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Course

On

Introductory Course in Real Analysis

By

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Lecture 52: Absolute Continuity and Related Theorems

Okay, if theorem is, if F is a mapping from A to R is uniformly continuous on a subset on A of R, and if XN is a Cauchy sequence in A, then F(xn) this sequence is a Cauchy sequence in R, it means if F is a uniformly continuous function, then it will transfer the Cauchy sequence to the Cauchy sequence. The proof is let XN be a Cauchy sequence in A okay, we wanted to, and F is given to be uniform continuous over A, so by definition of the continuity let for a given epsilon greater than 0 we can identify a delta, there exists a delta which depends only on epsilon greater than 0 such that the mod X-U is less than delta, such that for if X,U belongs to A satisfy this condition, then F(x)-F(u) this is less than epsilon, okay so let it be 1.

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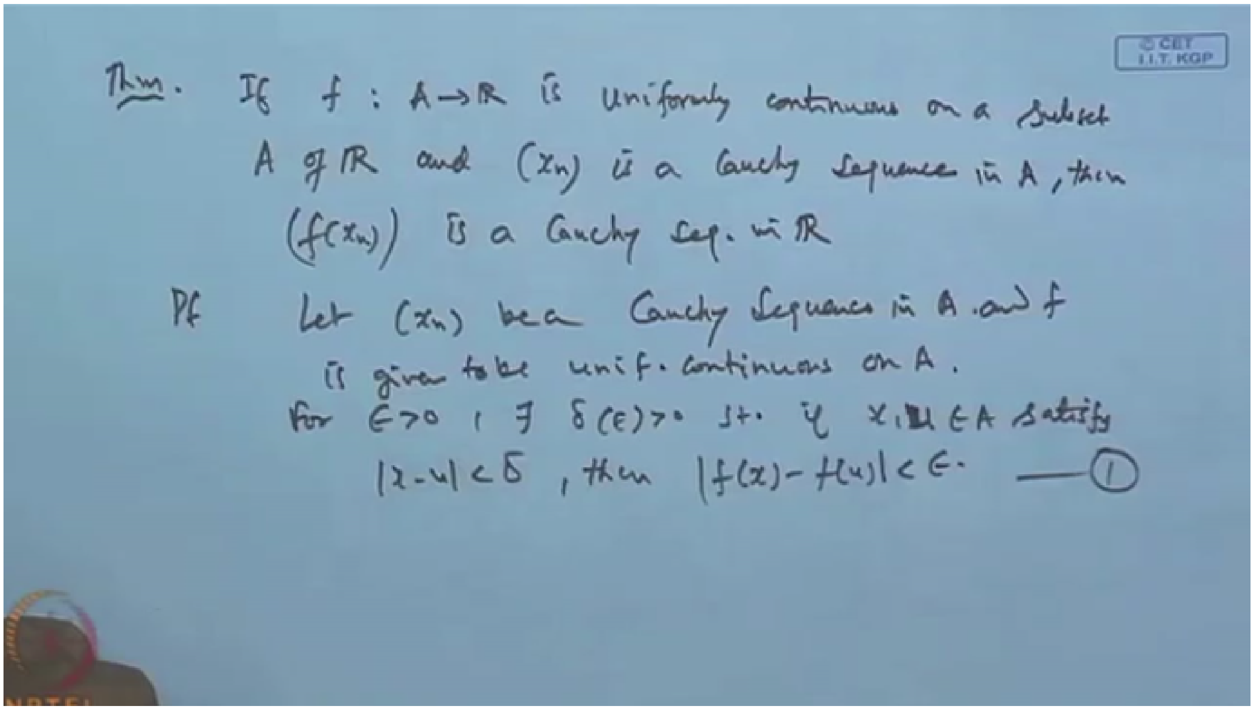

Now it is given the sequence XN is a Cauchy sequence, so since XN is a Cauchy sequence so by definition of Cauchy, so for a given epsilon greater than 0, for given delta so we can say they'll exist for given delta greater than 0 there exist an H which depends on delta such that the difference between any two arbitrary term of the sequence after a certain stage can be made less than delta for all M, N which are greater than equal to H, this is 2 okay.

Now by this same delta,
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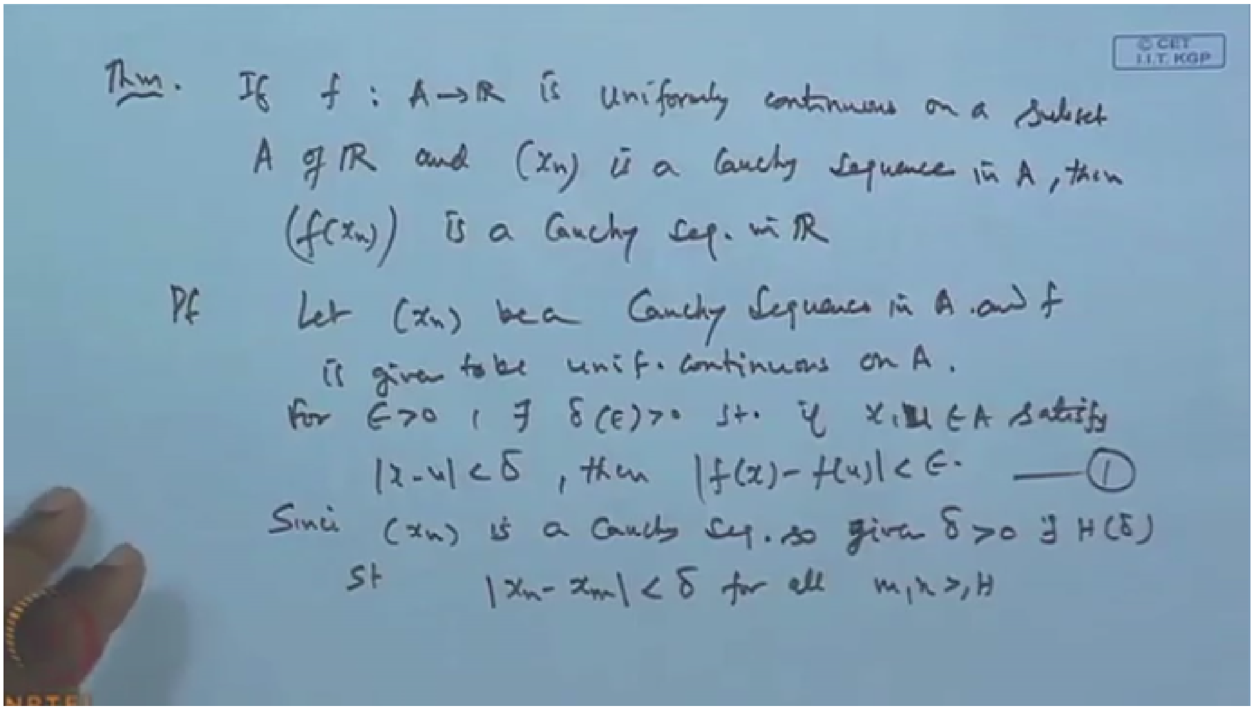

if they the same choice of delta since X, M, N is less than this so if you take X = XN, U = XM, so from 1 it follows that F(xn)-F(xm) mod of this will be less than epsilon for all N, M greater than equal to H, this shows the sequence F(xn) is a Cauchy sequence in R, so that's proof the result for, okay.

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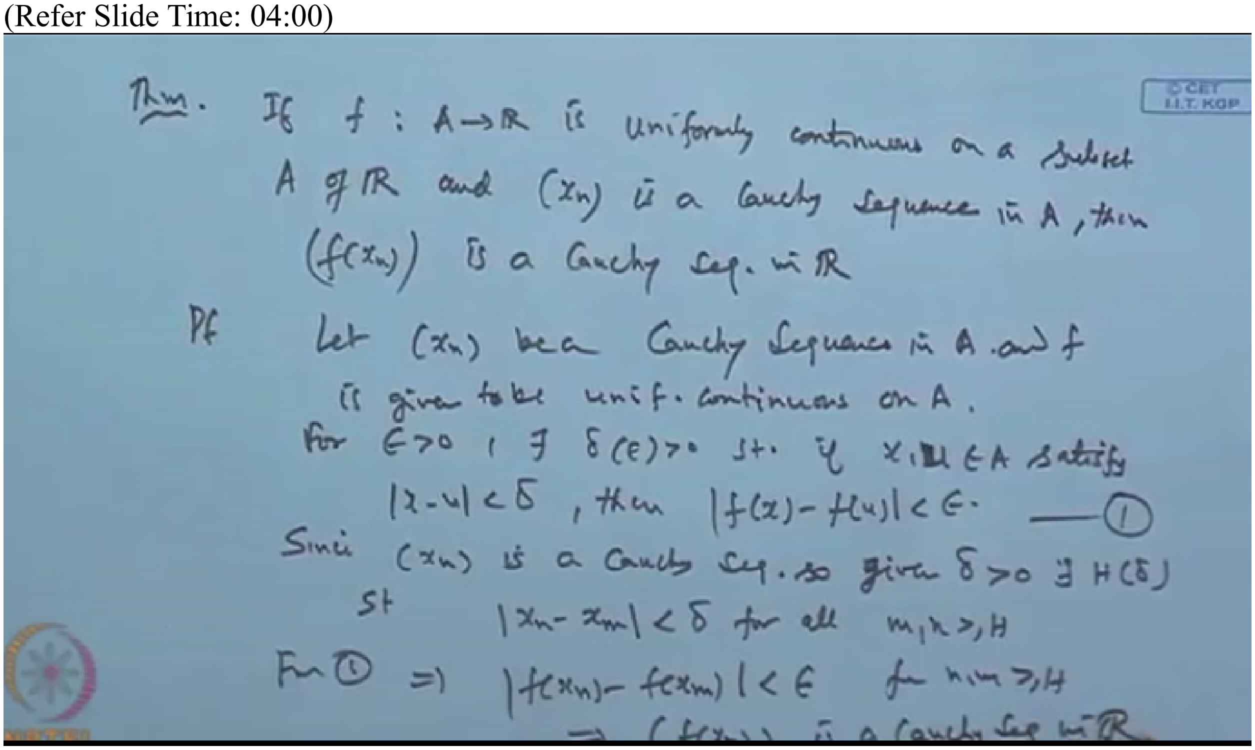

Now we come to the thing which is say our absolute continuity, absolute continuity means a function F(x) is said to be absolutely continuous in the interval say A, B if corresponding to N

arbitrarily choosing positive number epsilon, another positive number delta can be sine, can be determined such that for a sequence of non-overlapping intervals HR, KR open interval HR, KR define in A, B, the sigma of mod F(hr) – F(kr) is less than epsilon provided sigma of KR - HR this length is less than delta, so what is it?
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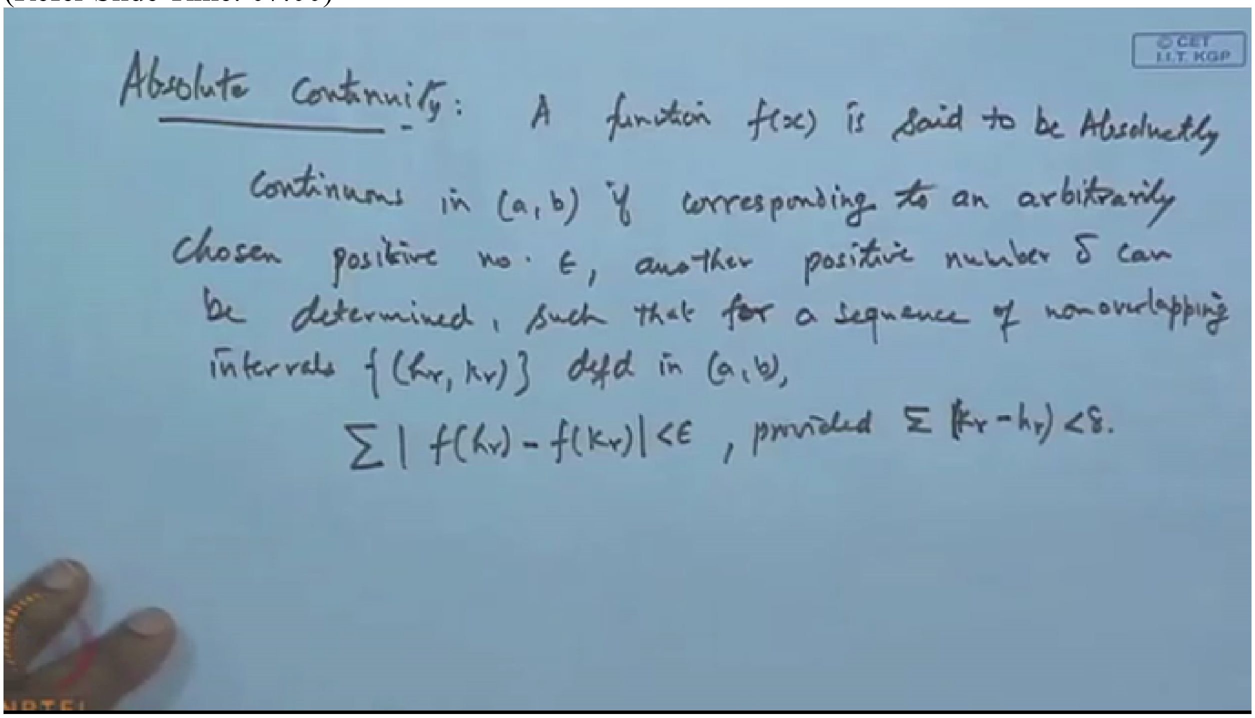

It means that is we can say, so we say the effect that is F(x) is absolutely continuous function, the meaning of this is absolute continuous over the interval A, B, if corresponding to a given epsilon a number delta exists such that in a countable set of non-overlapping intervals of total length, I am not using the measure of total length less than delta, the sum of the fluctuations of the function is less than epsilon.
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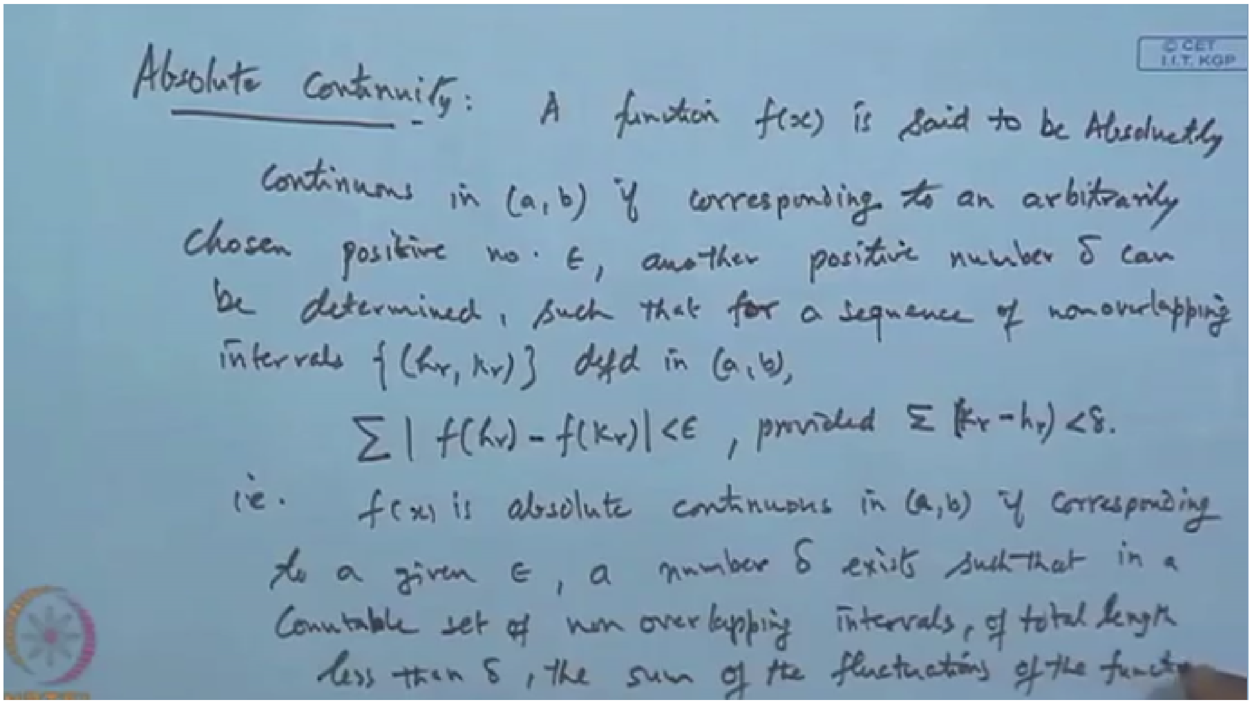
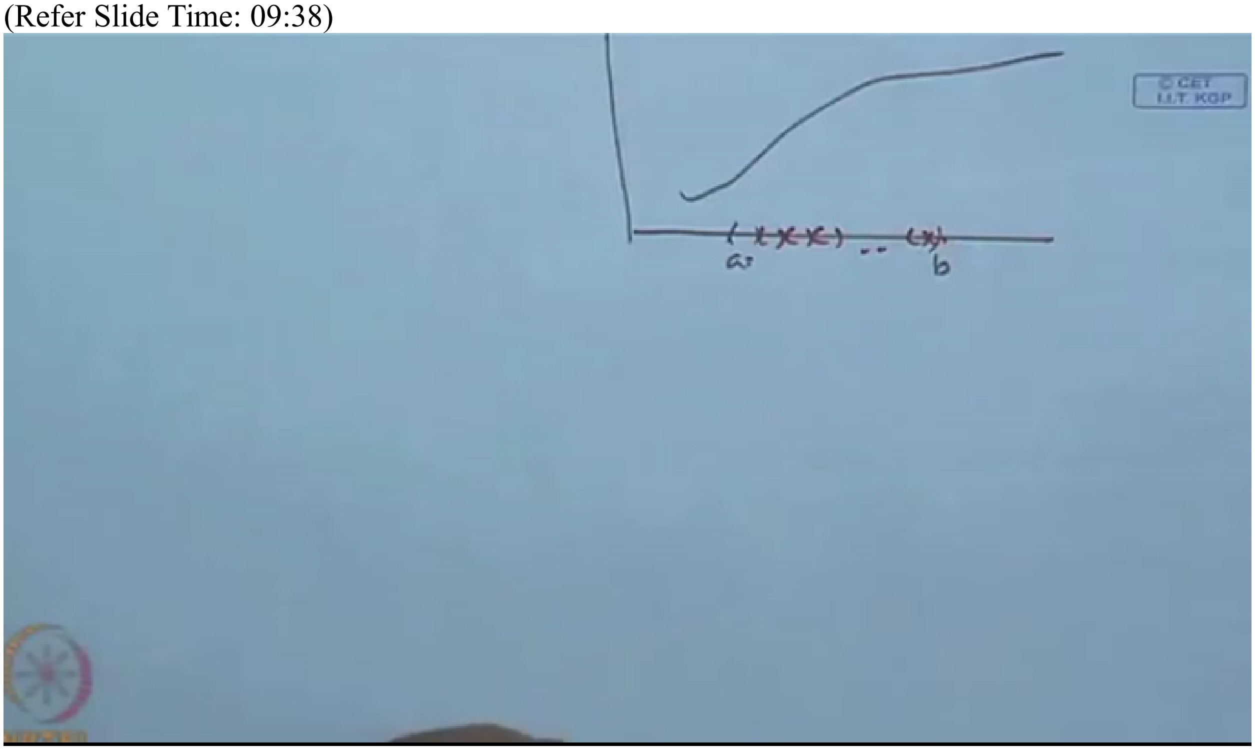


So let us see what's the meaning of this is? Suppose we have a interval A, B, and a function F is given, we say this function is absolutely continuous in the interval A, B, if B for a given epsilon greater than 0 if we have a non-overlapping intervals means divide this one say in this non overage say here this is like this, if I take this non-overlapping intervals and so on like this (Refer Slide Time: 09:38)



so if we take a countable number of non-overlapping interval of this whose length is less than delta so we have to take a small portion, suppose I take this small portion this, now this small

portion is total length is delta, total length of this is delta, so this small length we can over this small interval we can find a non-overlapping intervals such that sigma of this length KR - HR this length is less than delta, these are countable 1 to infinity, R is 1 to infinity, total length is less than delta, so for a given epsilon, say this epsilon is there, for this given epsilon we can find a delta such that whenever we have a non-overlapping intervals countable number of non-overlapping intervals whose length is less than delta, then corresponding fluctuation over these sub-intervals, the total sum of this corresponding this fluctuation should not exceed by epsilon. If so, then we say that is sigma of mod F(hr) – F(kr) this should be less than epsilon 1 to infinity.
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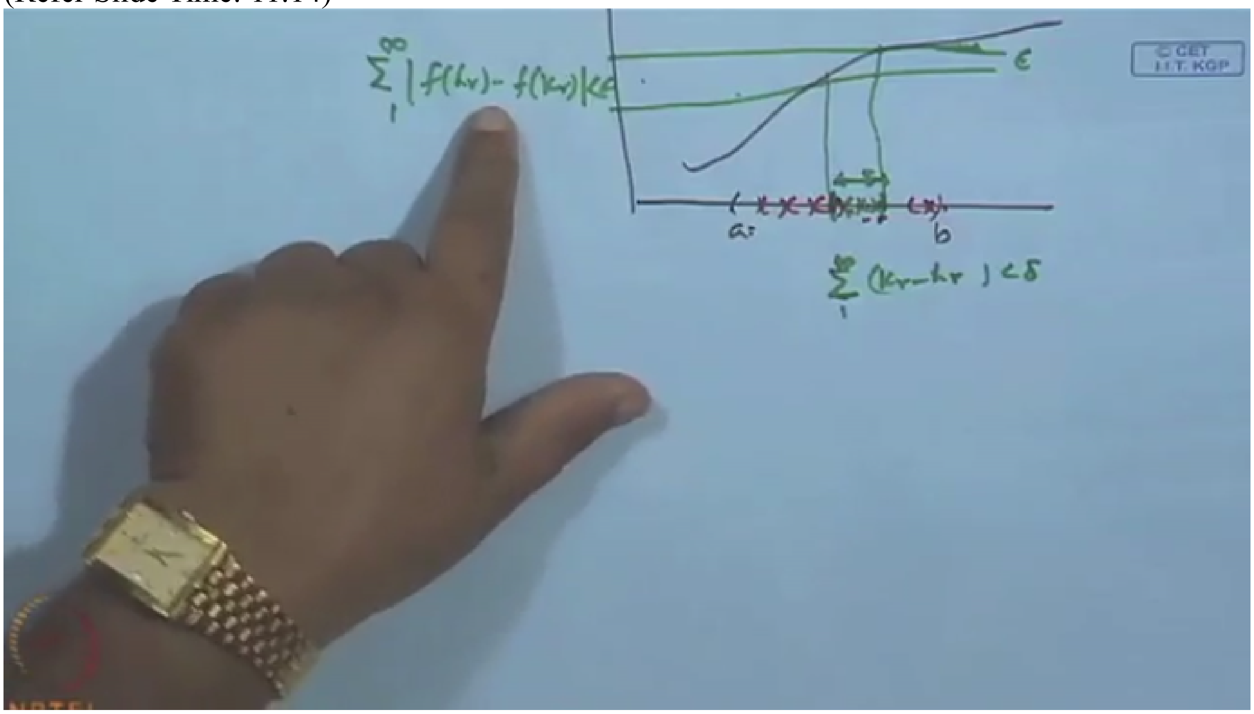

So if the total fluctuation of the function over this sub-intervals is less than epsilon whenever the points are in this intervals, in counter number interval whose length total sum is less than delta, then function is said to be an absolutely continuous function.

So obviously every absolutely continuous function is continuous, as a note we can say function which is absolutely continuous in the interval A, B is also continuous in accordance with Cauchy definition,
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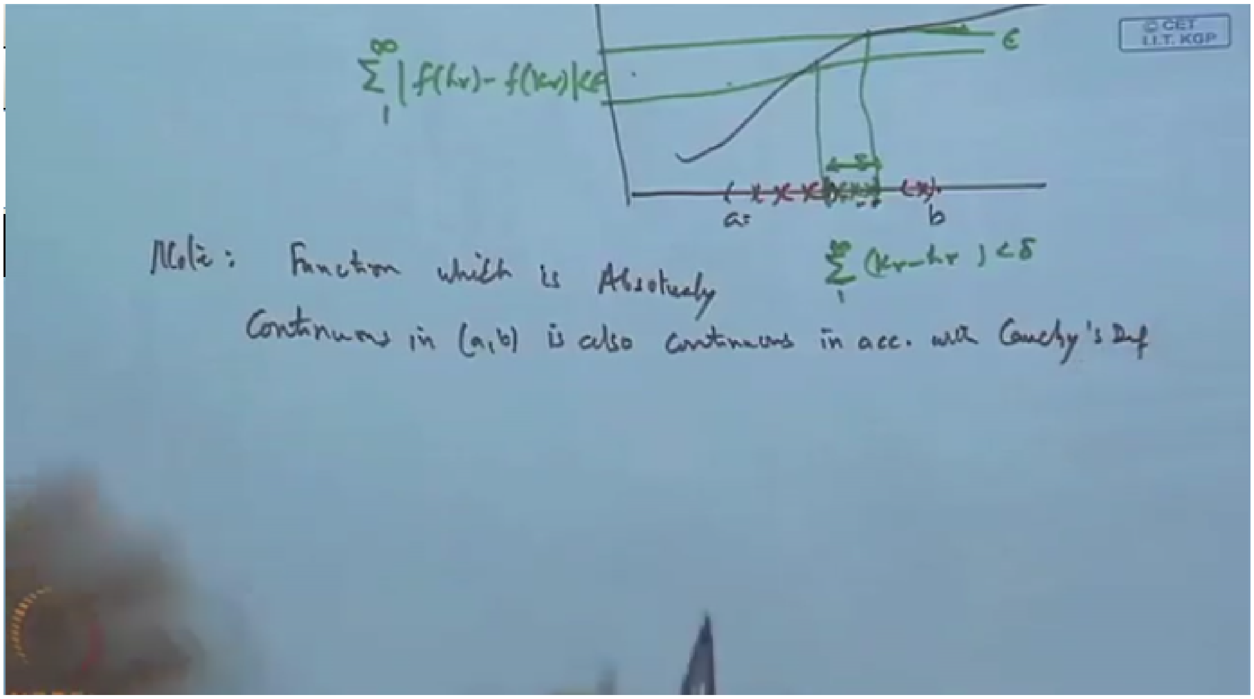

because what we do we shall replace it that total set KR we consider single interval, the sum of these we can replace by a small interval, total sum is less than delta, and then correspondingly here we get the total fluctuation is less than epsilon, so this will be obviously true consists of this, okay less than then the condition of uniform continuity.

Similarly if we can also say that we may take this interval, single interval length L then definition of uniform continuity is also satisfied over this, uniform continuity is also settled, so function which is absolutely continuous is a continuous A1 and 4B also we say it is uniformly continuous, and in uniform continuity condition is also satisfied, if we choose that KR to consider single interval of length delta, okay.

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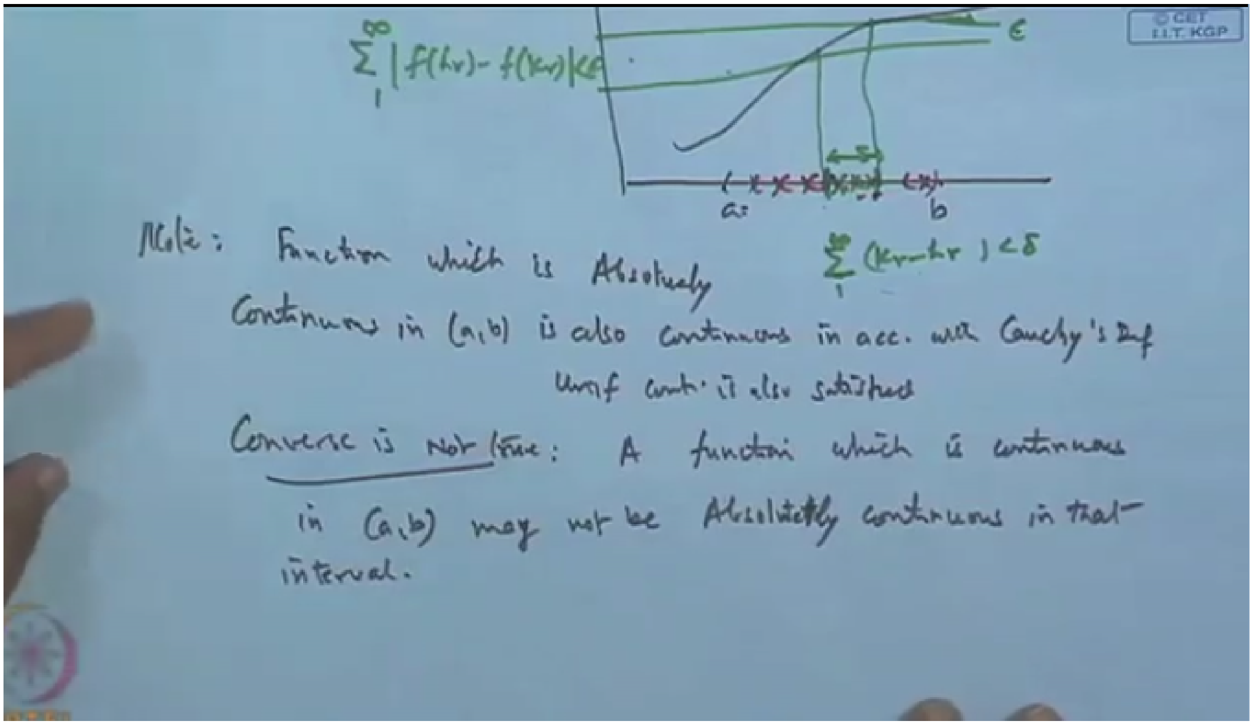

Converse is not true, that is a function which is continuous may not be absolutely continuous in that interval that is a function which is continuous in the interval A, B may not be absolutely, okay, absolutely continuous in that interval, okay. A function which is continuous may not be absolutely continuous, for example if we take the function F(x) which is defined as X sine X, sine 1/X, when X is different from 0, and 0 when X is 0 we know this function is a continuous function, this is a continuous till this we have seen, this function F(x) is continuous, this we have already seen, so nothing because the limit of this F(x) continuous at 0, okay, and otherwise also is continuous in the total, limit as X tends to 0 is the value of this function is coming to 0 which is F(0) and for other point at X belongs to 0, 1 interval, in fact entire interval it is continuous, because at the point 0 the value is coming to be this and continuity follows, okay.

Now it's not absolutely continuous, we claim but this function F(x) is not absolutely continuous on the interval 0, 1, why? It means the condition of the absolute condition is not settled that is if
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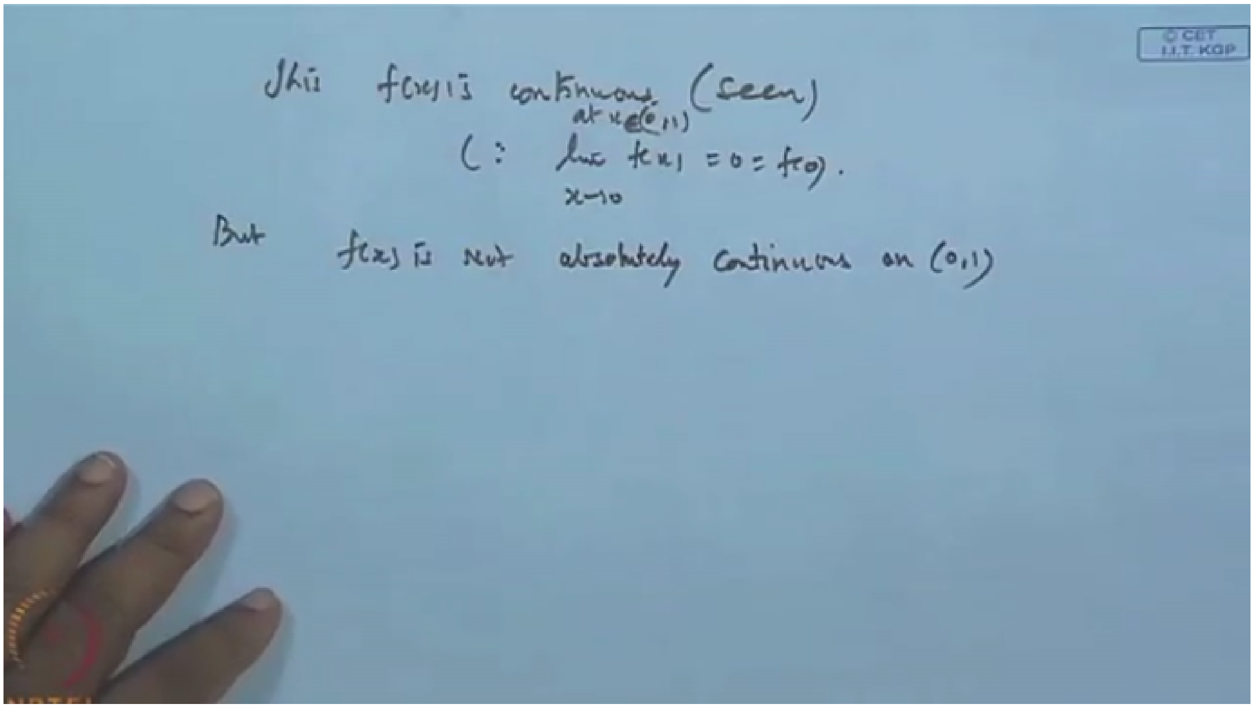

we choose the infinite number of, containment number of subintervals whose total length is less than delta, but the fluctuation may not be less than epsilon, so that's what, so suppose I divide the interval 0, 1 in each of these sub-intervals we get into sub-intervals say 1/R pi, 1/R+1 pi this sub-intervals where R is 1, 2, 3 and so on into sub intervals, and each intervals in the sub-intervals 1/R pi and 1/R+1 where R is 1, 2, 3 the fluctuation of the function F(x) which is X sine 1/X exceed by this number 1/R+1 pi, let's see why?

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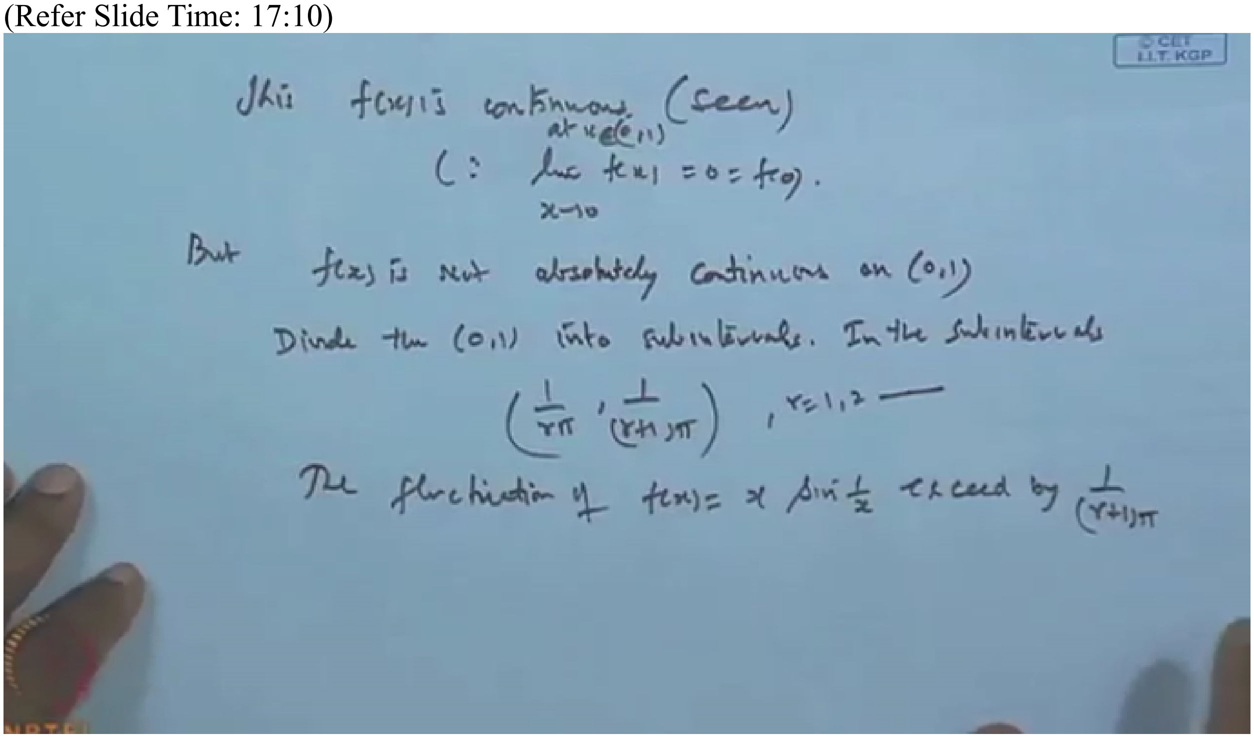

Over this function the reason is, over this function this is our interval 1/r pi this is our interval 1/R+1 pi, the function X sine 1/X if we write the function F(x) which is X sine 1/X at the end point the sine of this is 0, because it is integral multiple of pi, so basically the function will go like this, because at this endpoint R pi or R+1 pi this value will give the value 0, so the fluctuation of the function the value of this minus the value of this will be something which is greater than this number.
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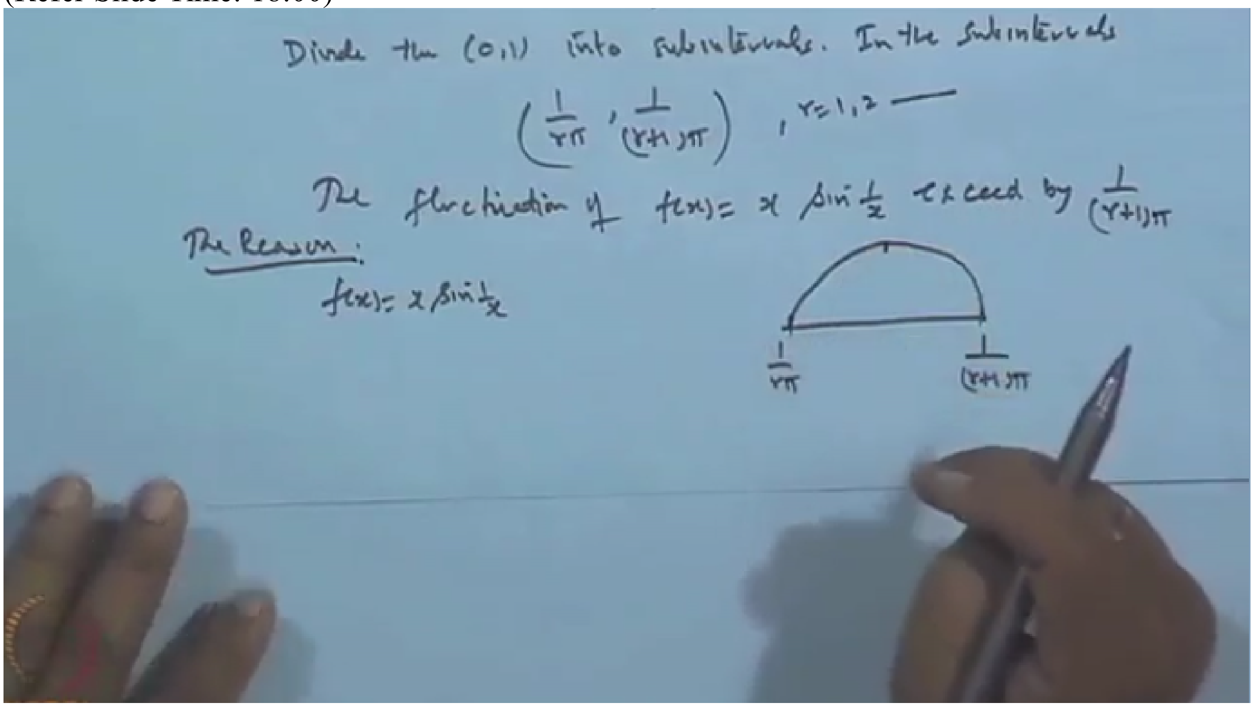

Now here the maximum value suppose I take X equal to say, let us choose X to be 2 over, 1/X to be 2R+1 pi/2 okay, 2R+1 pi, so X will be 1 upon this line say here, then what will be the sine of this value? Sine of 1/X is 1, we cover multiple of this 1, and this value will give the F(x) 2R+1 pi/2, they've been upon 2R+1 pi/2 so this will be greater than the value at this point R+1, so fluctuation will be exceed by this number which is greater than this, okay by this, clear, so we get from here is that the fluctuation is exceeding by this therefore the sum of the fluctuation, so some of the fluctuation in the sequence of intervals say 1/R pi then 1/R+1 pi, this sequence of interval corresponding to R = N, N+1 and so on is greater than, if I take the sum is greater than 1/pi 1/N+1, 1/N+2 and so on, but this series is a divergent series, so this cannot be made as small as we please, greater than, so it is greater than by assigned positive number A, therefore this large number cannot be, so it is not a, so the function F(x) which is X sine 1/X, X is different from 0, and 0, when X0 is not absolutely continuous, okay, so this so,

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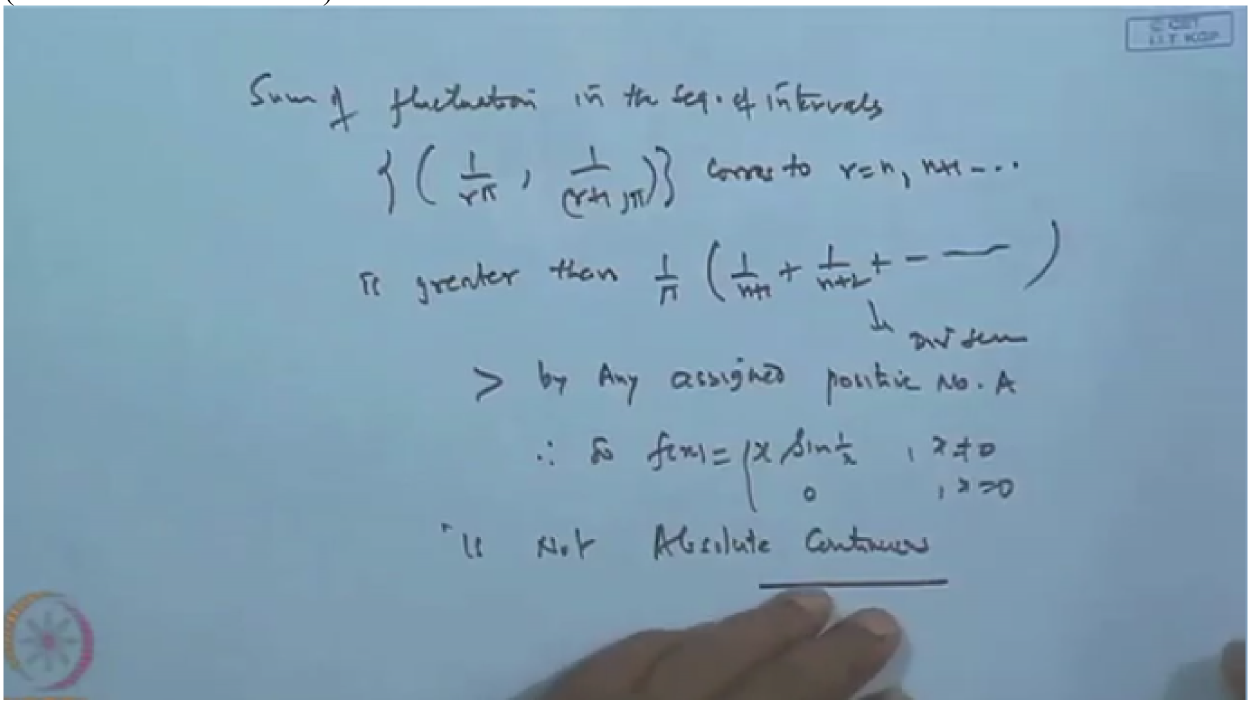

there results are, just I state that one result, the result says the some and the product of two absolutely continuous functions are absolutely continuous,

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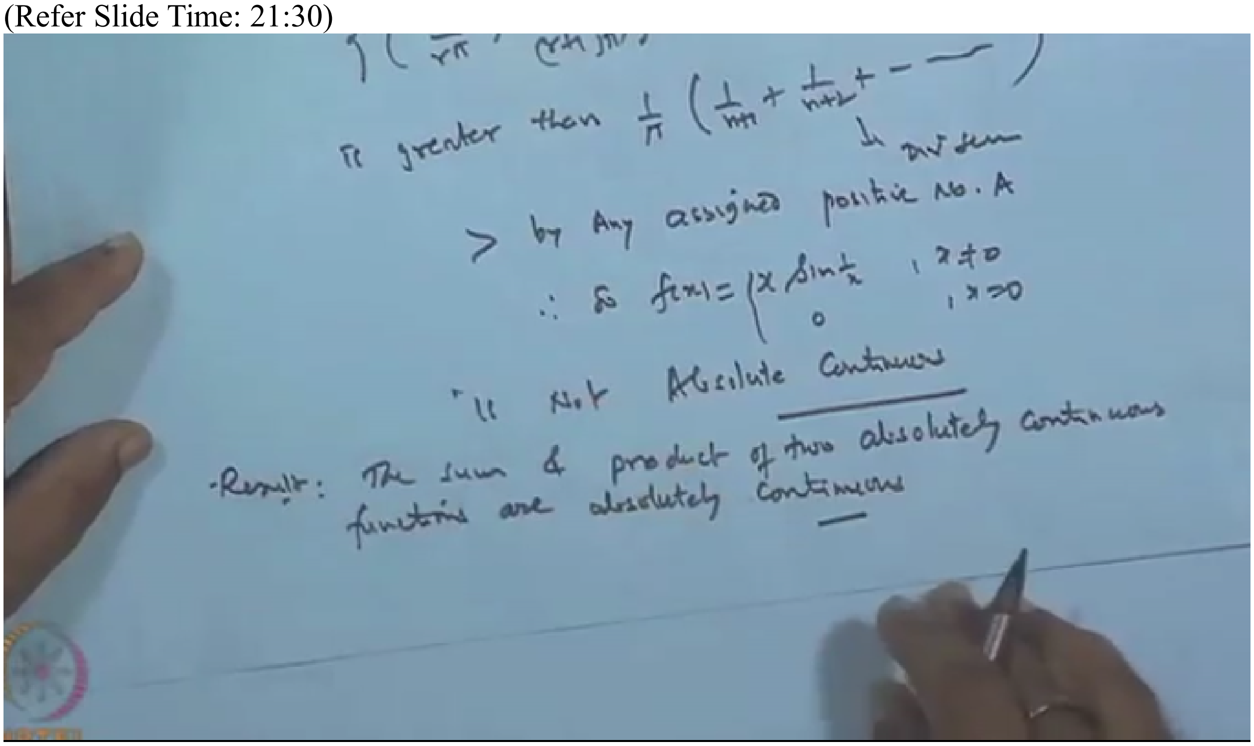

so this proof follows by the definition and one can go with the product. Thank you very much.