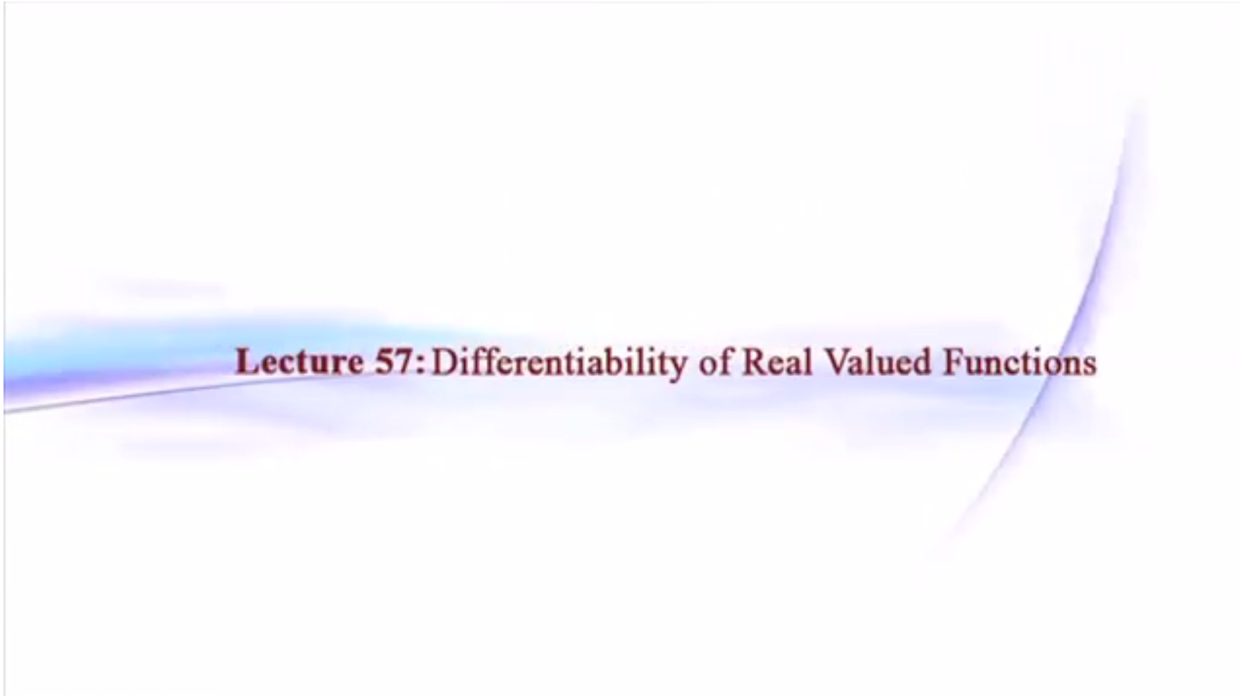
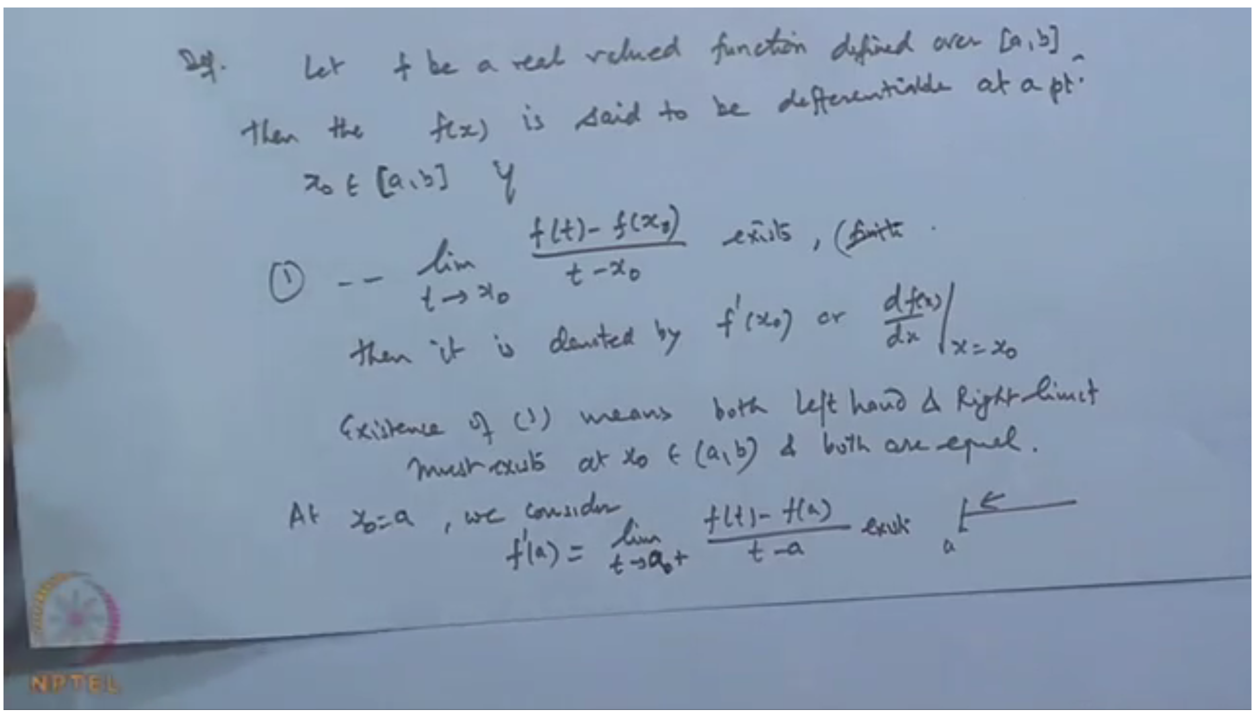
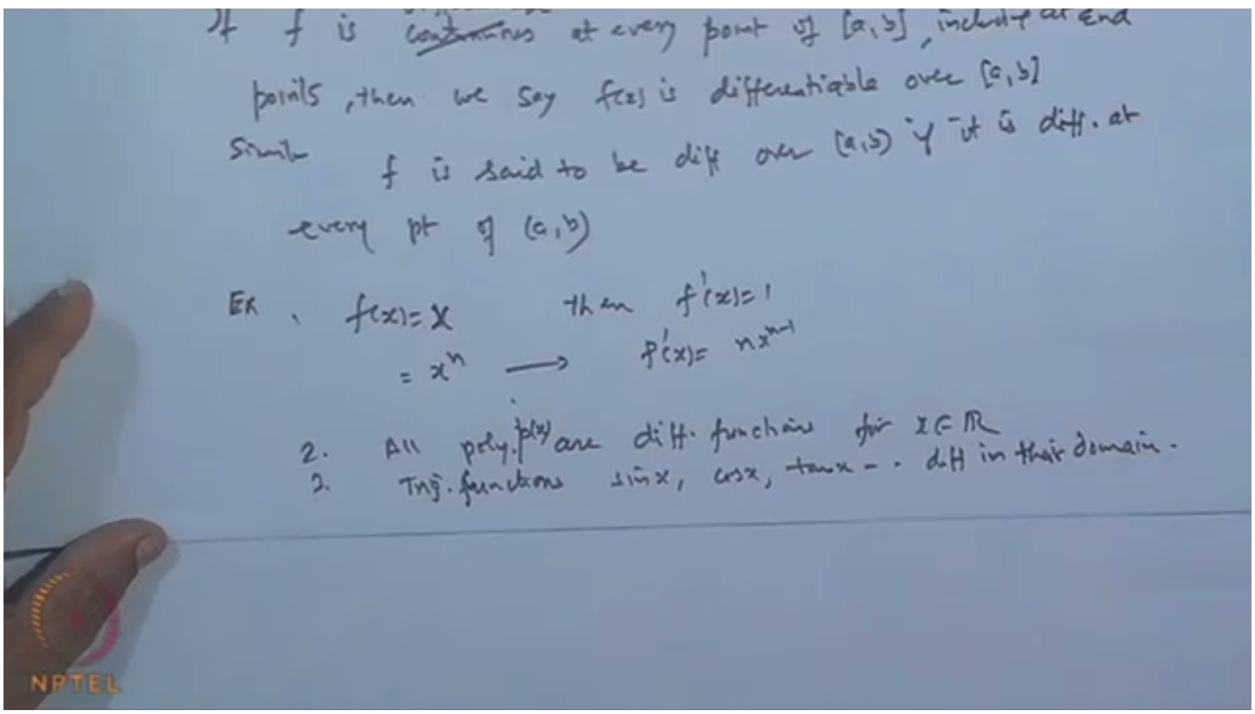




Prof. P. D. Srivastava: Yeah, so today we will discuss in this lecture the derivative of the function at a point as well as on the interval and also the mean value theorem, differentiability or derivative of the function f(x) of real variables. Let f(x) be a real variant. Let f be a real valued function, defined over the interval say [a, b], over the closing term say [a, b]. Then the function f(x) is said to be differentiable at a point say x_0 belonging to the interval [a, b]. If the limit of this function, $f(t) - f(x)_0/t-x_0$ limit t tends to x_0. If this limit exists and finite, exists and finite of course. In case infinite, we say, if the limit is infinite derivative, then the differentiability -- we can also discuss the case of infinity later on only, okay. So let us say just exist.

Then it is denoted by $f'(x)_0$ or we also say $d/dx\, f(x)|x=x_0$, that is also notation to use the derivative at this point. Now here, if the point is coinciding with a or b, then in that case we have a different concept. Now when we say the limit of this exist. That is 1, if this limit exists means, the existence of 1 means, then both the limit, both left hand and right hand limit must exist at a point x_0, which is in the open interval (a, b), and both are equal. Then we say the derivative of the function exists.

At the point, at the corner point, at x_0 = a, we consider only the right hand limit, that is we consider the limit t tends to x_0 -- say, x_0 = a, so here t tends to a_0 + f(t) - f(a)/t-a. So this is an interval (a, b). We will look only this side, the points which are approaching to a from the right hand side, right hand side of a. So this point t -> a_0+, so right hand limit of this, if it exists, if exists, then we say the derivative of the function at a point a.

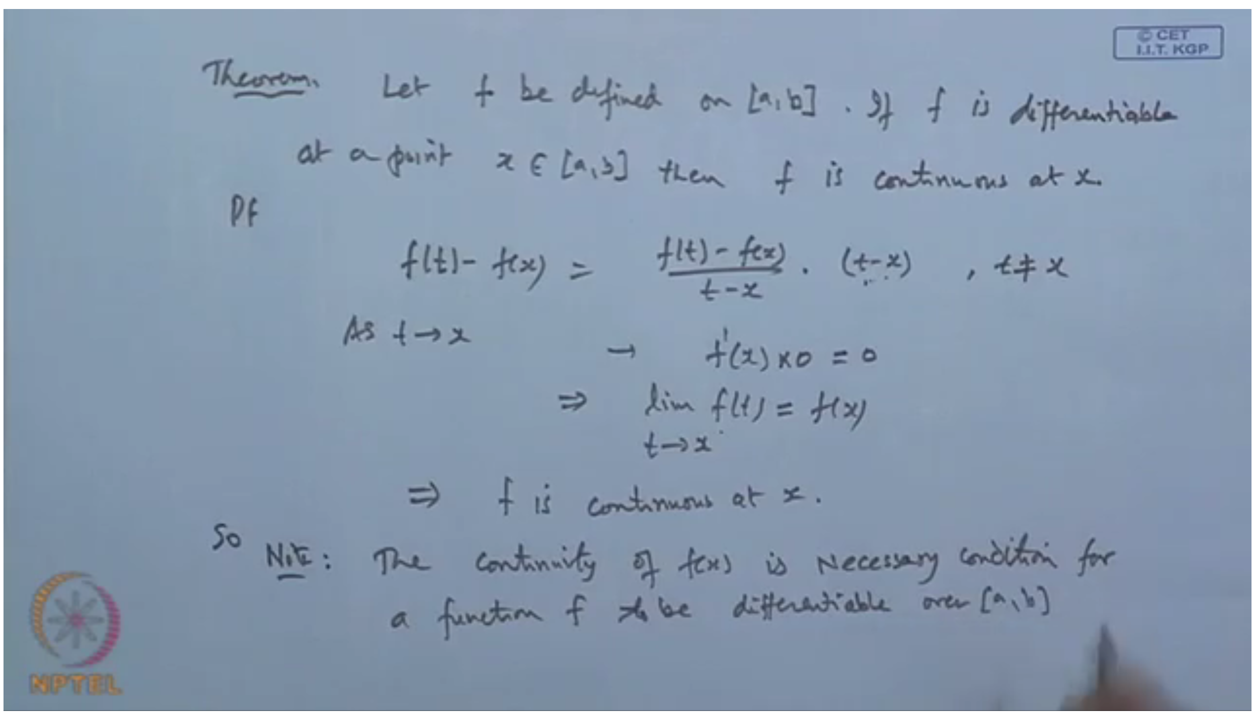


Similarly, the derivative at the point b, that is at the endpoints. This is the endpoint interval means limit t -> b- f(t) - f(b)/t-b, okay, when t -> b-. So if this limit, we are taking the point -- this is b and we are approaching towards this side, left of the b. All the points are taking consideration which are left to b and at this point we are taking the image of this when t -> b-, then we say the function has a limit at the point b.

So if the function f is continuous at every point of the closed interval (a, b) including at the endpoints, then we say the function f(x) is differentiable over the closed interval (a, b). Similarly, we say the function f -- sorry if f is differentiable not -- sorry it is differentiable at every point of the interval including endpoints, then it. Similarly, a function f is said to be differentiable over the open interval (a, b) if it is differentiable at each point, at every point of the interval (a, b) like this. So we can extend this definition to that.

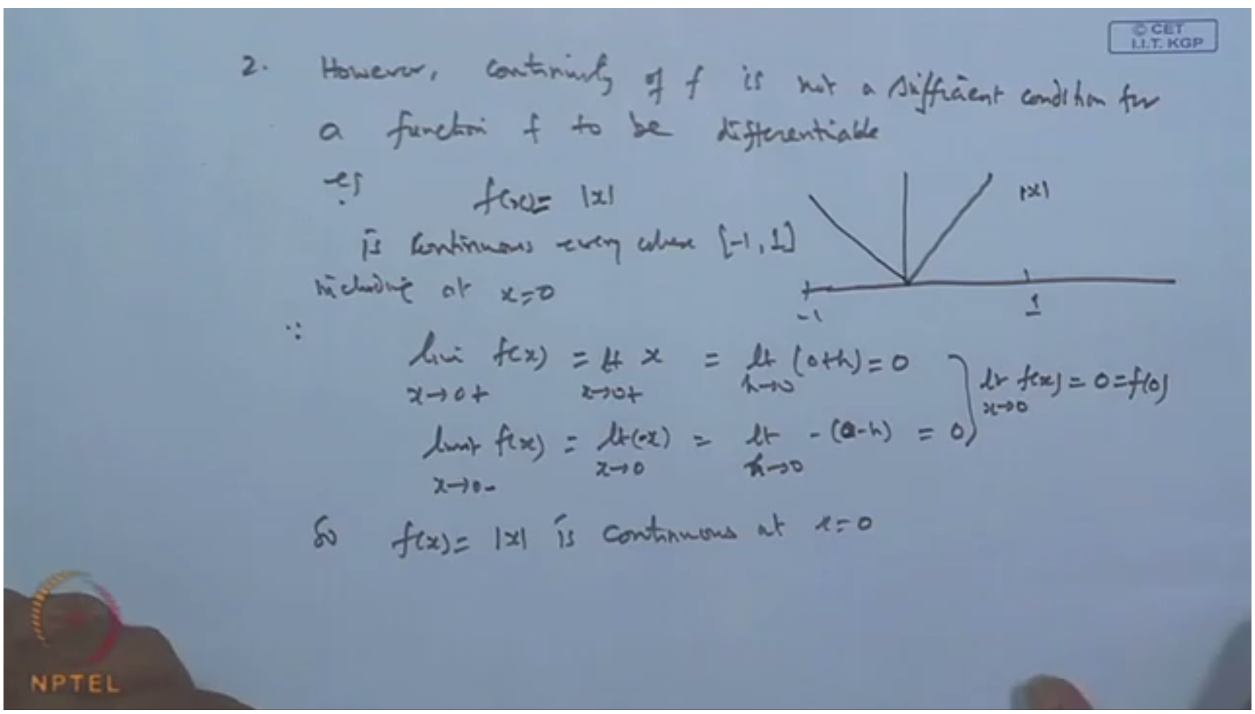
Now using this definition one can easily saw that if f(x) = 1 -- if f(x) equal to -- the derivative will be 0, if f(x) = x, then the derivative of this function will be 1. In general, if f(x) = x^n, the derivative will be equal to nx^{n-1}, which can be used which can be proved directly with the help of this results.

So all the polynomials functions are differentiable functions for each x belongs to r, then polynomial p(x), they are all differentiated functions. Then trigonometric functions, like sinx, cosx, tanx, these are all differentiable functions in their domains, okay, like this. So these are all which is just results which we know in the calculus the differentiability of variable functions and formulas folie, so we are not going in detail for that, but this one.



Now interesting result is that every differentiable function will be a continuous function, that is so let f be defined on the closed interval (a, b), on the closed interval (a, b). if f is differentiable at a point at a point say x belonging to the interval (a, b), then f is continuous at x, okay. The proof is just simple, okay. We want the continuity, so let us take the point t, which goes to a, okay. So f(t) - f(x). This can be written as f(t) - f(x)/t-x . (t-x) where the t is different from x, so obviously this is well-defined. Now as limit t -> x, this gives the derivative f’(x) by definition and this part will give the 0, so total will be 0. Therefore, limit of f(t) when t -> x either from the left hand side or from the right hand side, if this limit exists means both will be equal so it will always give the derivative and this will go to 0 whether we approach t -> x from the left or right. So this will always be when t -> x, the f(t) = f(x). That is the limit exists and equal to the value of the function at the point where the limit is required, this implies the function f is continuous at the point x, okay.

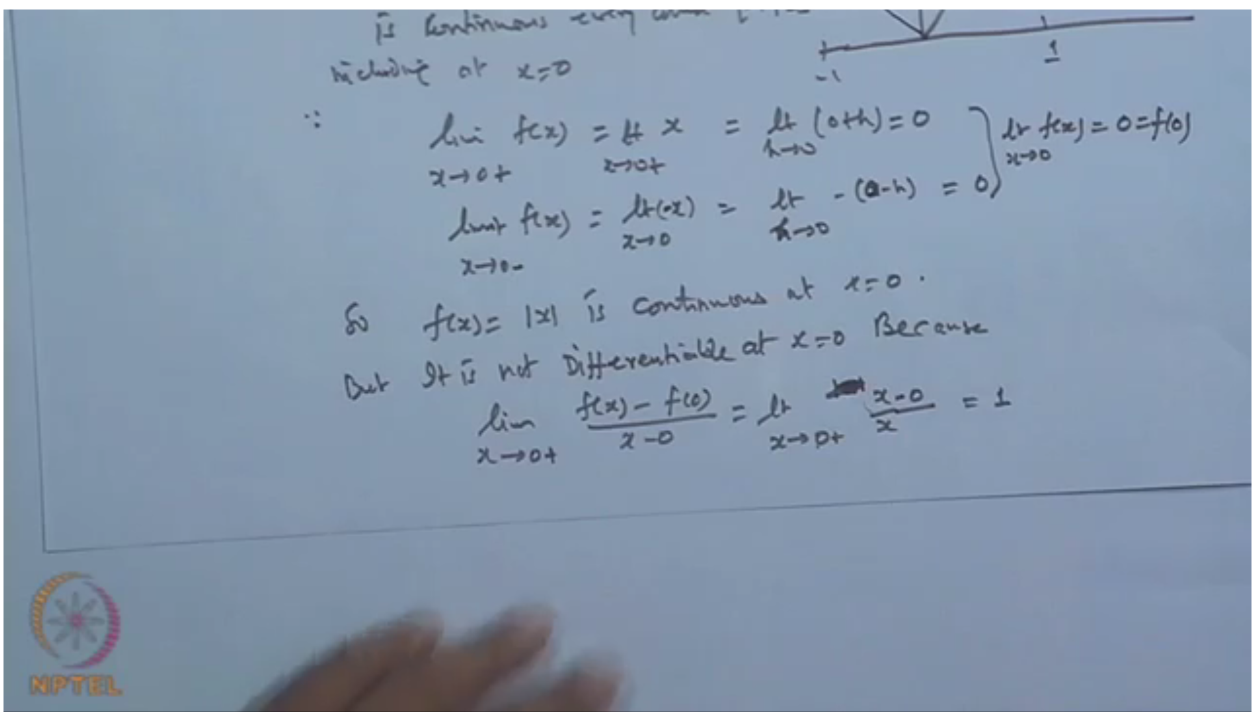
So we have discussed this thing, the differentiability, and for the differentiability just we need only the functions to be well defined in this. So what is required now, if we looked at that this example source or this result source that continuity is a necessary condition for a function to be differentiable, because every differentiable function we are getting continuous, okay. So continuity comes automatically when the function is differentiable, so it is a necessary condition for the functions to be differentiable, is it not. Then we say the converse of this so as a remark or not, the continuity of the function f(x) is necessary condition for a function f to be differentiable over the (a, b) or at any point x belongs to (a, b).



However, this condition -- however, continuity of the function f is not a sufficient condition for a function f to be differentiable. For example, if we look the function f(x), which is equal to |x|, the function, the curve of this function is like this, x is 0, it is 0; when x is positive, then y = x; when x is negative y = -x, so we are getting again this one. So this is the graph for |x|.

Now it's a very smooth curve, it's a continuous curve. In fact, the function f(x) is continuous everywhere, continuous say everywhere. I just take the interval for my convenience is 1 to 1, including at the point x = 0. The reason is because suppose I want this test at the point 0, what is the limit of the function f(x) when x approach to 0 from positive side, the limit of this is nothing but what, f(x) = x. So it is f(x) = x and then limit x -> 0 from positive side. This is the same as limit x -> 0, 0th = 0, and if you look the limit of this function f(x), when x -> 0 from the negative side, then we say it is the limit x, x -> 0, because all negative. So x is minus, |x| is -x, okay, so this is -x. x -> 0, that is x = x-h.

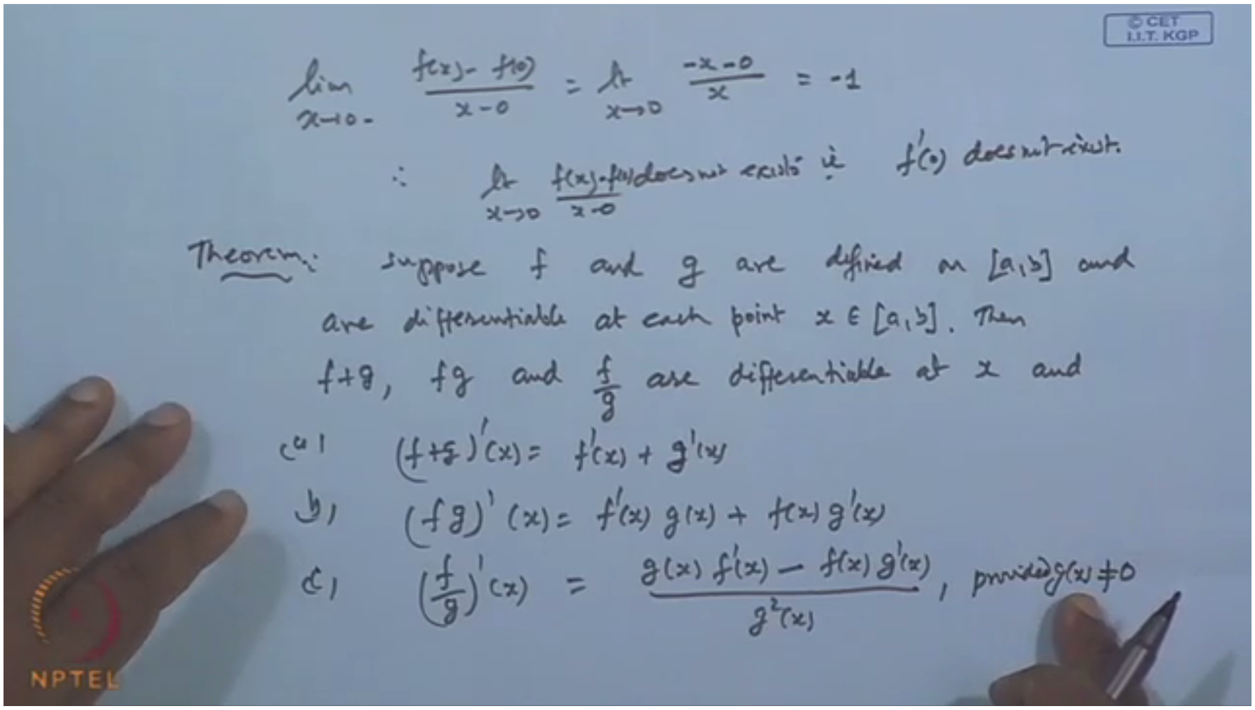
So we can write limit (x-h), a negative quantity, and then h -> 0, x=0. So we get again it is 0 and the value of the function is equal to f(0). So limit exists, limit of the function f(x), when x -> 0 is 0, which is same as f(0), so function is a continuous function at 0. So function f(x), which is |x| is continuous at x=0.



But it's not differentiable at x=0, because the reason is if you find the limit of this function f(x) - f(0)/x-0, x -> 0.

From the positive side, what we get, limit f(x) when positive side is |x|, |x| means it's simply x only, so it is nothing but the x only, because it's positive

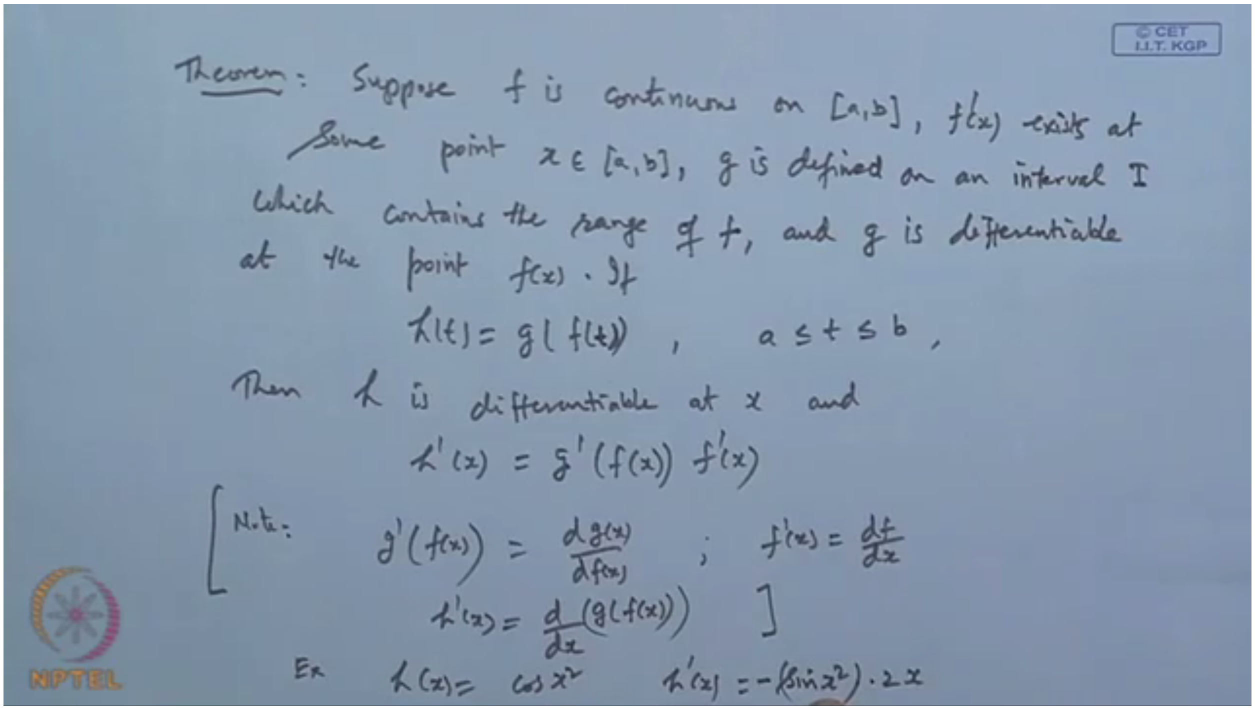
-0 and then divide by x, and x -> 0+, so this 0+ means when x is positive |x| = x, so limit comes out to be 1.



When x -> 0-, the f(x) - f(0)/x-0 = h. So f(x) = |x| when x is negative, so it is -x-0/x when x -> 0, but that comes out to you -1. So what you're getting is the right hand limit comes out to be 1, left hand limit comes out to -1. Therefore, limit of this function f(x) when x -> 0 does not exist. Sorry f(x) - f(0)/x-0 this does not exist. That is the derivative of the function at a point 0 does not exist. So function is not differentiable. It means the next continuity is no longer the sufficient condition, okay. Just by looking the function is continuous, you cannot say the function must be a differentiable function, no, but it is necessary, that is if a function is not continuous, we cannot talk about the differentiability of the function. Function cannot be a differentiable if it is not at all continuous there. So that's a very important result for that.

Then just like a limit case, we have proved that in case of the limit if all continuity -- if f and g are continuous, the addition of the two functions is also continuous, subtext is continuous, multiplication of the two continuous, functions is kind of like this. So similar results here also hold for differentiable functions. The results are, suppose f and g are defined, f and g are defined, on the interval say (a, b), close interval (a, b), and are differentiable at each point x belongs to (a, b), then f+g, fg and f/g are differentiable at x, and the values of the derivative, this x is the same as f’(x) + g’(x), (b) (fg) derivative at a point x is the derivative of the first function f(x) multiplied by the second. This is the product of the derivative of two functions. So derivative of the first function multiplied by second, plus the first function into the derivative of the second.

Similarly, when we say the derivative of the ratio of the two differentiable function at a point x, x is the denominator gx into the derivative of the numerator f’(x) minus numerator into the derivative of the denominator divided by the square of the denominator provided the function g is not 0 at any point where the differentiability is tested in this interval.

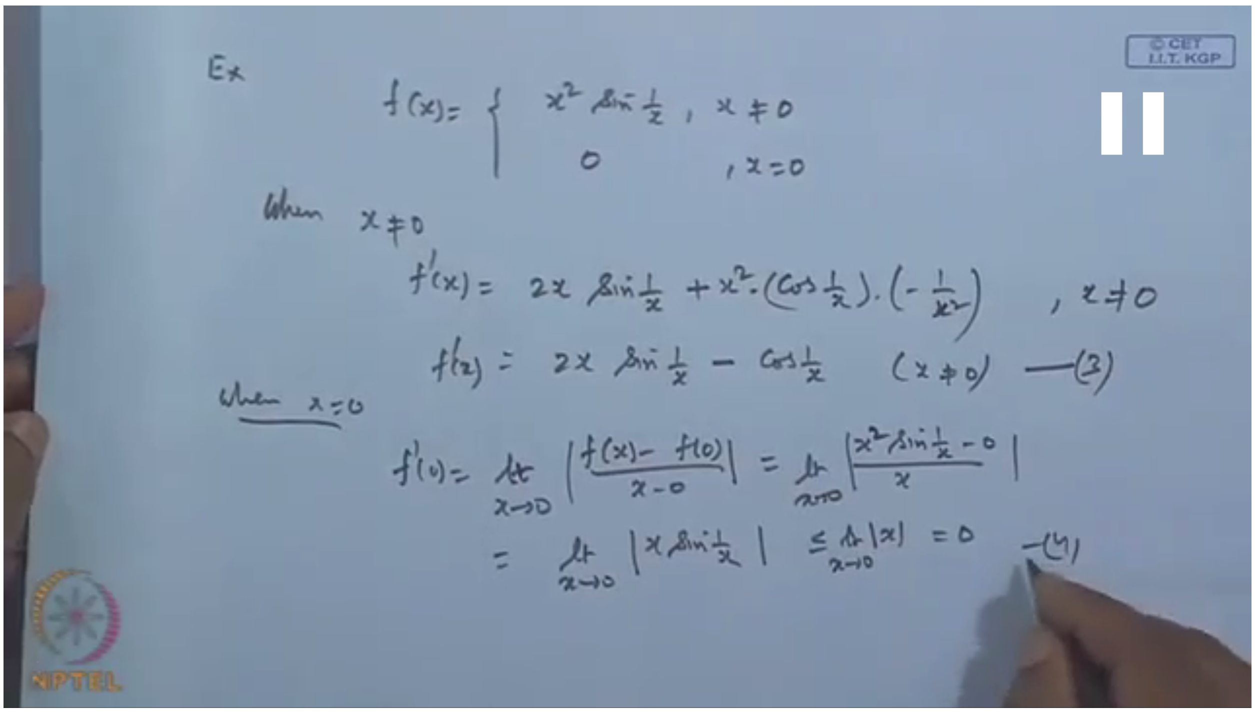


Now for the composite functions also the similar results for as we have in case of contingents, just like function f is continuous at certain point and g is also continuous at the point f(p) where the image of point where the function is continuous, then f composition g is continuous also. So just like a continuity, the same result follows for differentiability. So suppose f is continuous on the closed interval (a, b) and let f’(x) exists at some point say x of [a, b] belongs to [a, b], and g is defined on an interval I, which contains the range of f, and g is differentiable at the point f(x), okay.

Now if h(t) = g(f(t)) for t lying between this a to b, then then h is differentiable at x and the derivative of this is equal to the derivative of g with respect to this f(x) and then f with respect to x.

Now remember here, I will just explain, here we say note, then we say g’(f(x)), it means we are differentiating this g with respect to f, and then f with respect to x. So f’(x) = df/dx. So when we are saying the h’(x), it means we want to differentiate this composite function g(f(x)). This composite function we want to differentiate with respect to x. So since it is a composite function, g is a function of f. So first, we have to differentiate with respect to that composite function and that composite function with respect to x. That will be the idea for us, okay. So that is the value for this, okay.

So, suppose for example -- let me say just one thing. Suppose, for example I'd say the h function, $h(x) = \cos x^2$, suppose I take it this, $f(x) = x^2$ g is cos function. So when you differentiate this h with respect to x, it will define cos with as if $x^2 = t$, a function of t, then cos(a) is function t is the -sin -- derivative of cos is $-\sin x^2$ and then x^2 will be differentiated with respect to x so that will give to 2x, so that is the composition, okay. So this is all.

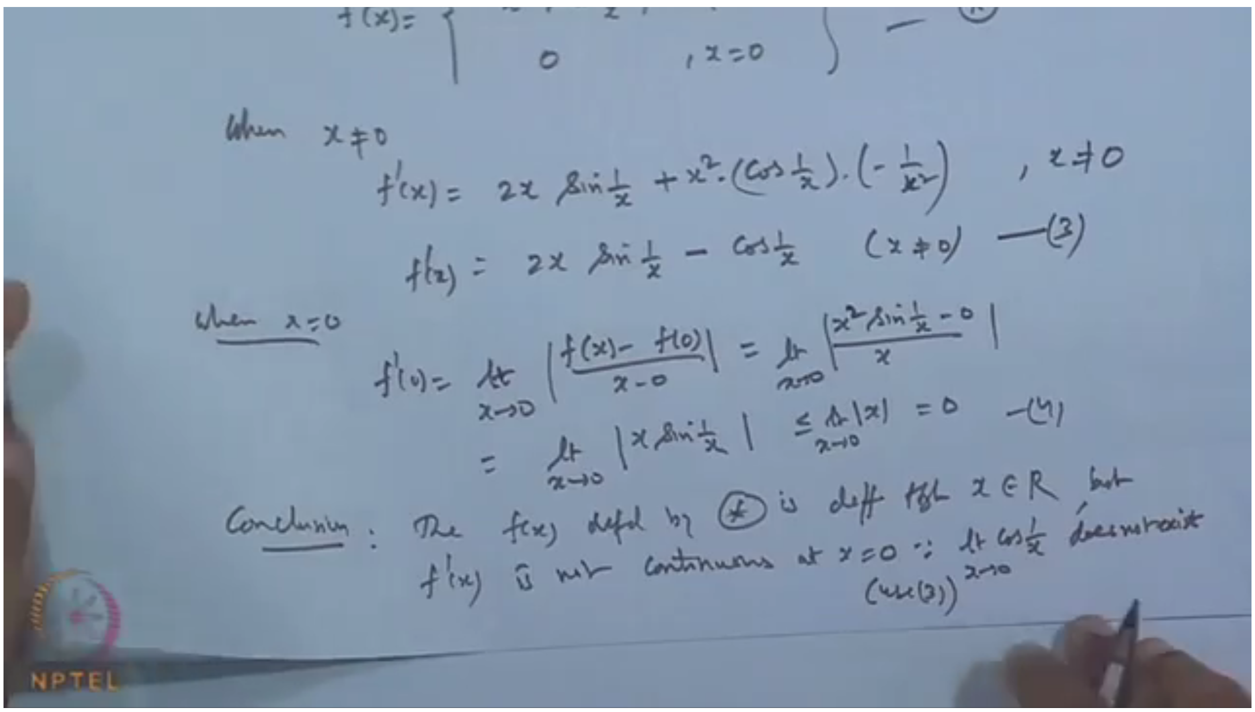


Now let's see the proof of it. -- and 0 when x=0, okay. Now when x is not equal to 0, what you are getting is, the function is defined in this fashion. Now x^2 is continuous function, it's also differentiable function, sin is a continuous and differentiable so far x is not equal to 0. So we can directly apply the formula and when we take $x \neq 0$, you can directly apply the formula product of the two functions and product of the two functions will give the derivative of the first 2x sin 1/x and then the second functions x^2 and the derivative of this is (cos 1/x) . ($-1/x^2$), because this is a composite function now.

Derivative of sin as a function 1/x is cos 1/x, but 1/x derivative is $-1/x^2$, so it is a composite function. So this holds when $x \neq 0$, that is the derivative of the function comes out to be 2x sin 1/x - cos 1/x for x which is different from 0, okay. But when x=0, what happens, the derivative of the function, we have to compute it, because at the point x=0, when you are looking for the derivative, then the value at this point applied the value which are different 0 is given by this one. So find the derivative of this function at the 0 by using the definition that is limit of this f(x) - f(0) over x-0, x-> 0, okay, and that comes out to be what, f(x) = x^2 sin1/x - 0/x when x -> 0. So that is equal to basically limit of this x - > 0 xsin 1/x.

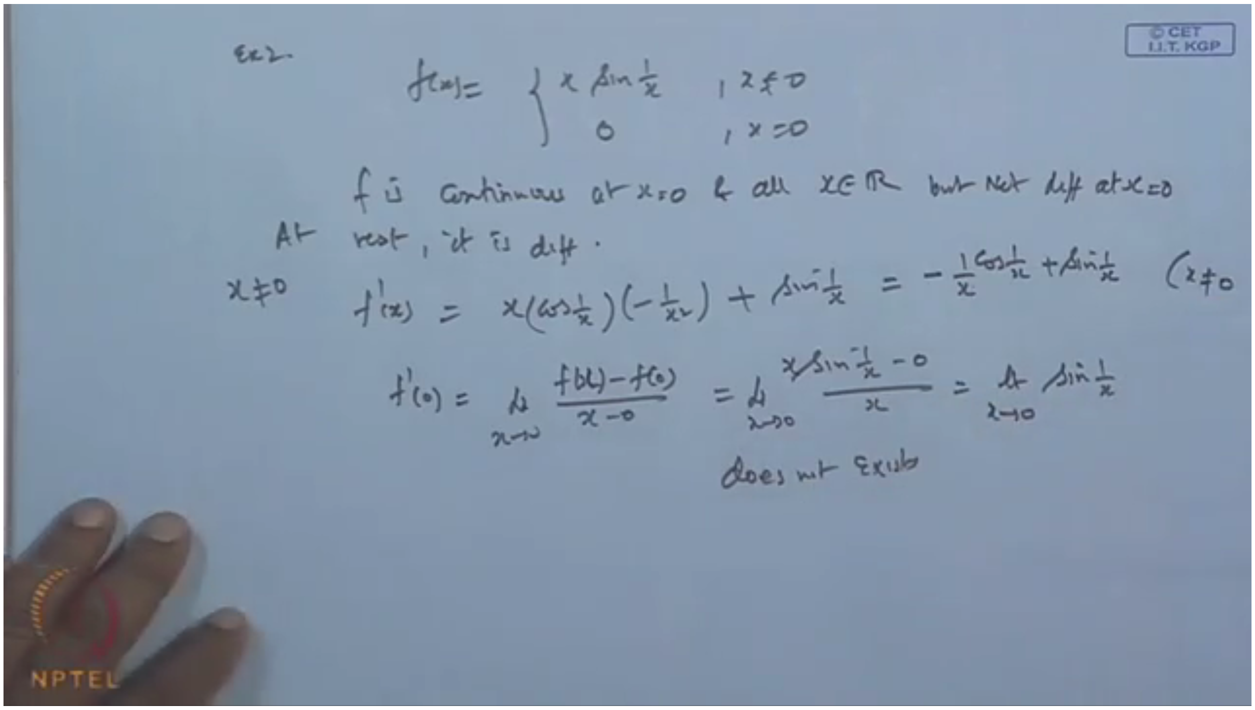
Now if we look that term xsin 1/x is always be dominated by |x|. So basically, when you are taking the mod of this, then this is mod of this, which is less than equal to |x| limit x -> 0, and this limit will always comes out to be 0, okay. This limit comes out to be 0. Therefore, the limit of this exists and the derivative at the point 0 will be 0, okay.

But what about the equation, let it be this equation 3, this is 4.



The function is differentiable, so conclusion is, the function f(x) defined by say here (*), defined by (*) is differentiable throughout x -> R including x=0, but the derivative of the function f(x), if you look, the derivative is not continuous at x=0, why, because when you take the derivative, derivative is involving cos 1/x. So when you take the limit as x -> 0, the limit does not exist, because limit of the cost 1/x as x -> 0 does not exist, which is available in 3, because use 3, okay. Then we have it.

So when the function is differentiable, you cannot say that the right function remains continuous, it may not be continuous. If it is continuous, then there is a possibility of going further, is it no, but if it is not continuous, we cannot talk about the derivative. So in this case, we cannot talk about the differentiability second derivative of the function at a point 0, okay.



If we look the f(x) is similarly xsin 1/x, we will see in a similar way, the limit does not exist. Similarly, example two, if we take the function f(x), xsin 1/x, x ≠ 0 and 0, then function is continuous, f is continuous because at 0, and also for all x belongs to R, in fact continuity follows, but not differentiable at x=0. It is difference. At rest of the point, it is differentiable, because when you take the derivative of x for x is different from 0, you can just apply the formula. First function x derivative of this is cos 1/x . - 1/x^2, then + 1/x. So the derivative would come out to be -cos 1/x divided by 1/x and then + 1/x, when x ≠ 0. Burt when you consider the derivative at a point 0, then as per the limit f(x) - f(0)/x-0, x -> 0. This comes out to be xsin 1/x - 0/x limit x -> 0, which turns out to be limit of 1/x as x -> 0, which does not exist.

So this function is continuous function, but is not differentiable at the point 0. It is differentiable at rest of the point, but not at the point 0, okay.