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NPTEL ONLINE CERTIFICATION COURSE
Introductory Course in Real Analysis
Lecture - 56
Relation between Continuity and Compact Sets

With
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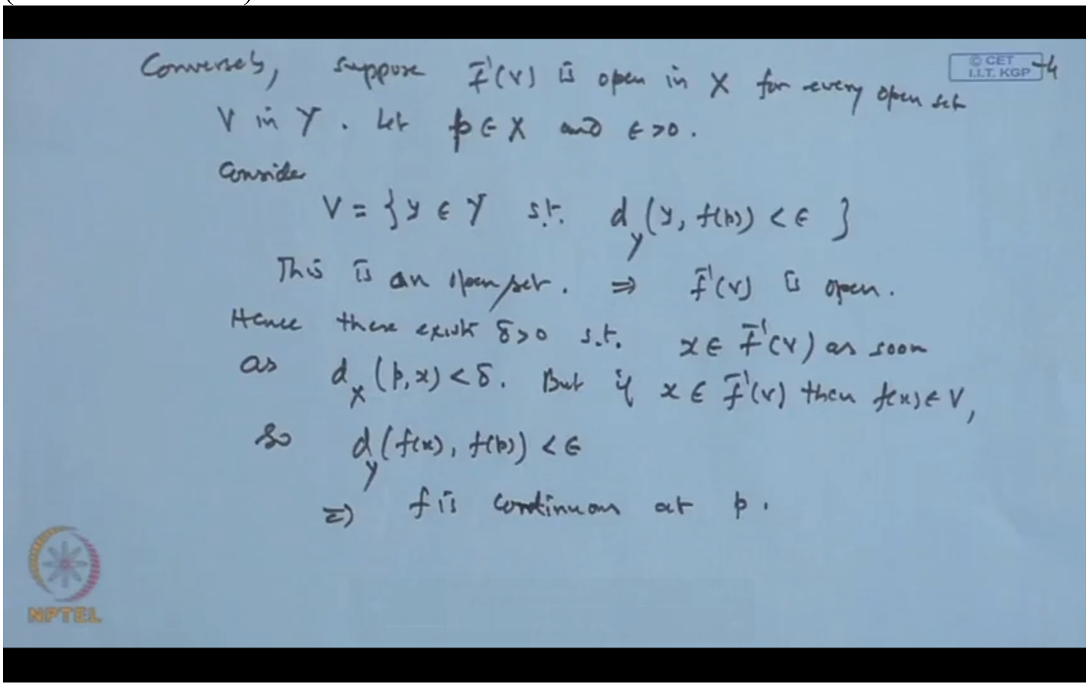

Okay so this particular result will be used for establishing the Relation between the Compactness and Continuity theorem. So, let's see. The result is relation between continuity and compactness. Suppose F is continuous, mapping of a compact metric space X. X is given to be compact, this is important. X is given to be a compact metric space okay. X into a metric Y, metric space Y. Then the result says F of X is compact. So, a very interesting result, the image of a compact set under a continuous function will always be compact. And in fact as a particular case we have seen that if you take any closed interval, the image of closed and bounded interval in R1, because closed and bounded interval in R1 is a compact set. So, image of the closed and bounded interval under F is coming to be compact. And even the arc itself, K cell in in space or in RK space is a compact set; K cell in RK space is a compact set, because RK space the case L is a compact set. So, closed and bounded interval will be compared. But if you take only the closed set okay and not bounded, it may not be a… image may not be a closed set okay, it might be different. That we have the various counter examples we have seen.

Okay so in order to show it is a compact set what we want to prove it that every open cover of this has a final subcover. And it's already given X is compact so with the help of this we will establish this result. So, let us suppose V alpha be an open cover of X okay. Now once it is open cover, it means each element -- each point of V alpha when alpha is the index set it's an open set. And since F is continuous so by the previous result the inverse image of the open set must be open. So, F inverse of V alpha for each alpha is an open set as F is continuous function okay. So, this is open, that must hold everything okay. Now what is given? It is given that X is compact; so, any open cover of X will have a finite subcover. F inverse V alpha is an open set in X, so every alpha is an open cover so correspondingly we can say this will behave in the alpha belongs to in there is if we choose this as an open cover for X then since X is compact there must be a finite sub cover for it. So, compact and the sequence of this is an open cover of X. So, there exists the finite sub cover, there exist indices say alpha 1, alpha 2, say alpha n, such that the finite union of this: F inverse V alpha 1 Union F inverse V alpha 2 Union F inverse V alpha n, will cover X, because X is compact okay.

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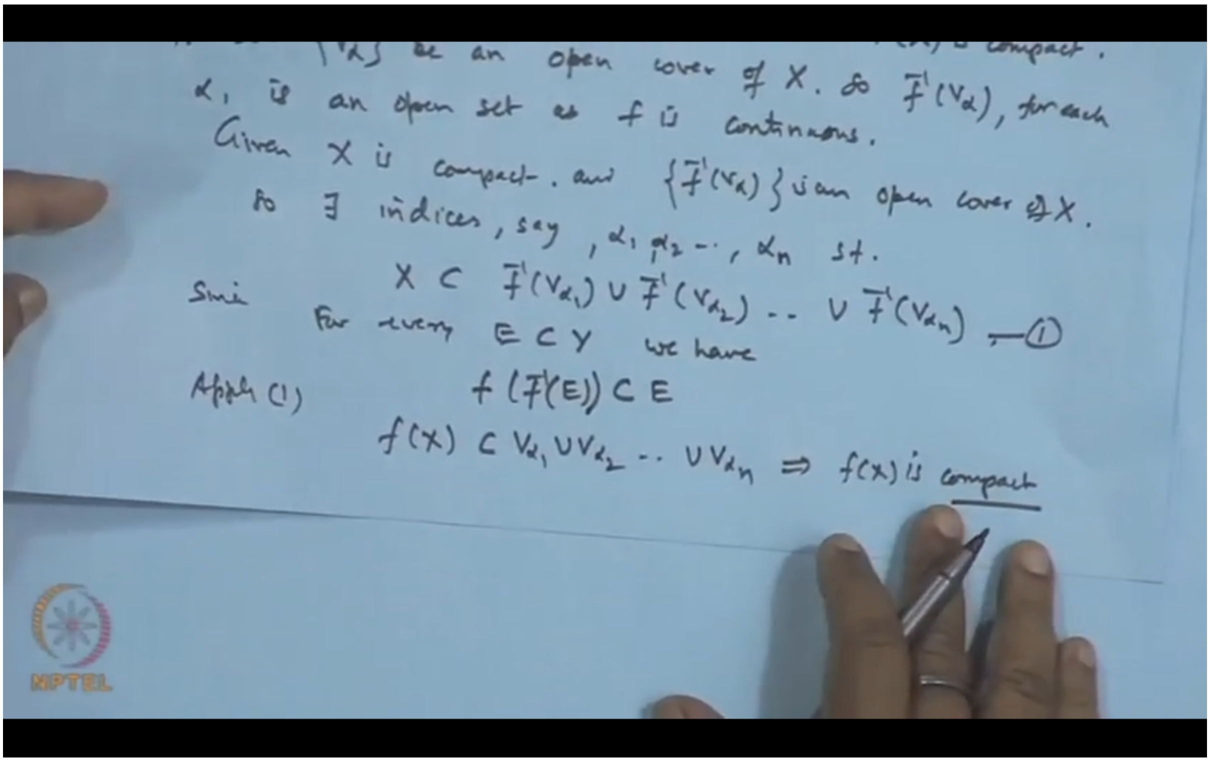

Now since our set E since for every E, which is a subset of Y, we have this F of F inverse E F of F inverse E is contained in E. This is true okay. If E is a subset of F this may not be true, it is the opposite direction, but this is true when you need a subset of Y okay. Then using this on one, so apply on one. What we get F of X is contained in V alpha 1 Union V alpha 2 Union V alpha n. So, FX is covered by a finite union of the open interval open sets, so from this open cover we can identify a finite circle which covers the FX. Therefore, FX is compact okay. So, that's a very interesting.

Now the next result we will show it the Relation between Continuity and Connectedness. So, this too we will write as theorem. The theorem is if F is a continuous mapping of a metric space, capital X into a metric space by into a metric space Y. And if E is a connected subset of E and E is a connected subset of X, sorry, subset of X, is a connected subset of X. Then F of E is connected. If F is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X then image of this connected subset will also be connected. Let's see okay. So, assume that contrary. Suppose FE is not connected, then we will reach a contradiction. So, if it is not connected means that is FE can be expressed as the union of the two sets A and B. Where A and B are non-empty, separated subsets of Y. That is A bar intersection B is empty, A intersection B bar is empty. That's by definition okay. So, that is of this part so when A bar intersection of this and this is now, okay.

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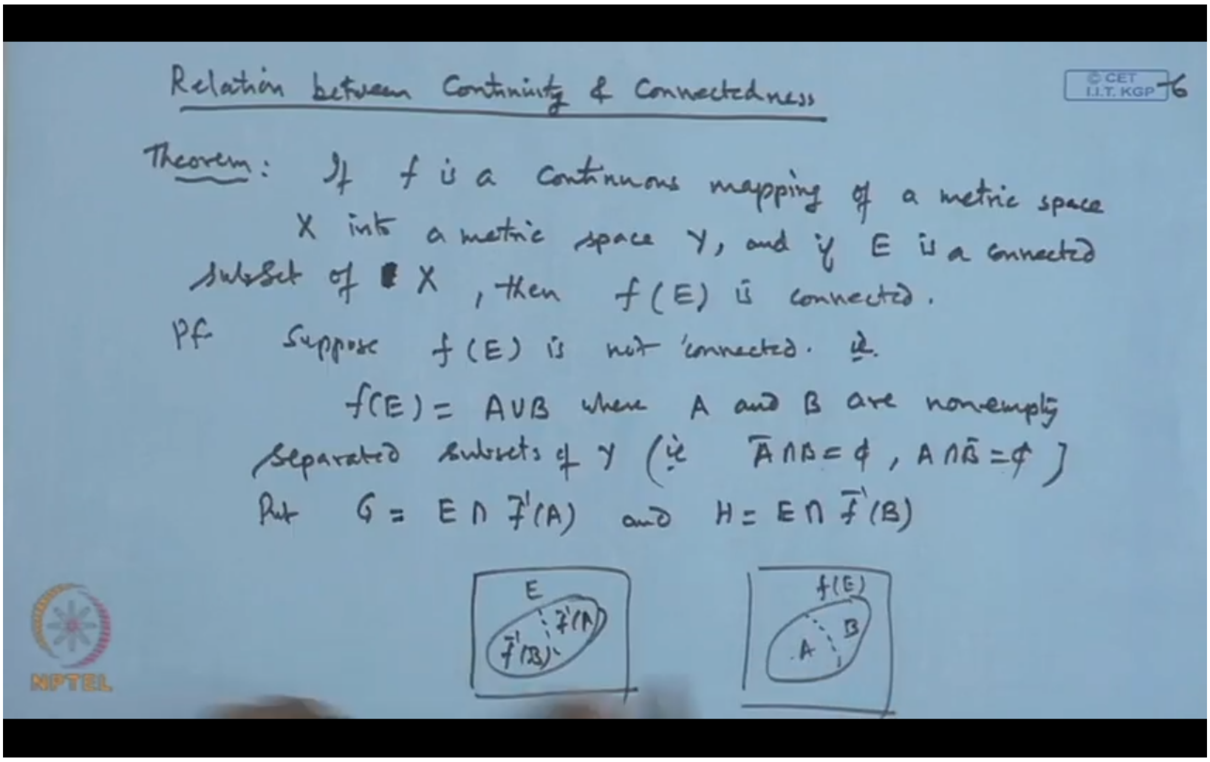

Now let us take then G we can write it put G as the E intersection F inverse A. And H as E intersection F inverse B, okay. Now if we take this, what is our FE? This is set E, this is it E. Here this is FE. We assume FE is not connected; it means there are the two sets, say A and B okay, such that this condition is satisfied. Now find out the inverse of this. So, we are getting F inverse A here, F inverse B here. Now if I find the intersection with AE intersection F inverse G, A and E intersection F inverse G. Then obviously G and H will be nonempty set, because F we have already assumed is a not connected set so there are the non-empty sets in B and we are non-empty, whose Union is FE and they are separated. So, inverse is intersection of this beauty non-empty. So, clearly G and H are non-empty first thing okay. So, there is no problem in it. And also, the union of GH is nothing but E, union of this is nothing but E, okay. They're non-infinite, means neither G is empty nor H is empty okay.

Now since our A is always contain its closed set, it's always contains its closed set. And F inverse image of this A closure -- which is a closed set, A bar, A bar, which is a closed set. So, inverse image of the open set is open, it can be extended to the closed set, inverse image of the closed set is closed if F is continuous. So, F inverse AG will be an inverse image will be a closed set okay. So, that's not. So, if it is closed and G is there, so we can say G is contained in; because G is already contained, G is a subset of this F inverse a. From here G is a subset of F inverse A, so if we replace this A by a bigger closure of it obviously G will remain as a subset of this. So, we can say G is further -- G is contained in F inverse A closure okay that's it.

Since this is closed set so obviously all the limit point of this must be here, so the limit point if I take then of G closure, if I take all the point set the point of G including the limit point then obviously it will also be contained inside it. So, this is correct. So, once it follows, then what does it mean? F of G closure is contained in A closure; this is one thing so let it be one, okay. Now what is our H? The H is this: E intersection F inverse B okay. So, F of H, since H is F intersection F inverse B. So, F of H -- sorry this is E, this is E intersection -- so F of H will be what and what is our E? E is -- we have taken this one, E is connected FE, sorry, F of

E is this: A union B. And A union B they are separately set to satisfy this condition, so if you find the F of H then it becomes FE intersection B. So, that is equal to FE intersection B and then when you find the intersection with this obviously it comes out to be B. So, F of H will be B.

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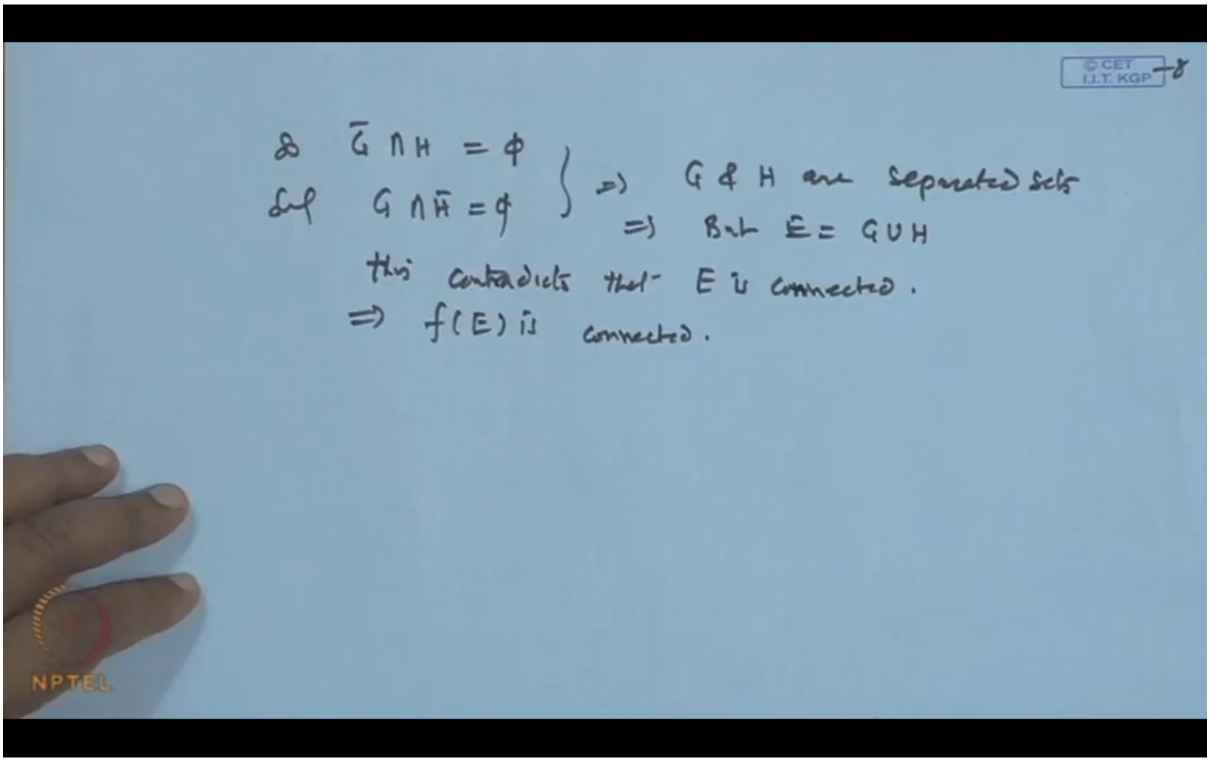

Now F of G inverse, G closure is contained in A bar. A bar intersection B is empty. A bar intersection B is empty and A bar intersection B is empty because A and B are separated set. These are separate set. So, this is empty set; therefore, when you find the intersection of G bar with our H what happened? Intersection of G bar with H that is nothing but an empty set; so, intersection of G bar and H is empty. Why? because G bar is contained in this: F inverse A okay, H is B. F inverse a is contained in A bar and this G bar is contained in F inverse A and this H is E intersection of this. So, when you find the G bar intersection H they will be disjoint and empty. So, we can put this. Similarly, we can say G intersection H bar is empty; so, this shows G and H separated sets. But E is the union of G and H is it not? So, it's a contradiction so it's not possible. Therefore, this contradicts that E is given to be connected set. E is connected. And this contradiction is because of a rogue assumption that we assumed that FE is not connected, so this implies F of E is connected okay. If so cannot because why it is? Because G bar is contained in this okay and H is B, where these two are disjoint; this two are so therefore the intersection with empty set okay so that's what. So, this shows our relation between the this one.

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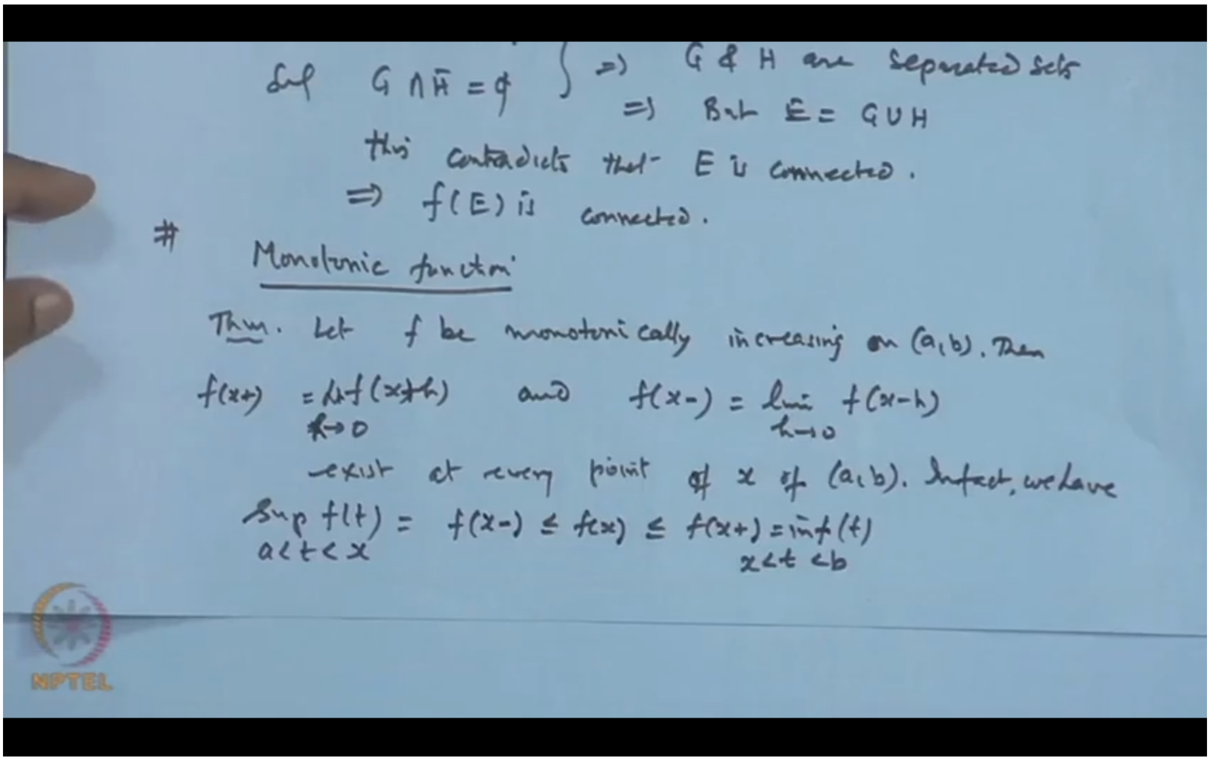

Now it's a similar relation we can also estimate for monotonic functions. So, the results between some monotonic conditions. So, let us see the monotonic functions. If F is continuous then what? So, the first result we'll see. I will not drive the result, let's just see. Let F be monotonically increasing function on the interval say AB. Then the left right-hand limit of this FX when X tends to plus side. X tends to plus, that is FX plus that is this is the same as X plus H, X tends to 0. So, FX plus and FX minus left-hand limit of this, that is equal to limit F of X minus H when S tends to 0. And then this exists if F is a monitoring increasing function left-hand limit and right limit will always exist. At every point of X, of the interval AB. And in fact, we have this inequality that supremum of F of T when T lies between A and X will be equal to the left-hand limit which is less than equal to FX, which is less than equal to the upper limit and which is equal to infimum of FT; when T lies between X and B. So, over the interval this scenario is there. Means the value of the function FX always lies between the lower limit and the left-hand limit and right hand.

Now if both are equal, then we say the continuity follows, if they are not equal, the point will be the point of discontinuity. The question is how many such points is possible over in the interval AB if the function is monotonic increasing function or monotonic decreasing function? Will this set of points where the monotonically increasing function is exist there in that, will it be countable or uncountable? The answer is that set of all points where the monotonically increasing or decreasing function is not continuous, forms a countable set. So, that is the main result which I wanted to show.

Okay. So, let us see what is that monotonically because this will help in getting the result, so I hope that this result we can prove it or let me see just proof of this result very fastly and then we can go for it okay. Suppose what is given is, this is given that F is a monotonically increasing function. What do you mean F is monotonically increasing? It means what? That is, if X and Y are the two points and if X is less than Y, then the corresponding image that is increasing on the interval AB means, that if A is less than X less than Y less than B, implies that F of X is less than or equal to FY. If we say it is strictly increasing then the sign strictly will follow, otherwise similarly for a decreasing the order reverses okay. So, now if you take this F to be monotonically increasing function, then this interval AB is this; now here is the point X so first I am taking this point T which lies between A and X. And then here I am taking and at the point when it lies here.

So, over this interval the function is an increasing function, so what will be the upper bound? Upper bound will be X, so in fact the least upper bound will be there. So, the set of according to the hypothesis or this, the set of numbers F of T, for T lying between VR is less than T less than X; this number is bounded above by FX. Because it is the monotonically increasing sequence and therefore it has at least upper bound, say A. A is the least upper bound okay. So, clearly there hence A will be less than equal to FX, this is true okay. Now we have to show the same okay. Now let epsilon greater than 0 is given. So, if I choose a number, it's slightly lower than A -- say A minus epsilon, then there exist a delta so that A minus X is so that A will not behave in -- A minus epsilon will not behave in upper bound.

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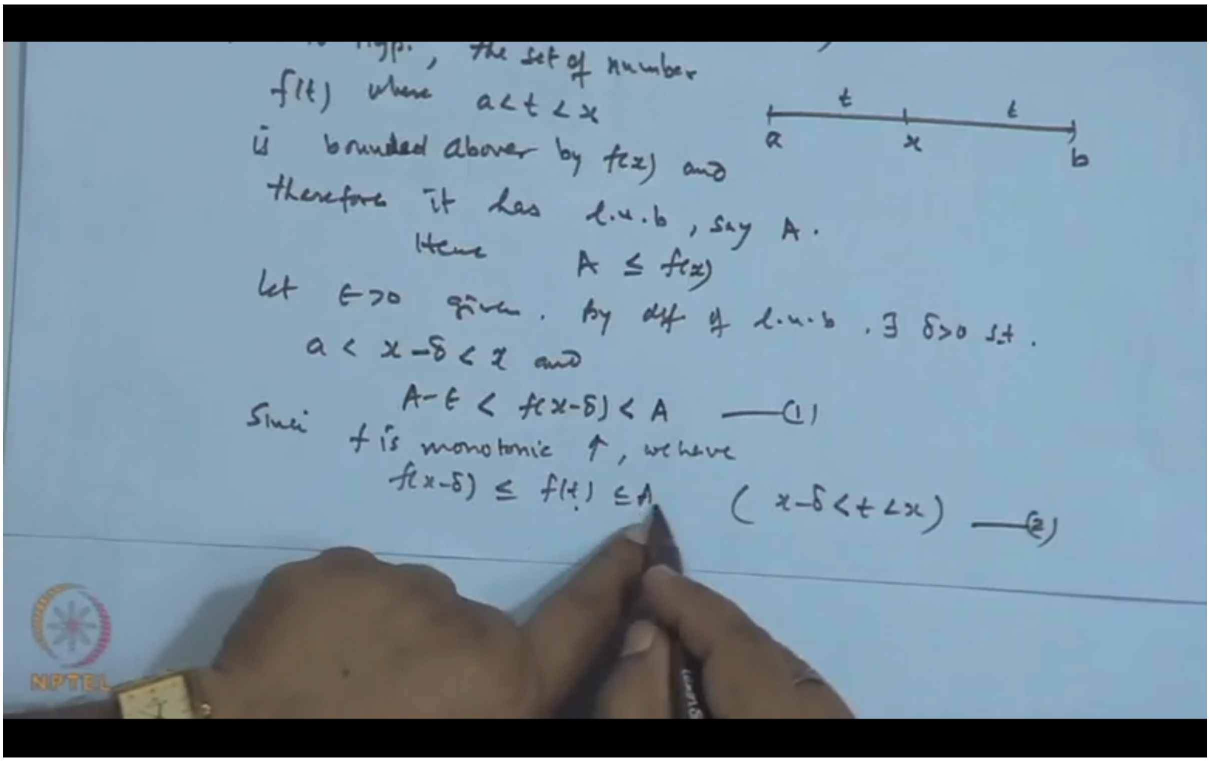

So, by definition of the upper bound -- by definition of least upper bound, there exists a delta greater than 0 such that for all X lying between A less than X minus Delta, less than X and this condition, and this A minus F will remain less than F of X minus Delta which is less than a because A. Because A is a least upper bound so if I take a number slightly lower than this, then there is a point lying X minus Delta, a point will be available such that the functional value will lie between them, okay. Now further since F is monotonically increasing function let it be equation one, okay? Let it be equation 1; since F is monotonic increasing function so we have that F of X minus Delta is less than equal to F of T is less than equal to A whenever X minus Delta is less than T less than X. This is by definition monotonic, let it be equation 2, okay.

So, if we combine 1 and 2 what happened? A minus X is less than this which is less than FT which is less than equal to A. So, combined 1 and 2 can you not say like this. So, combining 1 and 2 we get -- combining 1 and 2 what we get is, that mod we get modulus of FT minus A remains less than say epsilon whenever X minus Delta is less than T less than X okay. So, from the X-axis point here and here this is the neighbour -- left-hand neighbourhood of X, so we are approaching from this side, the T is somewhere here. So, what you're doing is the image of FT under A, with A is less than epsilon. It means the left-hand limit of the function when X approaches the plus 2x from the left-hand side is A, so this shows this implies the left-hand limit of the function is A. That's one. Similarly, we can prove the other side. So, similarly we can show if A is less than X less than Y less than V then we can see the right-hand limit of this is the infimum of FT -- where T lying between X and B, and this is the infimum of F of T when T lies between an X and Y, okay. So, using this and others we get the results okay. So, the last we obtained by combining these two over the set A, and then if we take over the set say AY, then we get this one F of Y minus is the supremum of FT, T lying between A and Y which is equal to supremum of FT when T lies between X and Y. So, this follows come comparing bigger than it.

But this result is interesting because the corollary of this. The corollary says monotonic functions have no discontinuity, no discontinuities of this second kind. Where the limit does not exist, but limit will always exist. Lower limit right-hand limit or left-hand it will always exist; they may not be equal, so the point of discontinuity may be there, but it is not of the second kind. This is clear from here. Now if the limit does not exist then there will be a point of discontinuity. And how many points are there? This can be shown by this following theorem. The theorem says let F be monotonic on the interval AB, then the set of points of AB, at which F is discontinuous is at most countable. Let's see the proof of it. Suppose for the sake of A… suppose F is case 1, when F is increasing function -- monotonically increasing function -- increasing monotonically, okay? F is a monotonic increasing function. And let E be the set of all points at which F is discontinuous. Now this point cannot discontinuity cannot be of second type, only first kind discontinuity there or removable discontinuities, okay?
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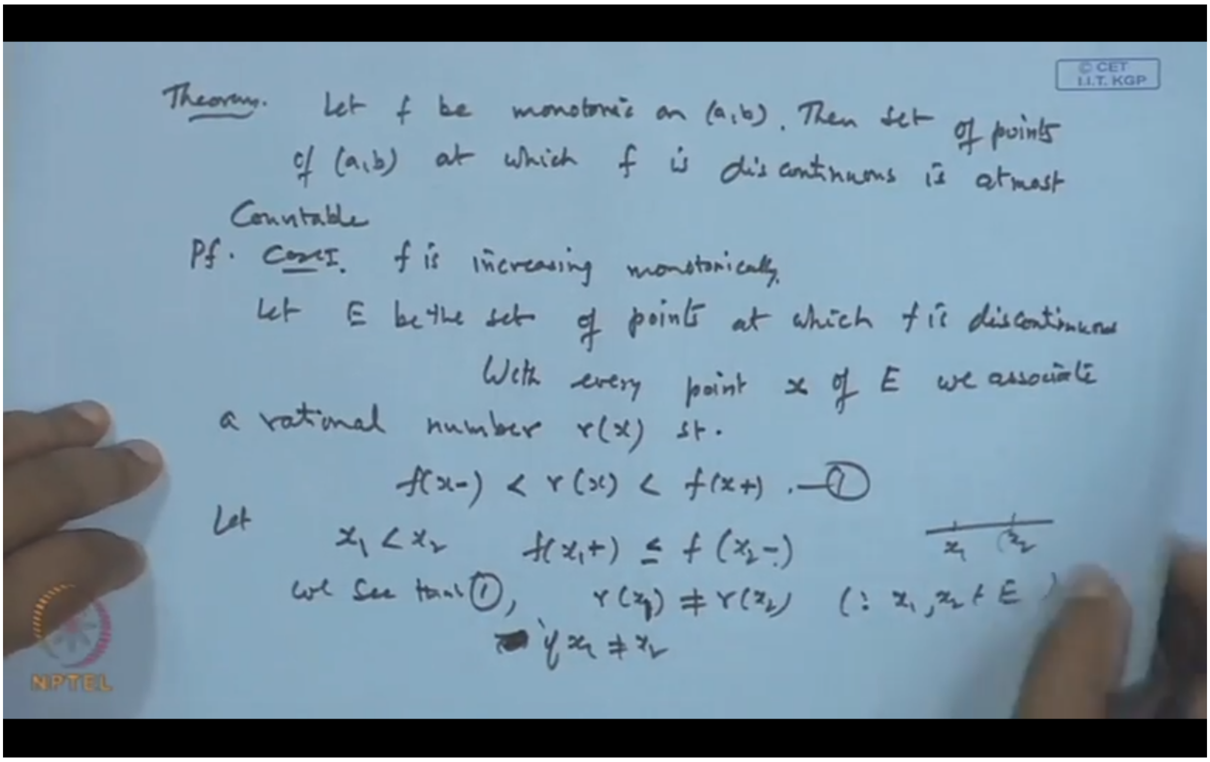

So, with every point X of A, now with every point X of E -- the discontinuous point -- we associate a rational number say RX, such that the lower limit left-hand limit of the function at the point X is strictly less than RX is the strictly less than A. It is of the first kind where the limit exists but both are not equal. So, we can identify a rational number in between so corresponding to each point we can identify rational number which is 2 okay? Now this is a - there is only one point okay, so let - since X is less than - let X1 and X2 are the two point okay. Now when X1 X2 are the two point then if the image of this. The right-hand limit of the X1 will -- if X1 is less than X2, then the right-hand limit of this will be either less than or equal to the left-hand limit of this; because X1 lies here X2 is here, so value of the point X2 will always be greater than equal to value of this. So, left-hand limit of X2 will at the most coincide with the right-hand limit okay.

But if X1is different from X2 and X1 and X2 both are the point of discontinuity, so we can identify the rational number RX1 and RX2, which are different. But we see here, we see that from first, that the rational number RX1is not equal to rational number RX2. Because these X1 and X2 these are the point of E, so this implies if X1 is different from X2 okay. So, if X1 is strictly less than X2 okay -- sorry if this is - RX1will be different from - if X1 is different from X2 then X1 is also point of discontinuity, so we can get the RX1corresponding to X2 we get a point X2, when X1and X2 are different point, then in that case we do not have this. It means that corresponding to each point X1, there is a rational number and vice versa. If the rational numbers are there which satisfy this condition X1 is not A, then the point X1 X2 we can identify which are the point at which the lower limit and the upper limit do not coincide. That is, it is a point of descent. So, there is a one-to-one correspondence -- so there exist one-to-one correspondence between the set E and a subset of the set of rational numbers. But set of rational number is a countable set. And there is a one-to-one correspondence between this -- this implies the set E is countable. So, what this shows E is the set of those, so this means that F continuously -- that is the set E of all points, where all points, where F is discontinuous and is a countable set. And that’s what we wanted to show, okay. So, this proves it. Thank you very much, thanks.