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NPTEL ONLINE CERTIFICATION COURSE

Introductory Course in Real Analysis

Lecture - 55

Types of Discontinuities (Contd.)

With

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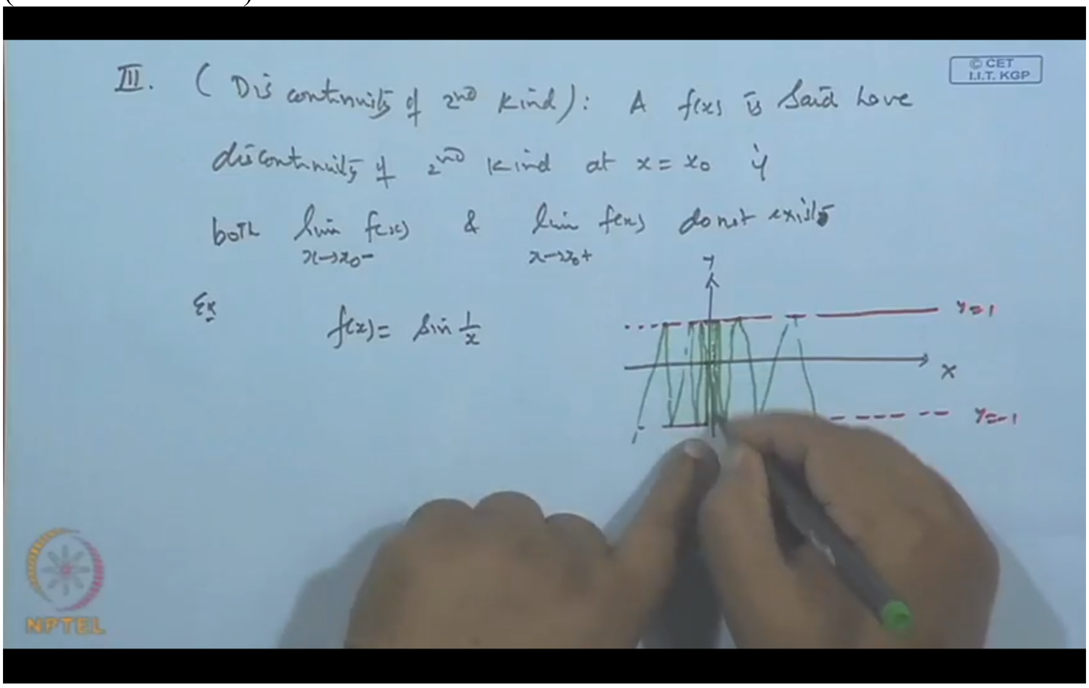

So, this is that. The third type of the continuity and discontinuity will be, we call it as a discontinuity of second kind. Second kind, okay? Now what is this discontinuity of the second kind? When the limit of this; left-hand limit, right-hand limit. What does not exist? That is a function FX is said to have a discontinuity of second kind, at a point of X equal to X naught, if the left-hand limit and right-hand limit. If left-hand limit or right-hand limit, both, if both, left-hand or right-hand limit, both do not exist, do not exist. Okay? There definitely. And left-hand limit does not exist, right-hand limit does not exist, in fact, so neither left exists nor light exists. So what happened then? For example, if we take that - okay FX equals to say Sine 1/X, this we have seen already this function, what is the behaviour of the function? If I look this function: here X this is Y and just I am taking say here Y equals 1, Y equals minus one, okay.

Now graph of this function. If we look the graph, the graph will be something like that. Say here, something like this, is it not? And as soon as it comes here it’s very closed, very very closed and like this, but it never touch it here okay. Similarly, here, this is very very closed okay? Really very closed for this and something like this, something like this. So only at this means around the point 0, it gets so much congested and frequently it's going up and down very fast. That is near to X equal to 0, it has a sudden jump. At this point it has value 1 immediately in the nearby point it has value minus 1. So, it has a sudden jump near at the point X equals 0 in the neighbourhood of G. So, let us see. We claim that this X equal to 0 is the point of discontinuity of second kind, let us see.

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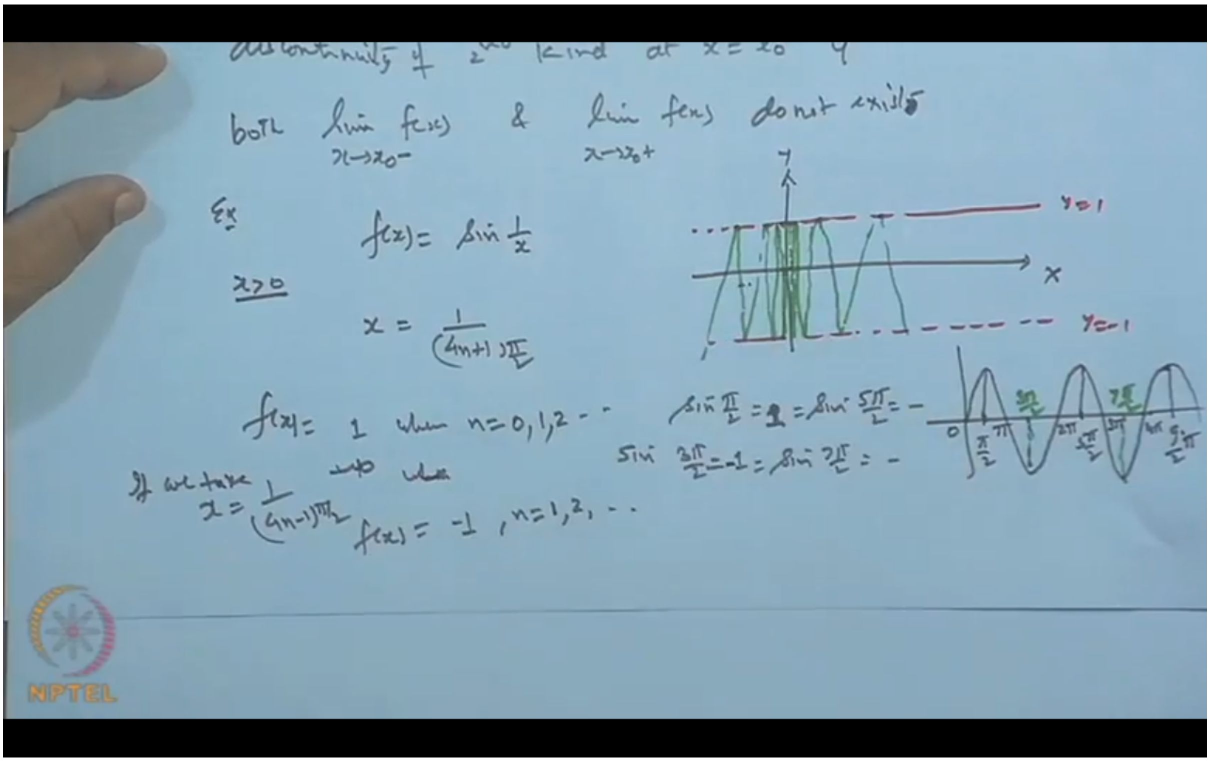

Suppose I take positive side first. You take X is greater than 0 okay, then if I take the number X equal to say 1 by 4 n plus 1 pi by 2 okay. So, what happened this Sine function we know Sine pi by 2 is 0 then because if we take the graph of the Sine function this is the graph of the Sine function Sine 0 is 0 like this, so Sine 0 is 0, Sine pi is 0, Sine 2 pi is 0, Sine 3 pi 0 and so on. But Sine pi by 2, this is the point, pi by 2 where it has a positive value 1. Sine of this thing 2 pi plus 5 by 2, that is 5 pi by 2, it has a 1 value. Then again, this 3 pi, 4 pi so it is 9 pi by 4 pi 9, then 9 by 2 pi. Then this value, the function has a value 1. So, when it is pi by 2 Sine, Sine of 3 pi by 2, Sine of 5 pi by 2 and so on, the value of this is, sorry is 1, is 1.

And then if we look the Sine 3 pi by 2. Sine 3 pi by 2, this is our 3 pi by 2, this value. 3 pi by 2 here it is 7. Is it not? 7 pi by 2. So, 3 pi by 2. Sine 7 pi by 2 and so on. The value is coming to be minus 1. It means when the point X, so 1 by X is this, so X is very very near any sufficiently large, the point is very close to 0. So, when it has the value n is the function, so this function FX has the value 1 when n is equal to what? n is 0 and n is one, n is say 2 and like this. And equal to minus 1 when n is what? n is… here say 3 pi by 2, so we are getting n is equal to… say if I choose the function FX, if we take X is equal to 1 by 4n minus 1 pi by 2, then the value of the function FX is minus 1 4n is equal to 1, n equal to 2, with 8 minus 7 pi by 2 and like this continually.

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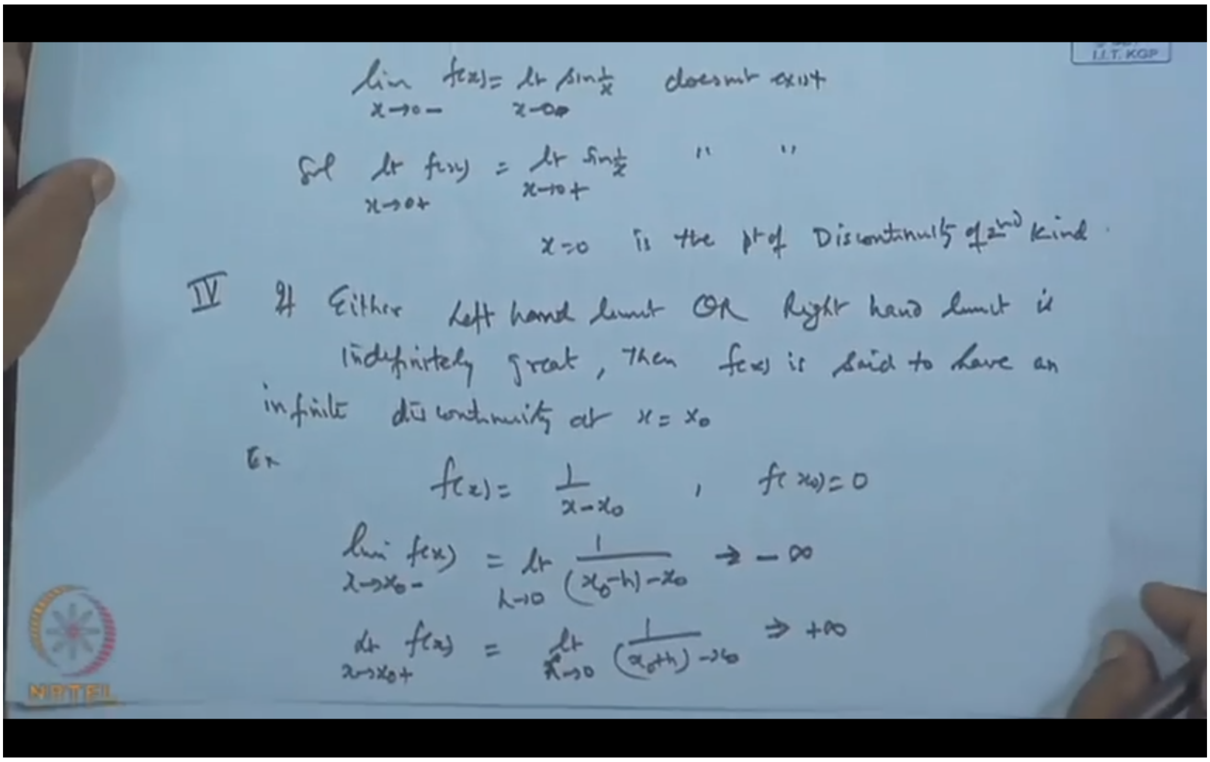

So, we have a sequence of the point approaching towards 0 where the function has a sudden jump. Plus 1 to minus 1 and like. Similarly, in the left-hand side. So, this function we say the limit of this function FX which is Sine X, when X tends to 0 minus, that is limit of this Sine 1 by X, when X tends to 0 minus does not exist. Similarly limit of this function FX when X tends to 0 plus does not exist. 0 plus, sorry 0 minus, here 0 plus, does not exist. So, X equal to 0 is the point of discontinuity of second type kind, second kind. So, this is what we have.

There are some other also functions we can go ahead. Now one more function we say that, of course this is not very important, but it still… if one or the more functions in finding that. If either left-hand limit or right-hand limit each indefinitely great; means very large, it goes to infinity when this left-hand or right-hand limit goes to infinity, then in that case the function FX is said to have an infinite discontinuity at the point X equal to X. For example, suppose I take the function FX which is 1 minus X minus, say X naught okay, and the value of this at the point X naught I am choosing to be 0. Now what happens? If you take the limit of this function FX, when X approaches to X naught from the negative side, then this is the same as 1 over X naught minus H, minus X naught limit is tends to 0, is it not? Now this is equal to what? This is nothing but tends to minus infinity. Tends to minus infinity. Because this value will go to basically X naught cancelled and H is sufficiently small, so it goes to minus infinity.

And when you go to the plus side FX, the limit X tends to X naught plus side. We have X naught, then what happens is limit X naught plus H minus X naught and then H tends to 0. So, this will be tending to plus infinity. So, the left-hand limit or right-hand or any, both of them, it goes to very large infinity. Then point X equal to X naught is a point of discontinuity for this okay? Infinite discontinuity of this, of X is equal to 0, okay.

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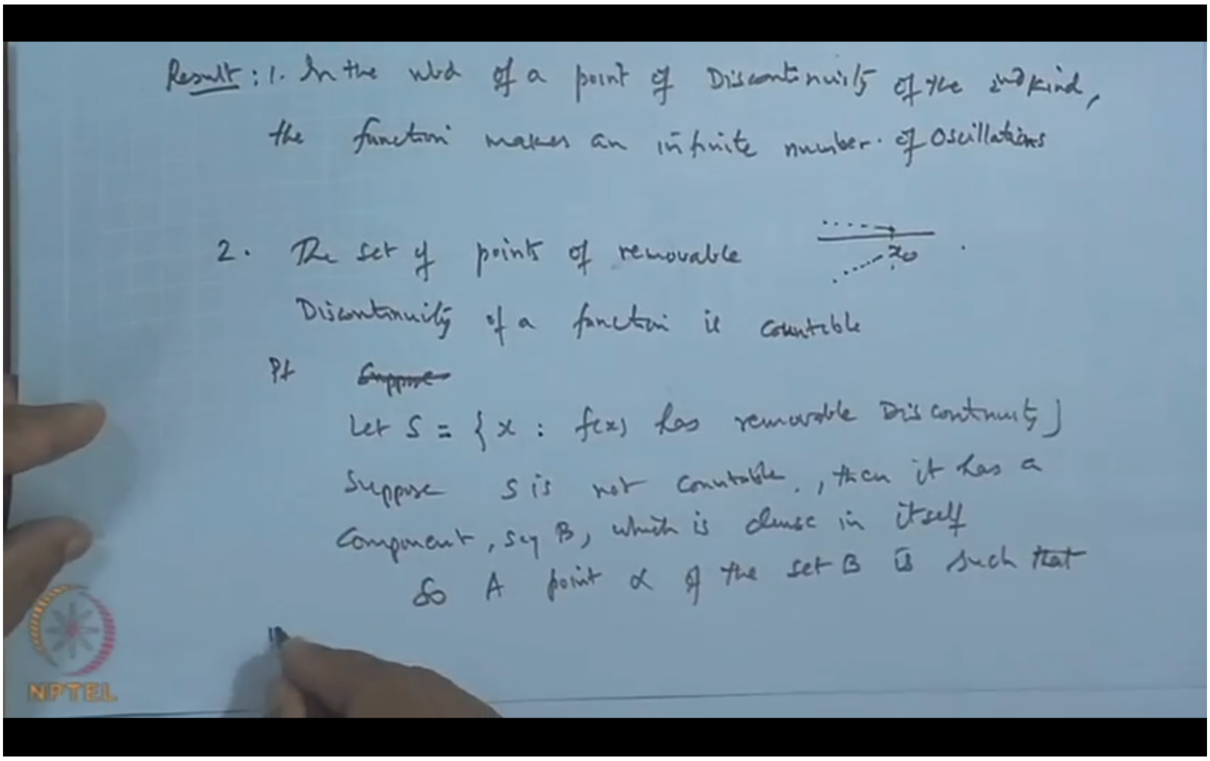

And then so almost this we have covered. One is more examples which we can do. The results of course without proof we can just say, in the neighbourhood of a point of discontinuity. Discontinuity in the neighbourhood of the point of discontinuity of the second kind, of the second kind. The function makes an infinite makes an infinite number of oscillations. Because the reason is very simple: when you take the limit of the function either from the left-hand side or right-hand side, then you get a sequence approaching toward that function, where along a different sequence you have a different value. So, in fact if X naught is the point of discontinuity of second kind, then we can get one sequence where the limit of the function will be, say Alpha 1, then another sequence will be obtained where the limit of the sequence is minus something, say Alpha two or something, so there is a sudden jump, makes an infinite number of oscillations. So that is this case of this.

Second is of course proof we can go through this very not difficult. The set of points of removable discontinuity, removable discontinuities of a function is countable, is countable okay. The reason is proof is really like this: suppose that if the set S, suppose the set S, which is the function here, a set of those function X where FX has removable discontinuity, has removable discontinuity, okay? let S be this. Suppose S is not countable. S is not countable. Then it has a component. Then it has a component, say B, which is dense in itself okay? That means a point Alpha B such that so a point Alpha we can find, a point Alpha of the set B is such that in any, so a point Alpha is such that, in any neighbourhood of Alpha, in any neighbourhood of Alpha, on one side, there are infinite number of points, at which the functional values limits are all equal, but are different from the functional value at that point. Therefore, Alpha is a point of discontinuity of second kind. Which is a contradiction, because we have assumed the Alpha as a point removable discontinuous point, therefore, our assumption is known, so the set S is countable. Similarly, another result is the set of points of

ordinary discontinuity, that is discontinuity of the first kind. Discontinuities of a function is countable.
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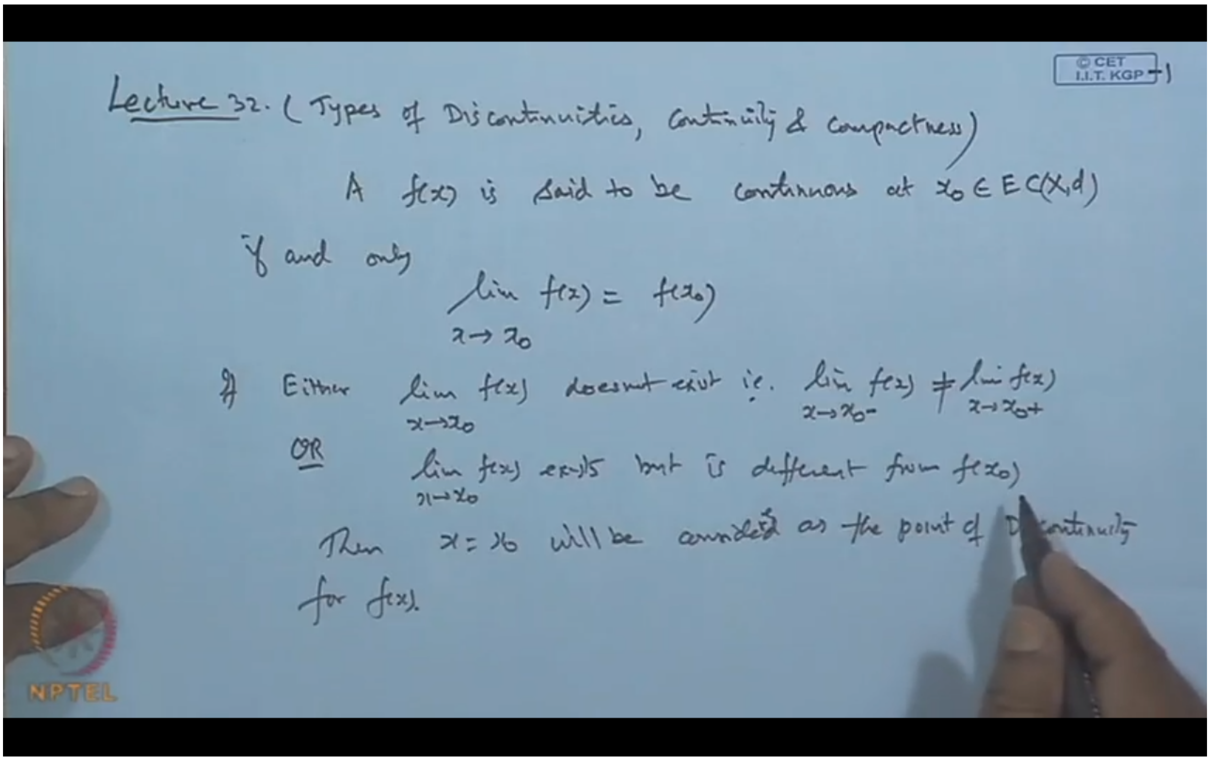

So today we will discuss the types, various types of discontinuities. And then we will see the relation between continuity and compactness okay. So, let’s see. First, we know a function FX is said to be continuous at a point X not belonging to the domain say E which subset of say XD, if and only if limit of this function FX, when X approaches to X naught and equal to the value of the function at a point X naught. Now, if this condition, when we are saying limit of FX X tends to X naught equals to FX naught, it means the function must have a limit when X approach to X naught, that is the left-hand limit, right-hand limit both exist, they are equal. And the value of the function at the point X naught must coincide the value of the limit FX when X approaches to X naught. Then only the function will be said to be a continuous function.

If any one of the conditions fails that is if either the limit does not exist, that is the left-hand limit of this function when X goes to X naught from the left-hand side, different from limit of the function FX when X tends to X not from positive side, then we say the limit does not exist at that point. Or maybe if the limit comes out to be infinity for that, in that case 1. Or limit exists but is different from the value of the function FX naught. So, when the limit does not exist left-hand, right-limit does not exist, then also the function will not be continuous. Or if the limit exists but this value is different from FX naught, then also the point X naught will be said to be continuous. Or it may sometime have been the limit of this function FX when X tends to X naught and Y tends to Y naught, both does not go to mean left-hand limit also does not exist, right-hand also does not exist. Then also we say the point X naught is a point of discontinuity.

So, if this case is there then X equal to X naught will be considered as the point of discontinuity for the function FX. So, obviously the point of discontinuity depends on the cases, whether limit exists but different from its naught, then we name this point of discontinuity to some like a removable discontinuity; when the limit does not exist then we give some other name to the discontinuity; or when the limit does not exist left-hand and right-hand limit does not exist, we also give some other name. Or sometimes the limit tends to infinity then also we give sometimes different names. So according to the situation the type of discontinuities are characterized.

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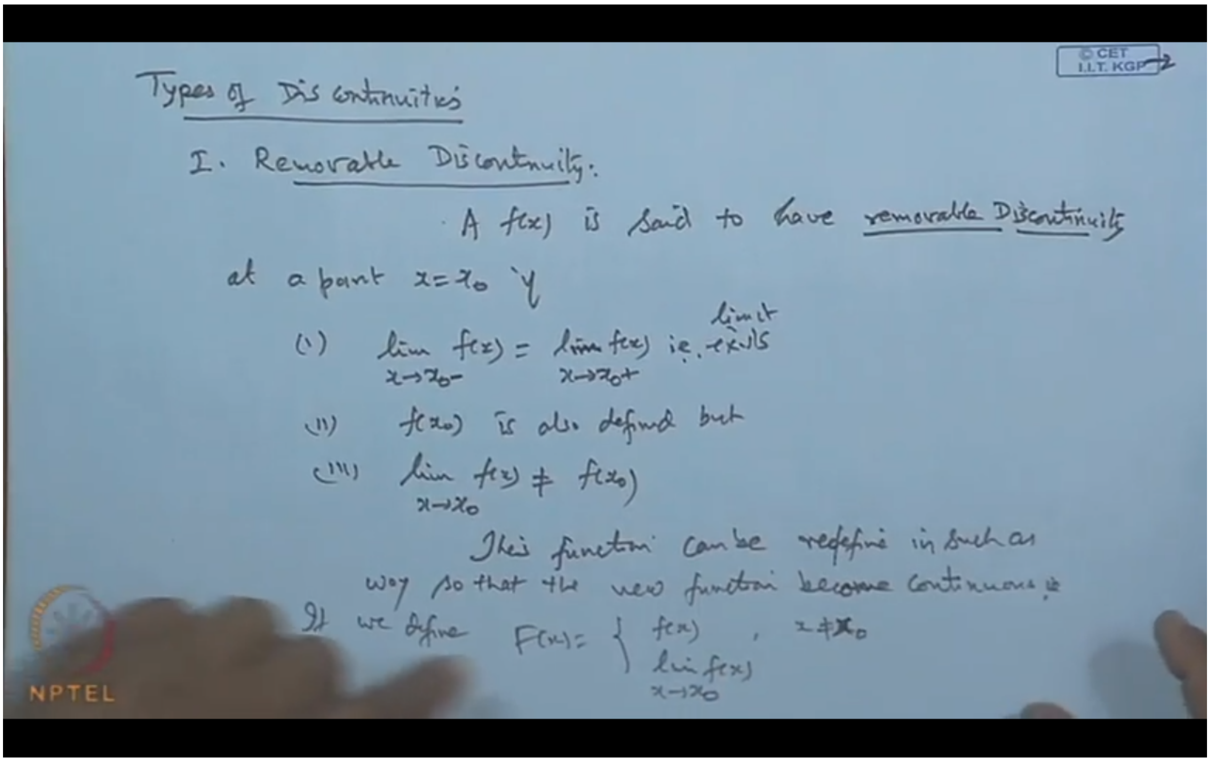

So firstly, let's see Types of Discontinuity. First is removable discontinuity okay. So, removable discontinuity, if the function of function FX is said to have removable discontinuity at a point say X equal to X naught in the domain, if the functional limit of FX when X tends to X naught from negative side, equal to the limit of this function. FX when X tends to X naught from the positive side, they exist and both are equal. Means left-hand limit is the same as the right-hand so limit exist. That is the limit exists.

And second one is the functional value F of X naught also defined, is also defined, but the third is the limit exists. Third is, limit of this function FX when X tends to X naught is not equal to the value of the function at a point X naught. Then we say X naught is a point of discontinuity and we call it as a removable discontinuity. Why removable discontinuity? Because we can redefine the function again so that it can be - the new function will be continuous. Because the reason is this function can be redefined in such a way so that the new function becomes continuous. That is if we define capital FX as our small fx, when X is different from a X naught and equal to the limit of the function FX when X tends to X naught, then the new function FX so obtained will be a continuous function. Then F will be continuous. FX as X naught as a continuous point. So, this function, if in this we have X naught as a discontinuous point can be converted to a continuous point by simply removing or redefining the function.

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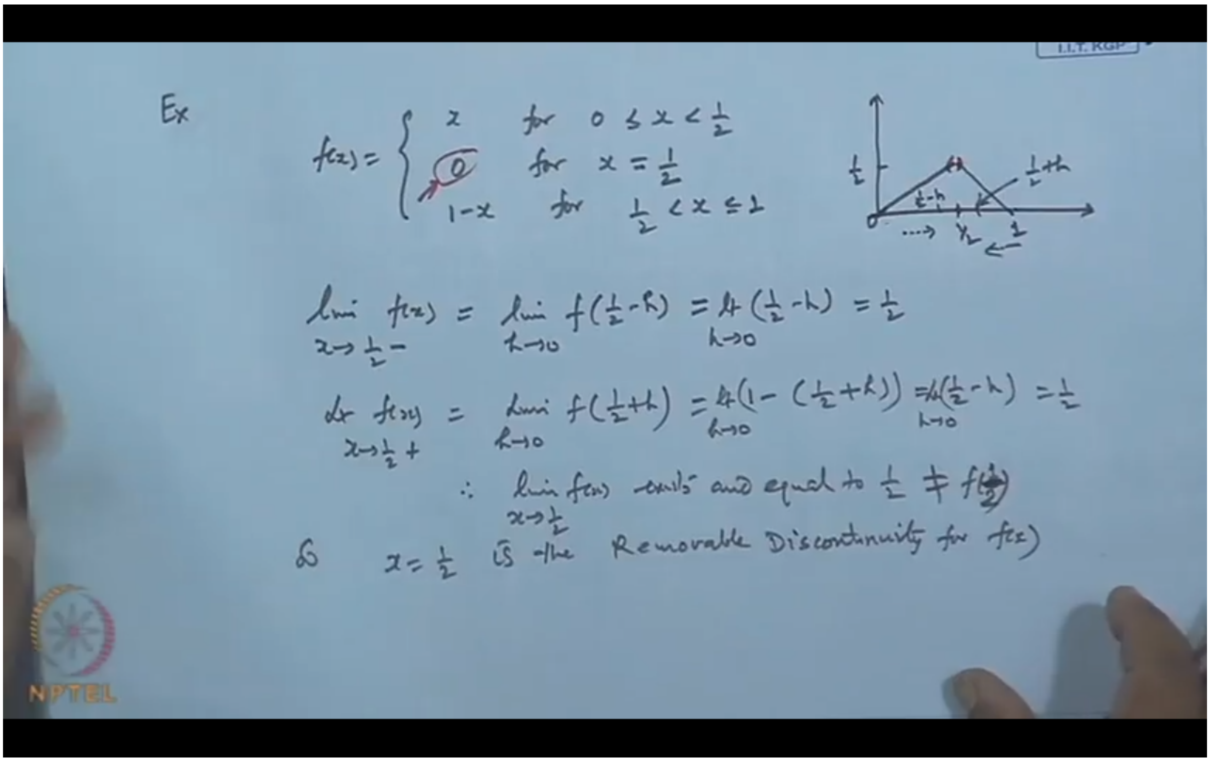

For example, if suppose we take the function FX is defined as say X for 0 less than equal to X less than half, equal to 0 for X is equal to half and equal to say 1 minus X for X lying between half, open at half closed at 1. So, over the interval 0 1 we are defining the function, so the function is of this type. This is 0. So, between 0 to half. The function is Y equal to FX, so it will be something like this, this is half, here is say half okay. And then at the point half it is coming to be 0. So basically, this half, this point is excluded. Here, you're not touching half here, it is coming down is it not? And then from this point when X is not half, just equal to half, it is start from here. So, we are getting this thing and then at the point 1 it is coming to be 0. This is 1. So, what we see, except at the point 1 it the function is very smooth function okay?

So here if we look, the limit of this function FX when X approaches to half minus, then this is the same as the limit of the function f half minus H. H tends to 0 from the left-hand side you are approaching -- from here you are approaching, so take the point X is ½ minus H. So, this is the point 1/2 minus H okay. So, consider this, now when the point is lying between 0 and half, the function is defined as X. So, this is equal to ½ minus H and then limit of this S tends to 0, which comes out to be 1/2. So, from the left-hand side the value of this limit is coming to be ½. Now from the right-hand side if you look, limit FX when X tends to 1/2 plus, then this is the same as to right limit H tends to 0; F of 1/2 plus H. Choose the point here which is 1/2 plus H and then approach about this side okay.

So when the point is lying between 1/2 and 1, this interval is taking consideration where the function is defined like 1 minus X. So, 1 minus this and then take the limit as H tends to 0. So, this comes out to be 1/2 minus H and then H tends to 0, which is equal to 1/2. So, right-hand limit and left-hand limit both are coming to be same as 1/2 so limit exists, therefore limit exists. The limit of this function when X tends to say half exist okay. And equal to half. But the value of the function at the point half is 0, which is different from the value of the function at a point half. Because the value at the point half is given to a 0; here we are limp value is coming to be half. It means this function has such a function where the limit exists, but the value of the function at that point is different from the limit. So, half becomes the point of removable discontinuity. So, X equal to half is the point is the removable discontinuity for this function FX, okay. So, if we redefine the function, instead of 0, if I did affine the function here, if I redefine here, with value at X equal to half, then our function becomes continuous, so that's why removable.

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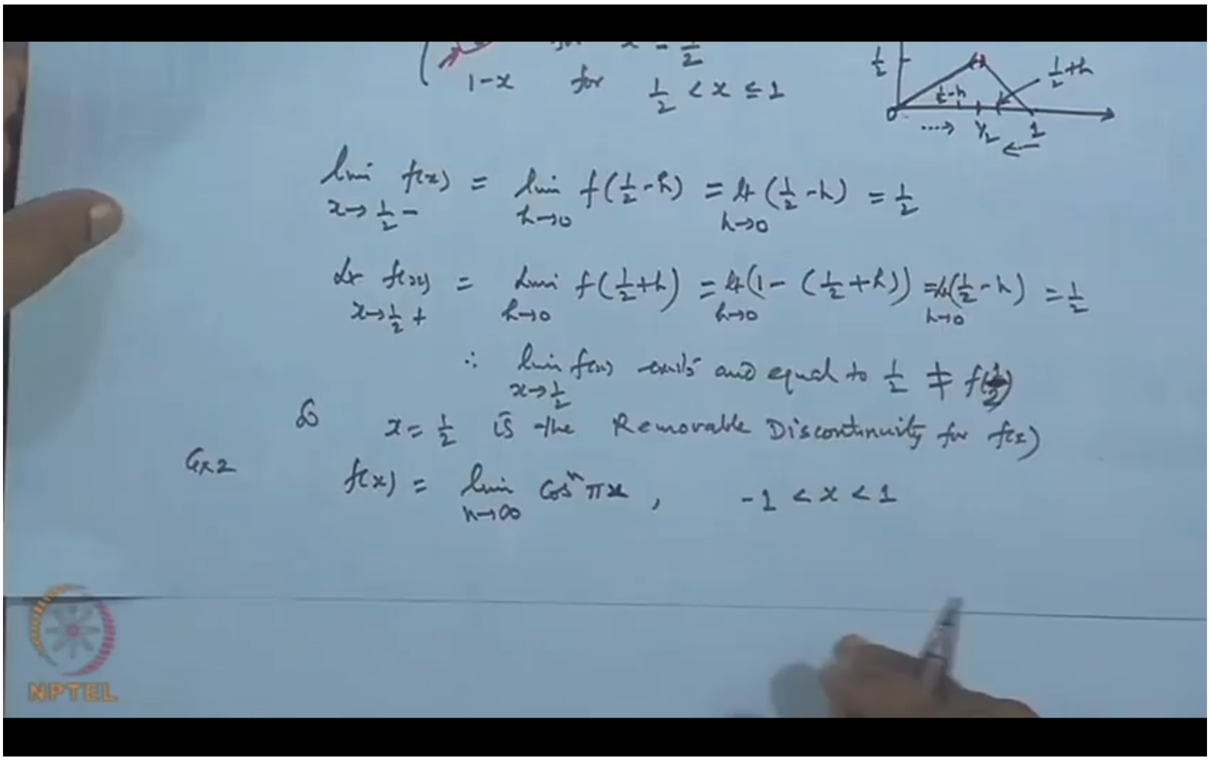

Now take another example. Let us define the function FX equal to limit of this, as n tends to infinity, goes to the power Cos to the power n pi X. We are x line pi X, okay. We have the X lies between minus 1 and 1 okay. It is not covering endpoints; open interval minus 1 to 1.

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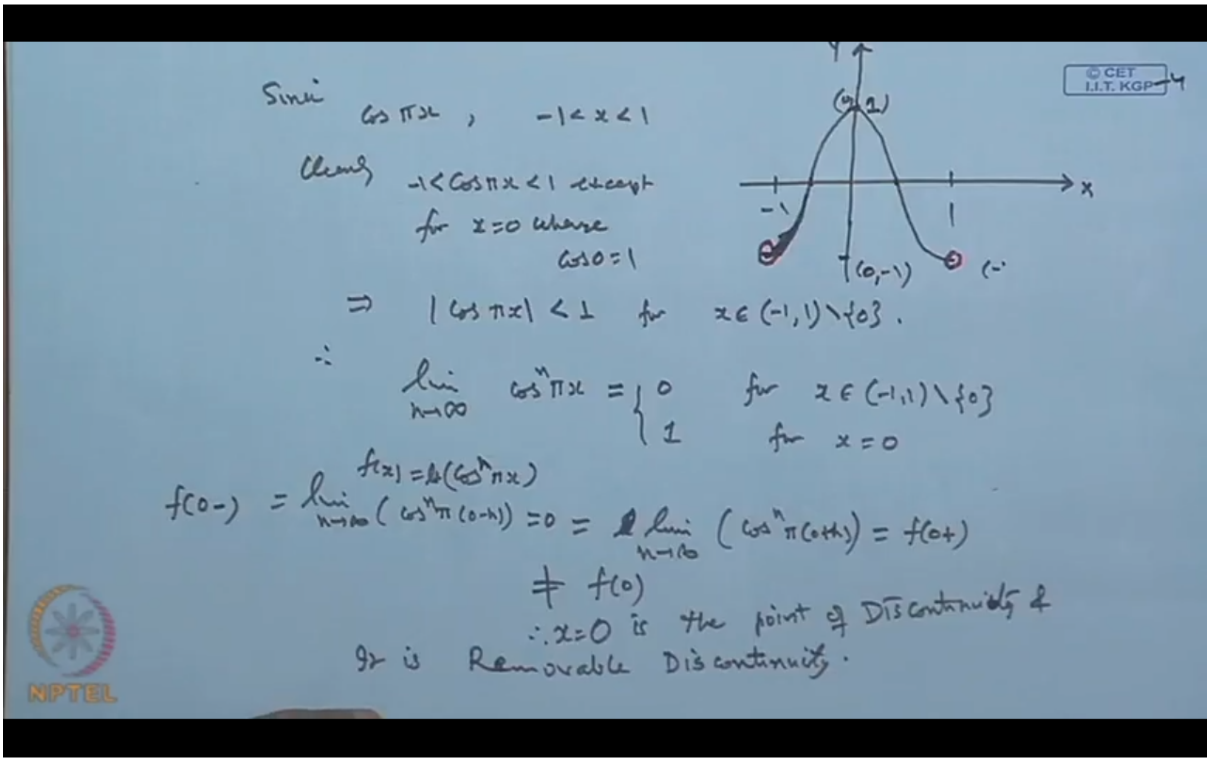

Now if we look the function. The function Cosine of this function; Cosine pi X. When X lying between minus 1 to 1, the function pi X has the graph like this: minus 1 to 1. When X is 0, Cosine 0 is 1 okay. So, when X is 1/2 pi by 2, the Cosine is 0. Similarly, minus pi by 2 it is 0, so it will be something like this okay. And when X equal to minus 1 but it's not touching minus 1, it's not touching. In fact, this point is not attained, okay. Similarly, this point is not attained, but when it is minus 1 and plus 1 the value is coming to be minus 1. So, this is the graph of this function; X and this YX. So, value of this -- this is 1 okay, X Y is 1, 0 1 in this is the 0 1 point and here this point is say, minus 1; here this point is 0 minus 1. So, Cosine of this thing for this function clearly Cosine of pi X always lies between minus 1 to plus 1, except for X equal to 0 where the value is Cosine, 0 is 1.

So, when X is different from 0 the value is always minus 1 to plus. 1 that is mod of Cosine X will be less than 1 and there. So, this shows that modern mod of Cosine pi X will remain less than 1 for all X belonging to the interval minus 1 to 1 minus singleton set 0. Hence, limit of this function Cosine pi X to the power n, when n tends to infinity, this limit will be 0, because this is always be less than 1 so n is infinity it goes to 0 for all X belongs to minus one to one, minus singleton set 0 and equal to 1 for X equal to 0. So, what then? Throughout this function attains the value 0 okay. Function is very smooth function, it is continuous except at the point 0. Because what happened is when you take the left-hand limit of this function FX is

Cosine n to the power pi X limit of this -- this is FX. So, when you take the F, X means 0 minus point, the left-hand side point.

It means the limit of this thing limit of Cos n pi 0 minus H. Just you take the 0 minus H and then here. So, this number will be less than 1 mod of this, so this will be 0. When you go from the outside, the limit of this -- sorry the function, limit of this function as n tends to infinity cos of n pi 0 plus h is also 0. That is f of 0 plus. But the value is entirely different from f of 0, because F of 0 is 1. Therefore 0, the function 0, X equals to 0, is the point of discontinuity and it is a removable discontinuity; that can be easily seen okay, at X equal to 0.

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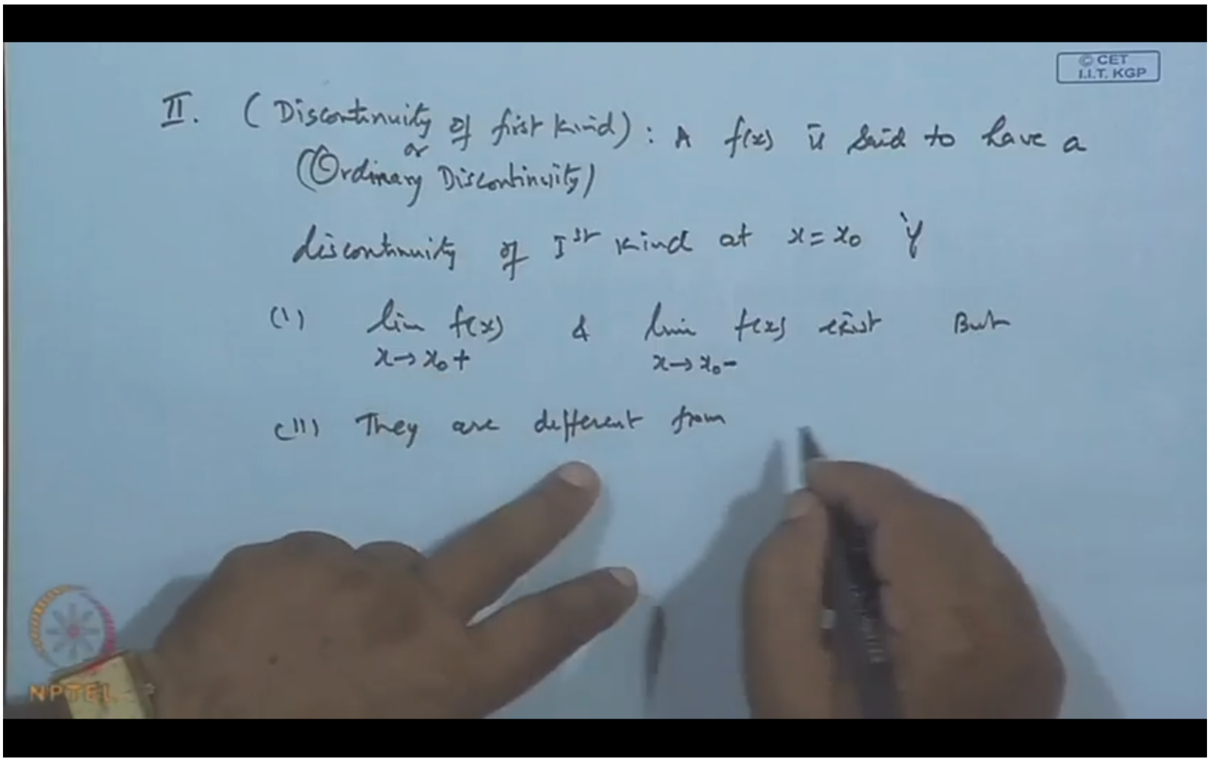

So, this is the first kind of discontinuity that is you know. The second type of discontinuity we call it as a Discontinuity of First Kind. Discontinuity of first kind or sometimes we also call it the Ordinary Discontinuity at this point okay. So, what we say is, if the limit exists, both the left and right limit exist but they are different. If a function FX is said to have discontinuity of first kind at the point X equal to X naught; if limit of this FX when X tends to X naught plus and limit of this FX when X tends to X naught minus, that's right-hand limit and left-hand limit exists. Both the limit exists, but they are different from…have different values; both exist but…