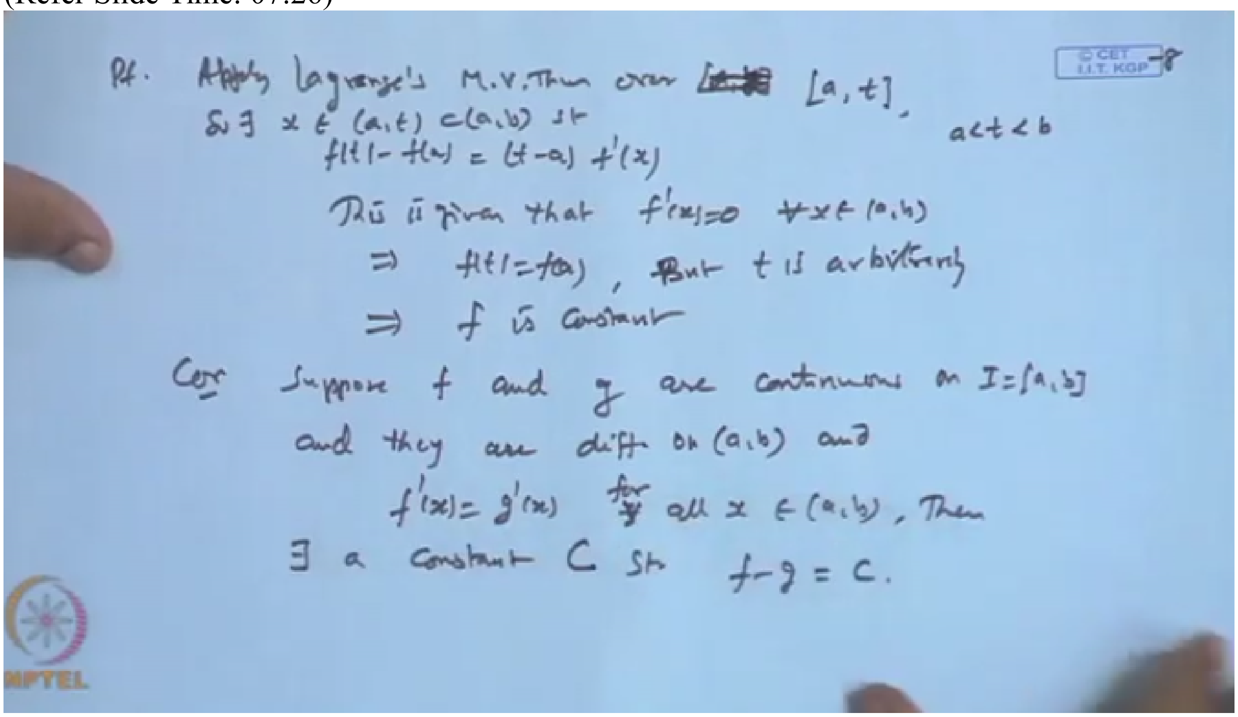

Now using these things we can also drive the results for the functions which are monotonically increasing and decreasing, so suppose that, suppose F is differentiable in the open interval say A, B then the following results holds, then number one, F if F prime X is greater than equal to 0 for all X belonging to the interval A, B, then F is monotonically increasing, B,
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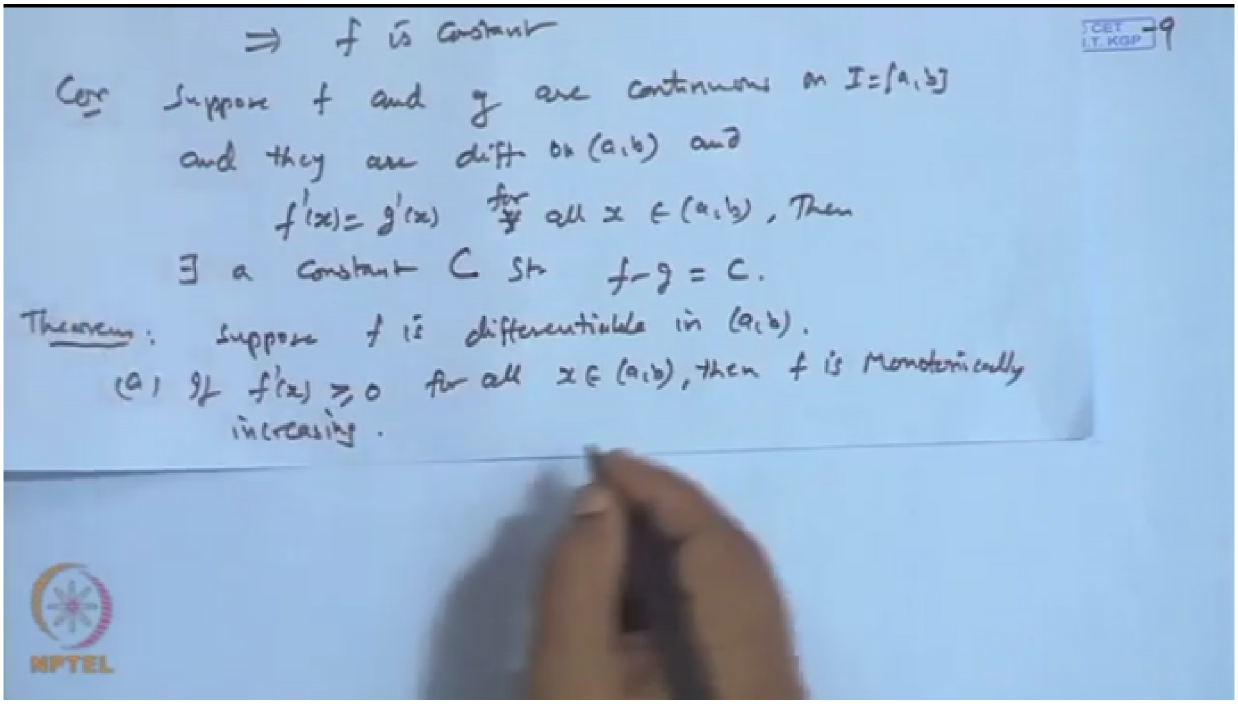

if derivative F prime X = 0 for all X belonging to the A, B then F is constant, which is already shown, and C if the derivative of the function is less than equal to 0 for all X belonging to A, B then F is monotonically decreasing, okay. The converse is also true here if I take here the converse also hold that if F is monotonically decreasing then the derivative will be negative, the converses also hold, and here also converse hold, okay, so proof is very simple just we take the choose the interval say X1, X2, choose X1, X2 where X1 and X2 these are the points of the interval A, B for some X being exponent, okay. Then apply Lagrange's Mean Value theorem then F(x2) this is proof for A, F(x2) –F(x1) = X2-X1 derivative of the function here for some X belonging to X1, X2 okay.

Now it is given the derivative is greater than 0 for all X so this is given, the derivative F prime X is greater than 0 for all X in the interval A, B so this implies that F(x2) is greater than or equal to, so it is greater than or equal to F(x1) when for all X1 satisfying this condition, X2 is greater than X1 okay,

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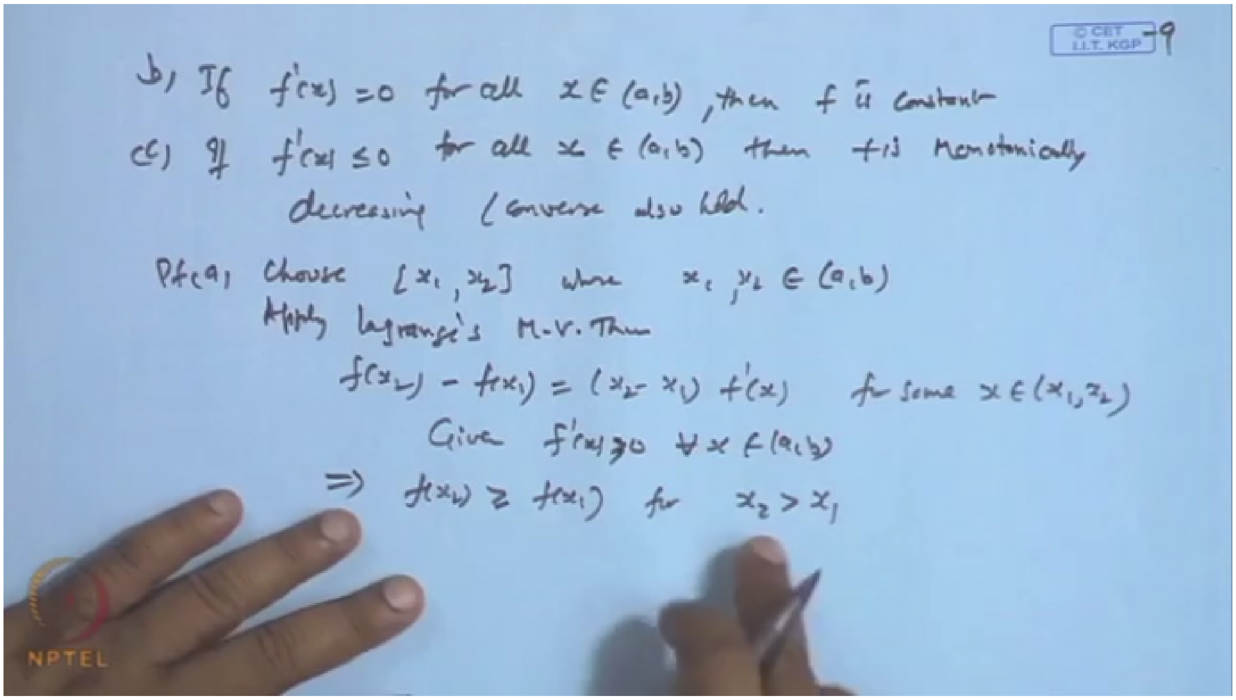

so this shows the function F is monotonically increasing function, okay, monotonically increasing function.

The converse of this also true, the converse also holds, why? Because suppose F is differentiable, suppose that F is differentiable, F is differentiable and increasing, monotonically increasing in the interval I, on the interval I which is say our A, B, I is the interval A, B, okay, on the interval I.

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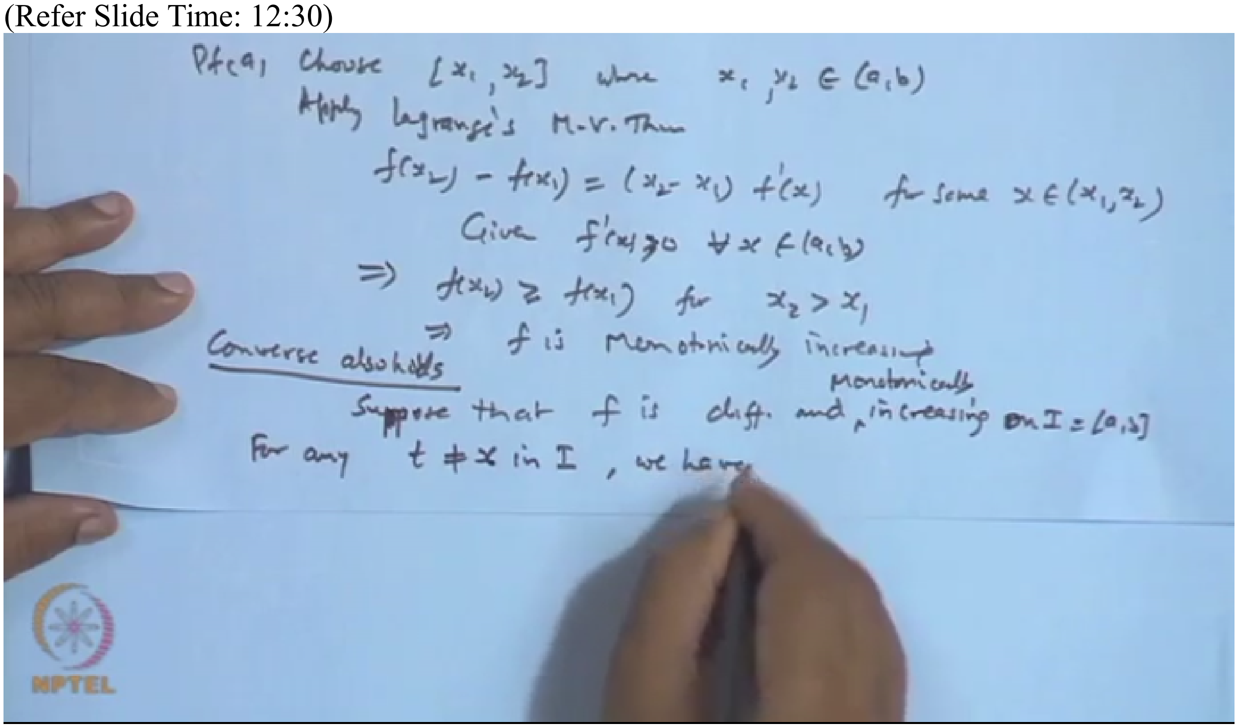

Now take a point, for any point T which is different from X in I we have F(t) – F(x)/t-x okay, now if I take this, this is the point X, here I am taking an interval suppose and T is point somewhere here, if T is in this interval then X is greater than T, so T-X will be negative, is it not? T-X will be, sorry this limit of this, limit of this when T tends to X because this limit exists it is differentiable and it is the derivative of the function at a point X, and since it is a monotonically increasing, so F(t) will be less than F(x), so since F is monotonically increasing, so F(t) is less than F(x) for T lying between this, say interval A, B here okay, so this is less than 0, this one will be less than 0, so this entire thing will be greater than equal to 0, so it is greater than equal to 0, if T lies between this interval, then what happens? This will be if when T lies in this interval then already T - X that is equal to what? This will be greater than equal to 0 because X will be here, is it not? So we can write it F(t) is a, F(x) is less than F(t) so this is positive, this is positive so this hold, so basically this holds, therefore the result follows.

Similarly for the second case we can go, so we are not going to, now here is a remark we can say a function F is said to be strictly increasing on an interval say I, if for any points X1, X2 in I such that X1 is strictly less than X2, then we have F(x1) is strictly less than F(x2) then we say the function is strictly increasing.
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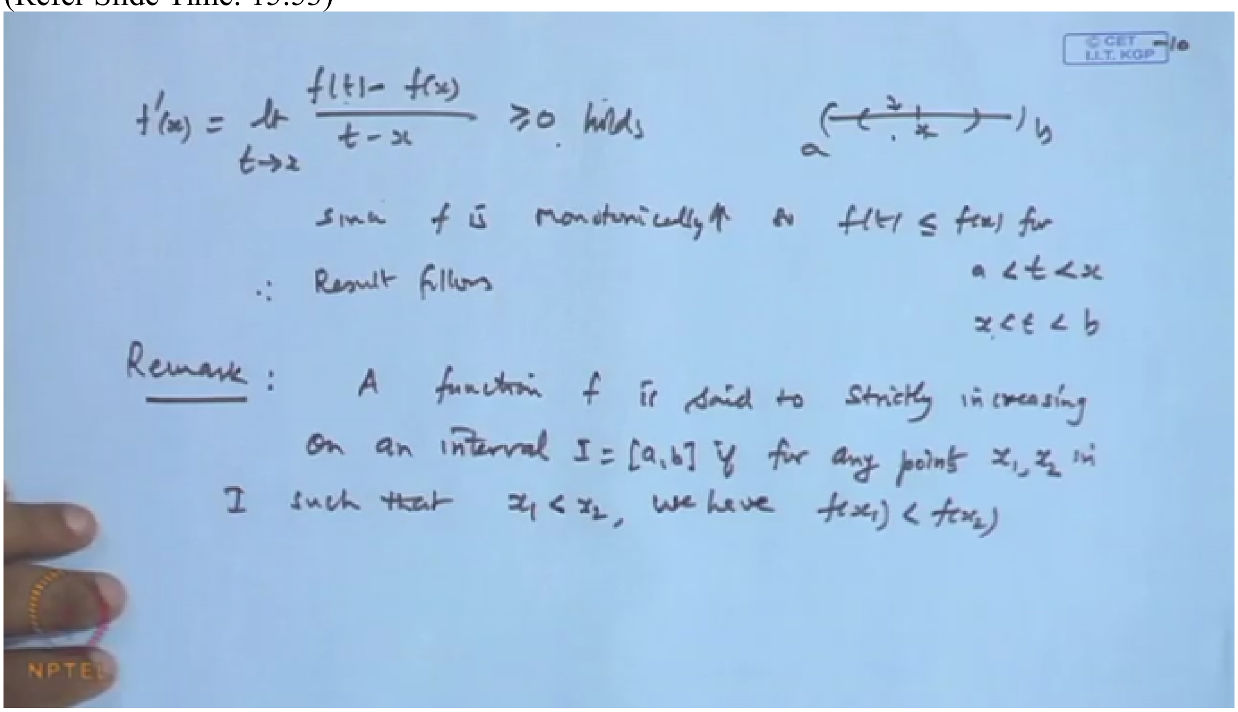

Now when we prove the converse part of the previous result, the note is the converse part of the previous theorem is not true, that is if the function is strictly increasing function that you cannot say that derivative will be strictly greater than 0, okay, but strictly greater than 0. For example if we take the function F(x) which is X cube from R to R is strictly increasing on R, but the derivative of the function at a point 0 is 0, okay, so what we say? The function will not be strictly,
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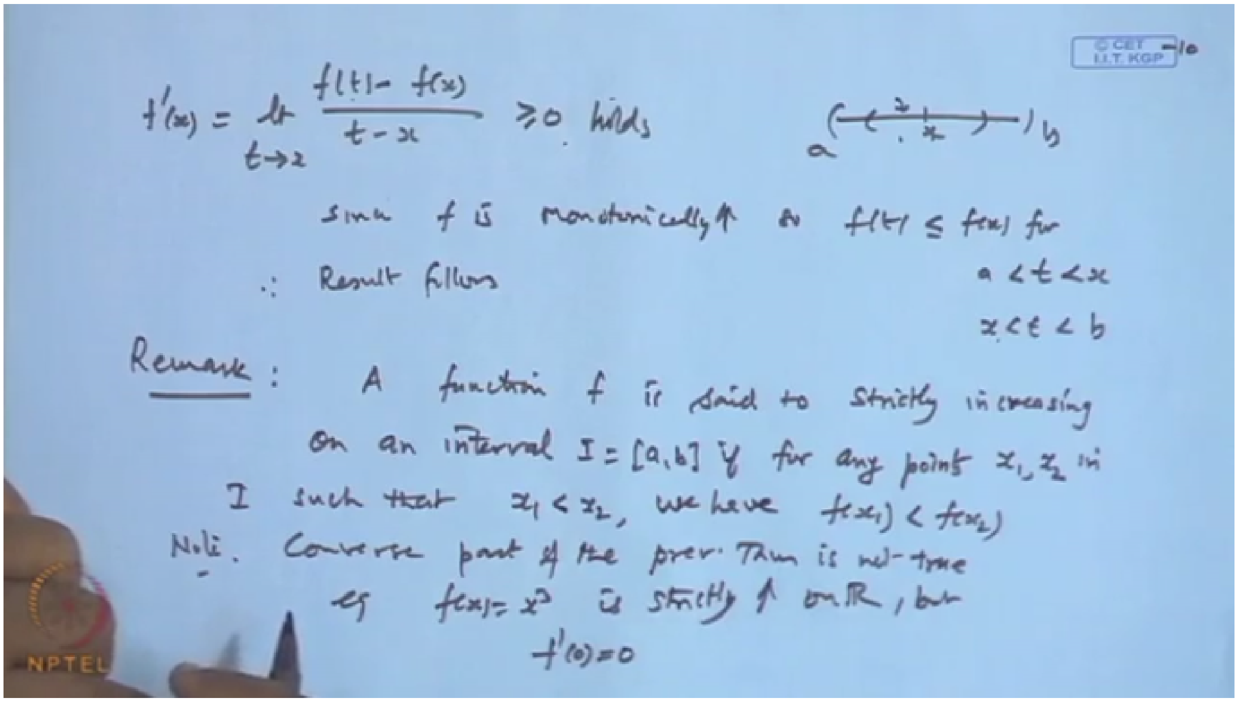

so here the function F prime X is strictly greater than is not satisfied, though the function is strictly increasing function, so that's the important point which I make.

Second remark which I wanted to give it and it is interesting also, the remarks says when we say the function is increasing at a point, then it has no meaning, then it means there exists some neighborhood in which the function is increasing, so when we say a function is said to be increasing at a point if there is a neighborhood of the point on which the function is increasing, on which the function is increasing, okay. So an increasing means that the derivative is strictly positive then the function is interior at this point, but just by looking the derivative at a point we cannot decide, but just by looking the derivative of the function F(x) at the point we cannot decide, it is increasing or decreasing character that is if when we say, that is when we say the function increasing its point, so if the derivative is strictly positive at a point, then the function is increasing at this point is this supposition is false, then one cannot say that the function is strictly increasing, function is increasing at that point.

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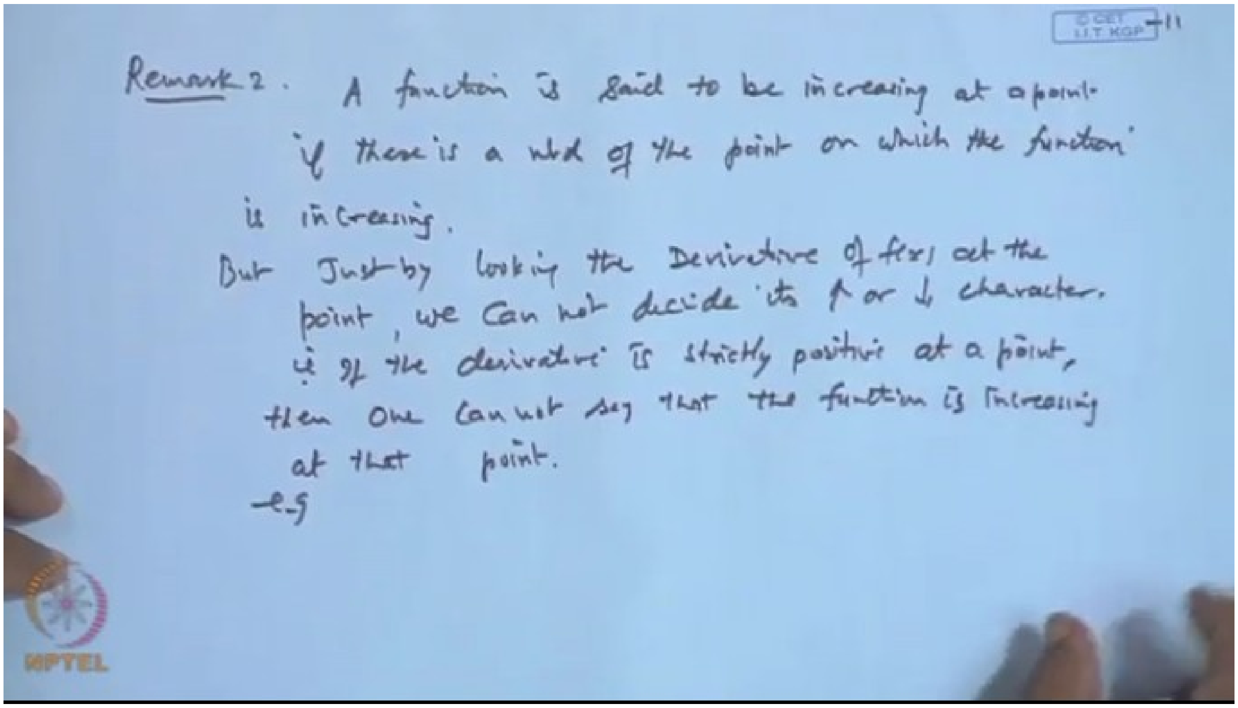

For example suppose I take a function G(x) which is defined as X+2X square sine of 1/X, if X is not equal to 0, and equal to 0 if X is equal to 0, okay.

Now this function the derivative of the function G prime 0, the derivative G prime 0 is G(x)-G(0)/X-0 limit X tends to 0, so that comes out to be what? 1+ limit of this X sine 1/X as X tends to 0 and this limit comes out to 0, so we get G prime 0, so G prime 0 is 0 which is strictly positive, but this function G is not increasing in any neighborhood of 0, okay, it cannot be, this can be seen easily, suppose this is the neighborhood of 0 and I take this neighborhood, consider the point say here the point I am choosing as X = 1/2N pi and also here another points I am taking as, so sorry it is positive, so let us take its okay, let's take another point X = 1/4N+1 pi/2, now both this point clearly choose X such that X is either this or this, then the X goes to, X tends to 0 as N tends to infinity, so these are the point in the neighborhood of 0, so X belongs to the neighborhood of 0 with a suitable radius say delta that I am not, but what is our function? The function, derivative of the function, the derivative G prime X, what is this value? If we look the derivative the function G prime 1/2 n pi, what is the function is?
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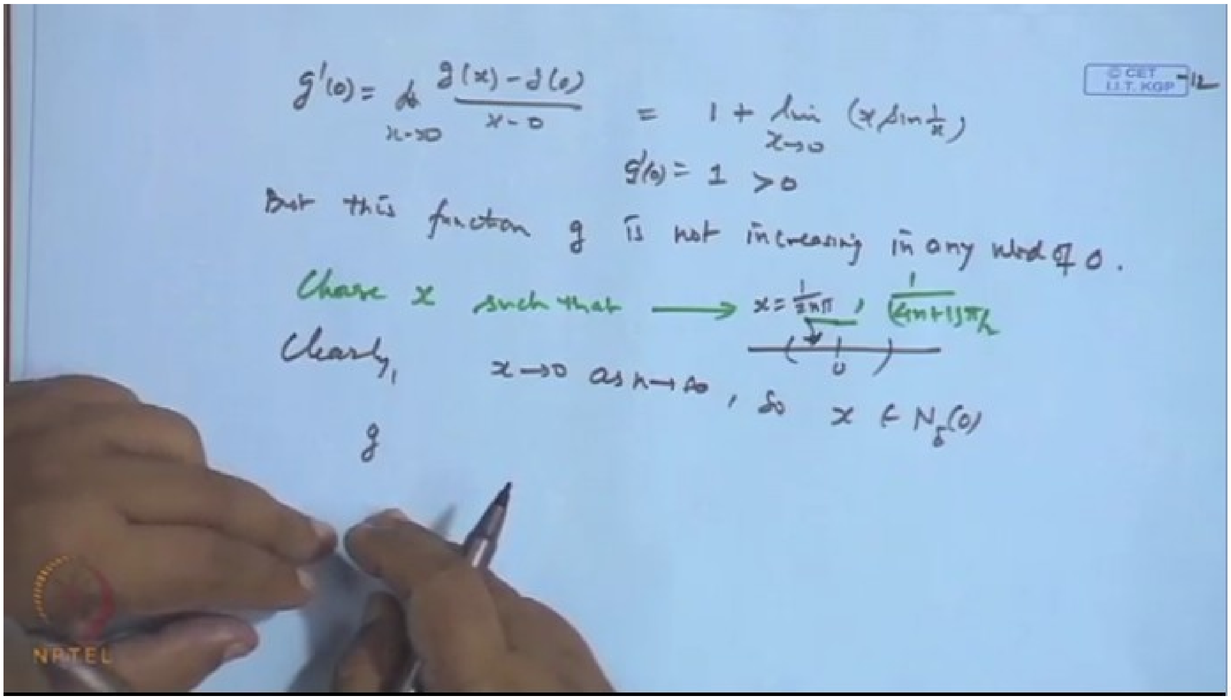

This okay, so when at this point when you find the derivative of this G dash X, G dash X is function because the derivative G is this function, X+2x square sine 1/X when X is not equal to 0, so when you differentiate it directly you get 1+4X sine 1/X+2X square cosine of 1/X into -1/X square, is it not? Now the value this G prime X comes out to be 1+4X sine 1/X and -2 times cos 1/X okay.

Now you see if I take at X = 1/2N pi, the derivative G prime X this is the any integral multiple of sine is 0, so sine 1/X becomes 0, so this part is not there, cos of 1/X when you take the multiple of 2 pi, then it is always be cos 0 is 0, cos 2 pi is 1, so it is always be 1, so value will be -1, so it will be negative for at this point. And if we take X = 1/4N+1 pi/2 then in that case what happens is at this point the function G dash(x) here, G has a derivative, negative, so at this point the function G has derivative, that is G prime X will be positive,

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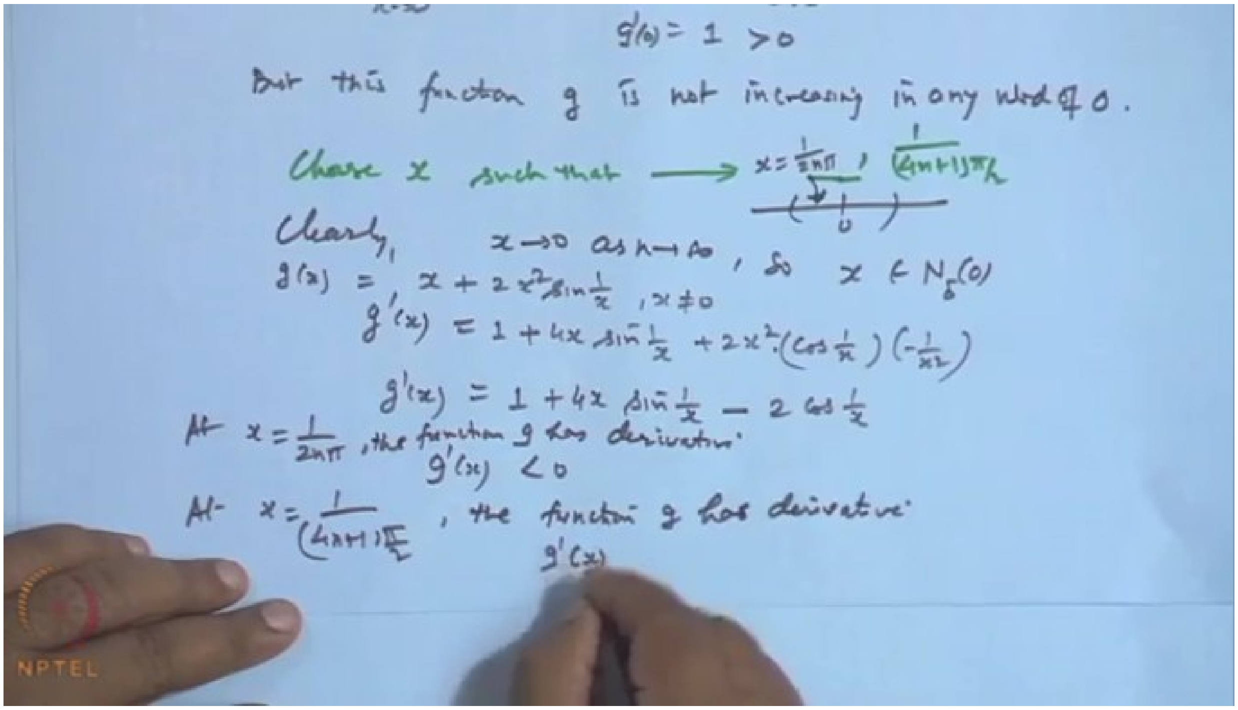

why it is positive let's see. If you look that 4, or multiple of pi/2 with N = 1, 2, 3 so in fact when N is one 5 pi/2, 7 pi/2 and so on, so this will be or multiple of pi/2 this will go to 0 so there.

And here it will give the positive values 1, is it not? So we are always getting the positive value, so this is always good positive when N is a positive integer, N belongs to N, so it is positive, therefore in the neighborhood of the 0 the derivative is negative as well as positive, so neither you can say it is increasing function nor it is decreasing function, although the function which we have taken G prime is 0,

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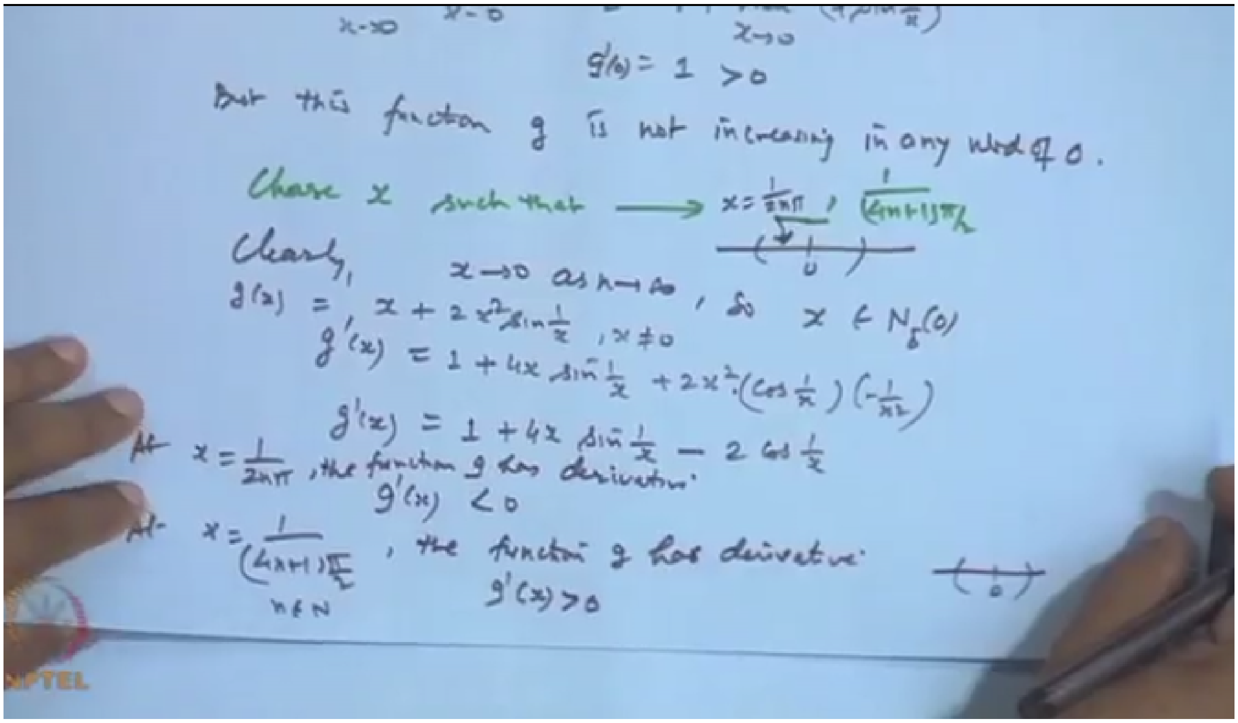

at the point 0, so what conclusion is that? If the derivative of function at certain point is positive or negative we cannot conclude its increasing nature or decreasing nature until we are sure that in the neighborhood the function has a character for increasing or decreasing, that's all. Thank you very much.