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Course

On

Introductory Course in Real Analysis

By

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Lecture 51: Uniform Continuity and Related Theorems

Okay, so today we will discuss uniform continuity and absolute continuity, okay. So let's see the first uniform, we have already seen the continuity definition, let A which is a subset of R and let F a mapping from A to R set of all real numbers A to R, then the following two condition, then following statements are equivalent, that is the first statements is that F is continuous at every point U belongs to A, and second is given epsilon greater than 0 and U which in element of A, there each are delta which depends on epsilon as well as the point U and greater than 0 such that for all X belonging to A, for all X such that X belongs to A and satisfy the condition mod X -U is less than delta which is, which depends on epsilon U, then the mod of F(x) – F(u) is less than epsilon.

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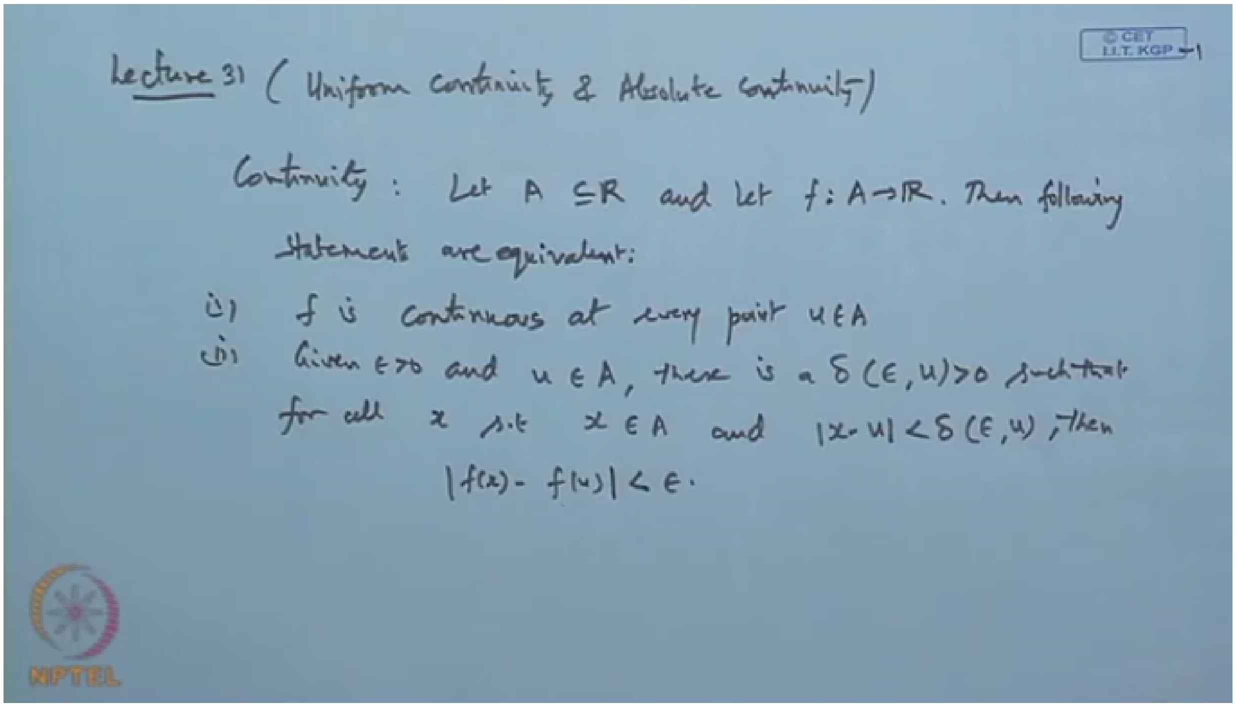

So this we have discussed already, a function is continuous at an point U, if for a given if epsilon there exists a delta such that mod of F(x) – F(u) less than epsilon whenever mod of X-U less than delta, and here we have seen that this delta depends on the point, if I change the point the correspondingly delta will change, okay, so what this shows? This shows the function F changes its behavior when the point changes, may be the way we are support, for some point the function is very slowly in changing their values and for some, or near to some point it changes very rapidly, for example if we take the sine 1/X, when X is not equal to 0 if we look this function then this function is changing very rapidly and when the point is very close to 0,

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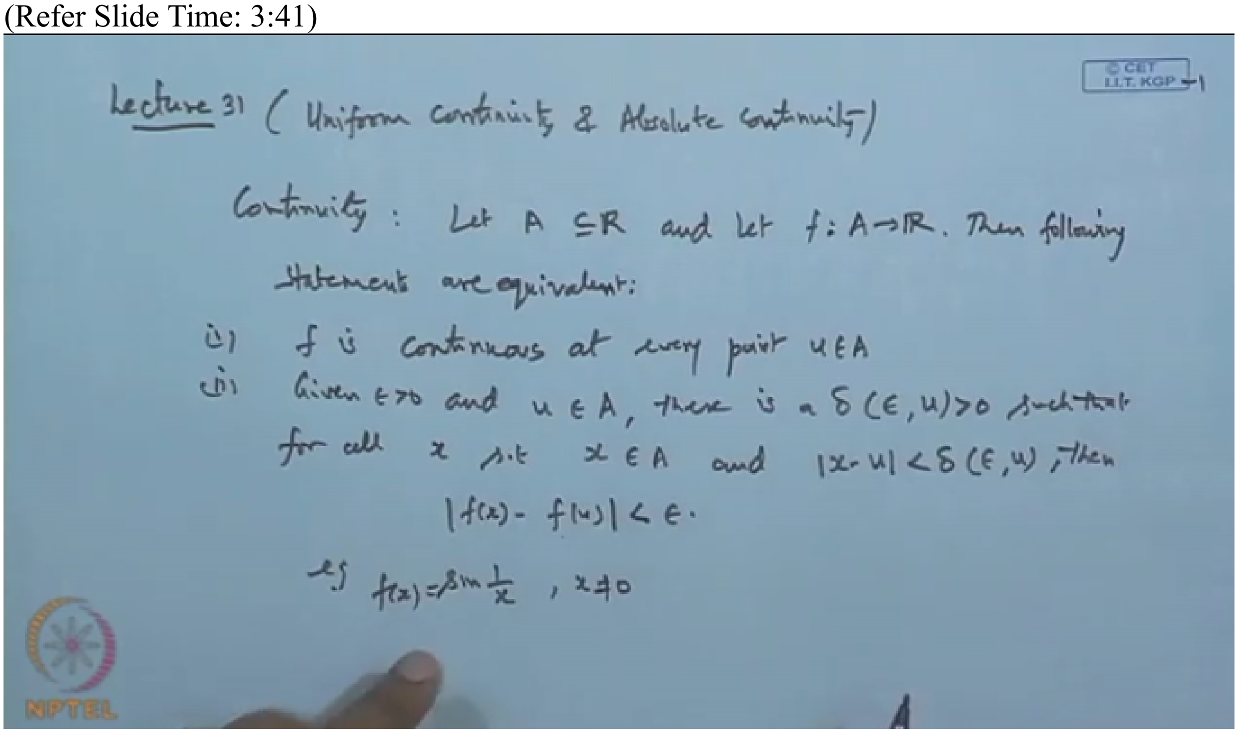

it goes very up-and-down from -1 to +1 and very rapidly it goes, so we are interested in such type of free functions, the change is smooth set, okay or we can say that we are interested the delta which is independent of U and that reads the concept of uniform continuity okay.

Though the function we say it is continuous point, Y is here but point wise means delta will depend on the U, so we wanted a definition which the delta is independent of U and that risk to the definition or concept of uniform continuity.

So let's see the definition of uniform continuous function, let A be a nonempty subset of R, and let F is a mapping from A to R, we say F is uniformly continuous,
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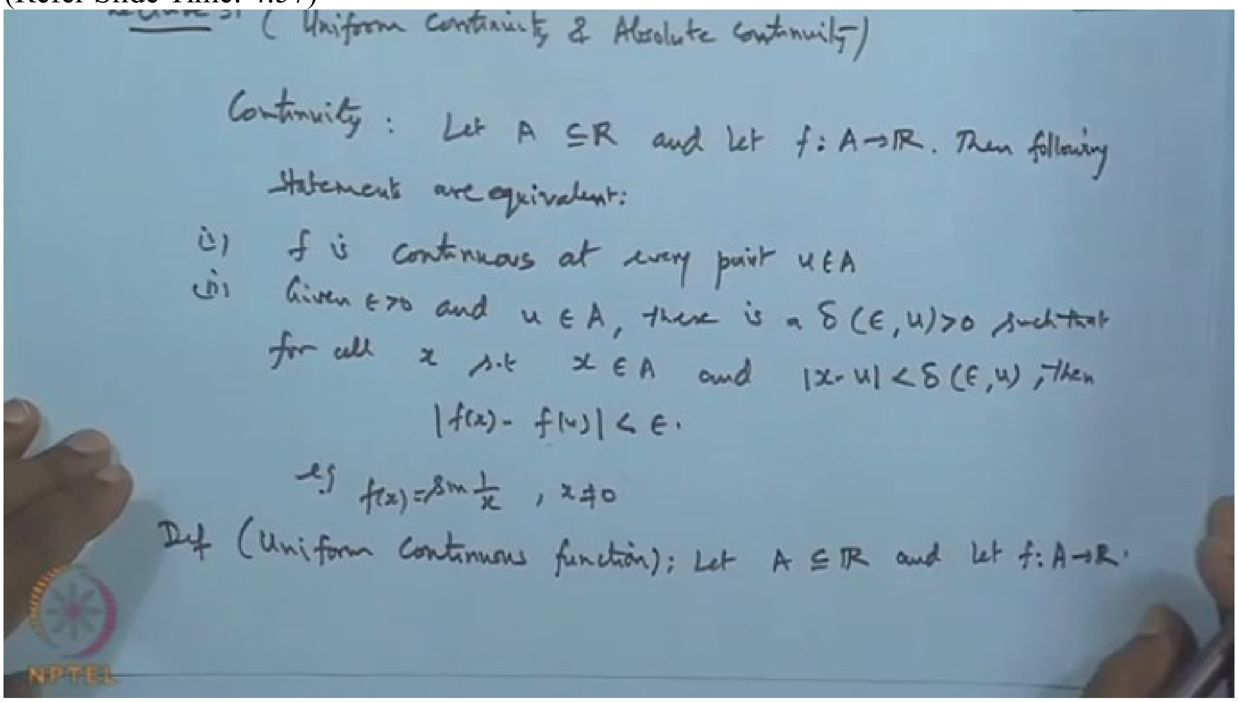

F is said to be uniformly continuous on the set A, if for each epsilon greater than 0 there is a delta which depends only on epsilon a positive delta which depends only on epsilon and independent of the point U of A such that if X and U are any two arbitrary point of A, or any number satisfying the condition mod of X-U is less than delta depends on epsilon only, then mod of F(x) – F(u) is less than epsilon, so what this shows is that if function is said to be uniformly continuous over the set A, remember when we say the function is continuous then we can say function is continuous at a point,
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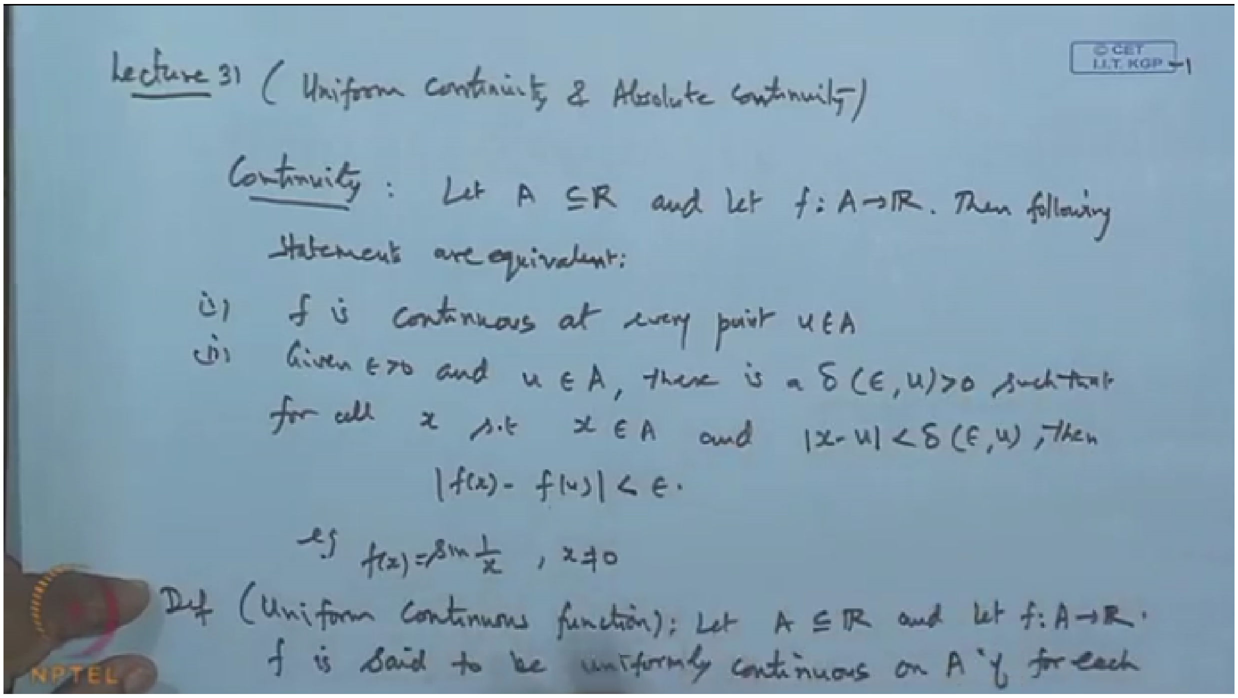

so at a point we can identify a delta which depends on, but when we say the function is uniformly continuous then saying uniform continuous at a point is meaningless, it will be continuous over a set, so a function is said to be uniform continuous over the set A that mean that if for any epsilon greater than 0 if we are able to get a delta which is independent of the points of the set A such that if whenever we pick up

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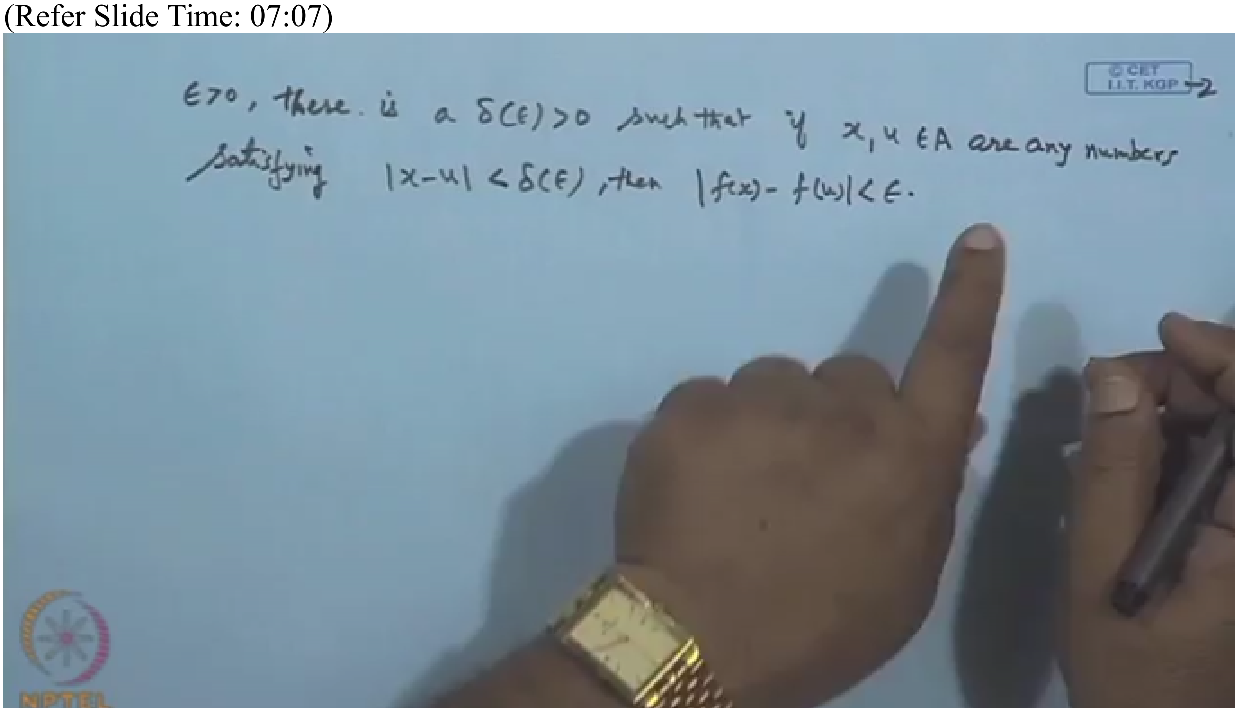

any two arbitrary point set in the delta neighborhood of this then corresponding fluctuation F(x) – F(u) will remain less than epsilon, the value of this.

For example if we take the function F(x) equal to say 2X okay, and this F(x) = 2X for all X belongs to the say real number R, okay. Now if I consider mod F(x)-F(u) where U and V are the point, let X and U these are the points in R, okay, and satisfy that condition, so consider this, this is equal to 2X - 2U which is equal to 2 times X-U, so if we take if we choose delta which depends on say epsilon U, and epsilon/2, so obviously this delta is independent of U, because this is basically we are taking delta to be epsilon/2 whatever the U may be, so it is independent of this, so if I take delta this then obviously this part for all epsilon greater than 0, then this condition holds less than epsilon for all X and U belongs to R such that mod of this X-U less than delta, is it okay? And this is independent of, delta is independent of U and positive quantity, so this function will be considered F(x) = 2X we can say it is uniformly continuous over the entire real line, okay.

However, there are all deep functions which are only continuous but not uniformly, for example if we take the function there so F(x) = 2X each uniformly continuous over any set A which is subset of R, or any subset of R, okay, or entirely. Now take a function G(x) say 1/X for X belonging to the set A which is the set of those points of real number such that X is positive, okay.

Now we claim that this function G is continuous but not uniformly over A, not uniform, continuous at each point of A but not uniformly, let's see how?

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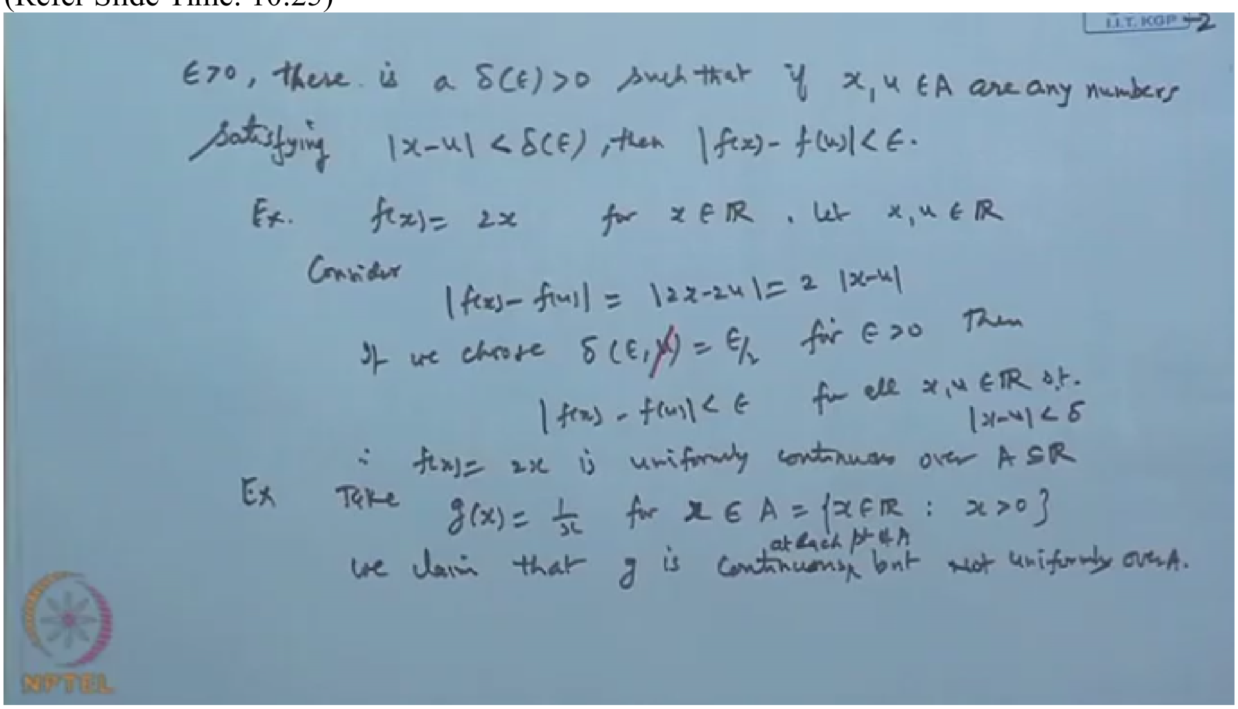

Let us consider G(x) – G(u), G(x) – G(u) this is equal to what? 1/X – 1/U which is the same as U-X over UX, okay.

Now if U belongs to this, if U belongs to A suppose, okay, U belongs to A is given, we wanted to test the continuity is given, so if we take delta which depends on U as the infimum of U/2 and U square epsilon/2, so let epsilon greater than 0 we given, here we let us write let epsilon greater than 0 be given, and let us picked up the point U at which the continuity is tested, so not choose the delta is this, so when you take this delta then if mod of X-U less than delta, okay, and delta which is depending on epsilon U, then I can choose so suppose first it is less than U/2 then so picked up take delta as U/2, so what happen is this shows that X-U is less than U/2 or this implies that X lies between 3/2U and 1/2U, because X-U less than U/2 so it becomes less than 3/2, and then U-S is that is X-U is U-U/2, X greater than this so this is greater than this, so it lies bound for this, therefore the bond for this, therefore 1/X can be X is greater than U/2, so 1/X will be greater than, sorry X is less than so 1/X is less than, because it will be X is greater than this, so 1/X will remain less than 2/U from here, once it is 2/U then the condition, in the condition which we have taken as G(x) – G(u) in this case what we get mod of this, this is less than equal to U-X mod over UX, so U into X, X means 1/X so it is 2 into U, so it can becomes the 2/U square into U-X.

Now further choose delta to be this thing, U square epsilon/2, okay, so once I am taking delta to be this, another one I am checking this, so what we get is from here this shows that this part mod of G(x) – G(u) is less than equal to delta mean mod of U-X is less than delta, so this is less than 2/U square into U square/2 epsilon that is epsilon, okay,

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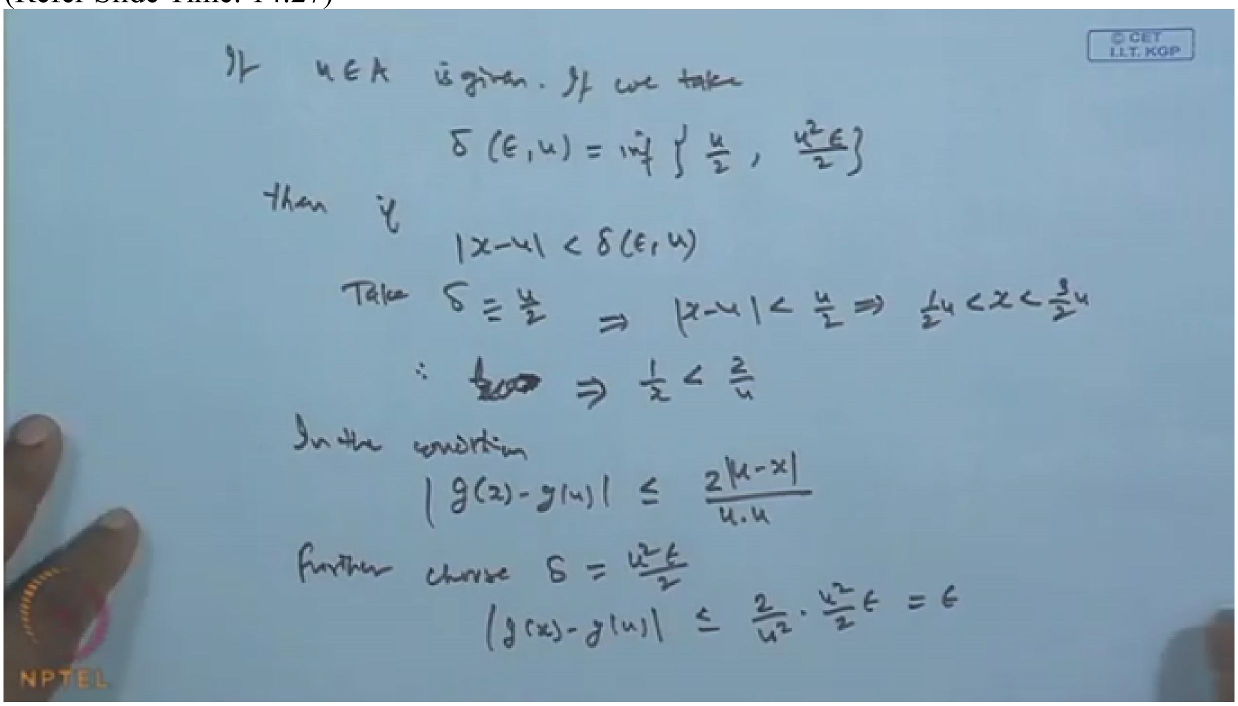

so this holds that if X-U is less than delta for all such X then G(u) X-G(u) will be less than epsilon, and this is true so this shows that G is continuous at U belongs to A, clear? But here the delta which you are choosing is coming with depends on U, it depends on U, it's positive quantity, okay, but what happens to this?

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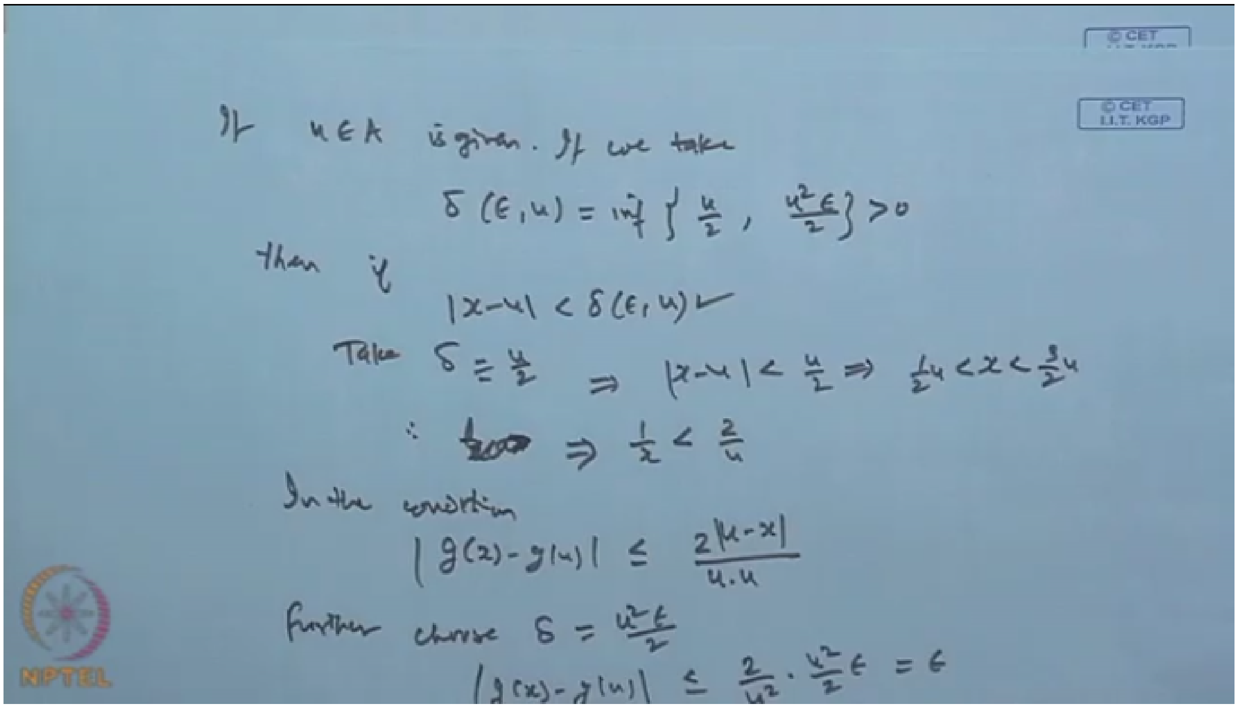

Sorry this is not, each individual delta is positive when U is taking to be there, but when you take the infimum value of this, when you take the delta as the infimum of all such, then what happens to that? Infimum delta here delta depends on epsilon U, so G is point wise continuous, but what is the infimum of all such deltas? Infimum over such delta which depends on epsilon U, and U is greater than 0.

The infimum value of this is coming to be 0, why? Because each delta here is nothing but either U/2 or U square/2, so U is greater than 0, so each delta is greater than 0, but when you take the infimum value of this delta over U then this infimum will come out to be 0, so we are not getting a delta which is greater than 0, so for a given epsilon greater than 0 we cannot get delta which depends only on epsilon, and greater than 0, such that condition mod of G(x) –G(u) is less than epsilon provided mod of X-U less than delta hold this we cannot get, therefore G is not uniformly continuous, okay. So we have seen the example where the function is continuous, and point wise and the function is uniformly continuous.

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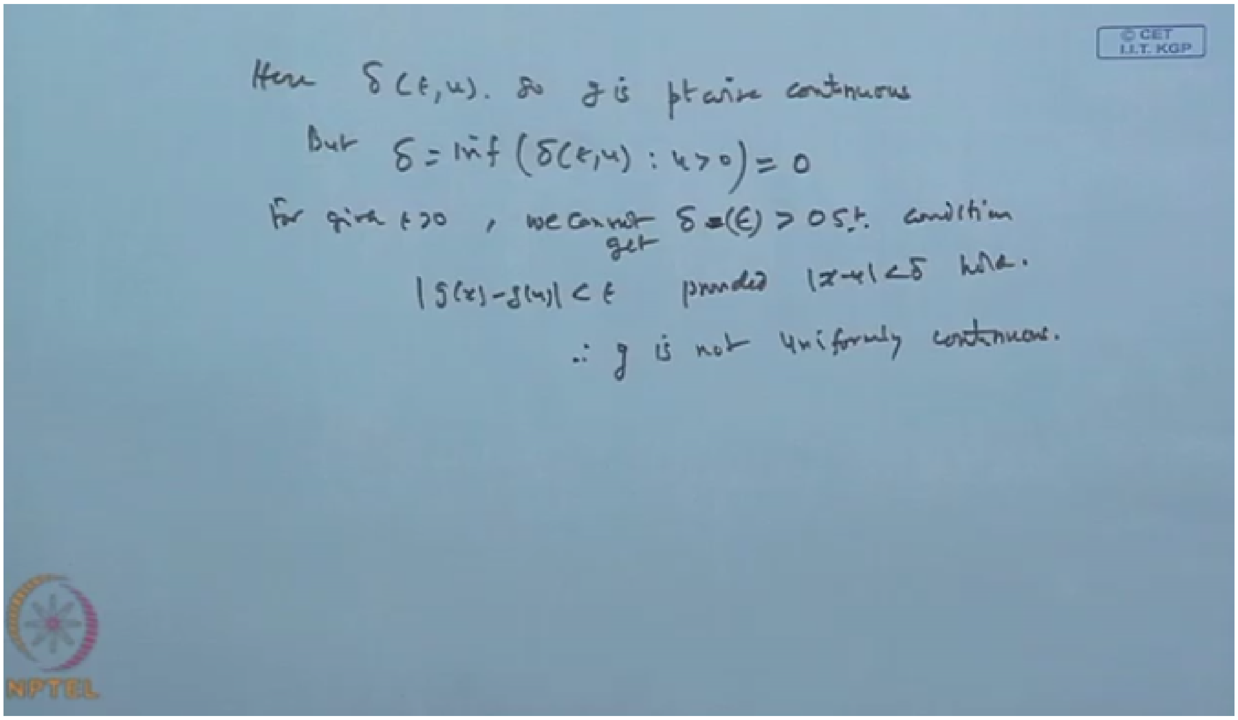

Now to show the uniform continuity we require the delta, we are independent of the point over the entire set, so that's not that easy, so what we can develop thus test which will give the, at least sufficient criteria when the function is not uniformly continuous, so we will just state those results without proof, the criteria for the non-uniform continuity, so the non-uniform continuity criteria, this will be needed, so proof we are just dropping but it can be easily done with the help of previous knowledge.

So let A be a nonempty subset of R, and let F is a mapping from A to R, then the following statements are equivalent, the first statement says F is not uniformly continuous on A. Second statement says that there exists an epsilon naught greater than 0 such that for every delta greater than 0 there are points say X depends on U, X depends on say delta not U,
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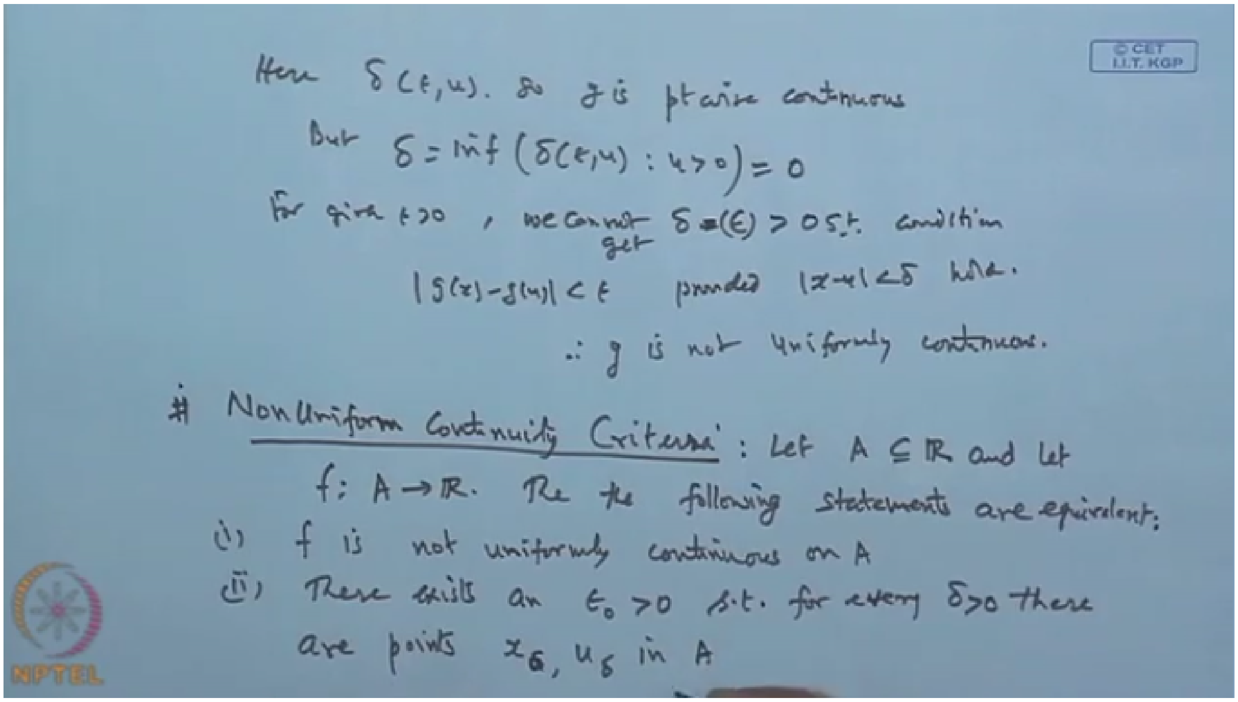

X depend on delta and then U depend on delta in A such that the mod of X delta -U delta less than delta and mod of F(X delta) –F(U delta) is greater than or equal to epsilon naught.

And third statement says there exists an epsilon naught greater than 0 such that, and the two sequences say XN and UN in A such that limit of XN-UN over N is 0, and mod of F(xn) – F(un) is greater than equal to epsilon naught for all N, N belongs to capital N.
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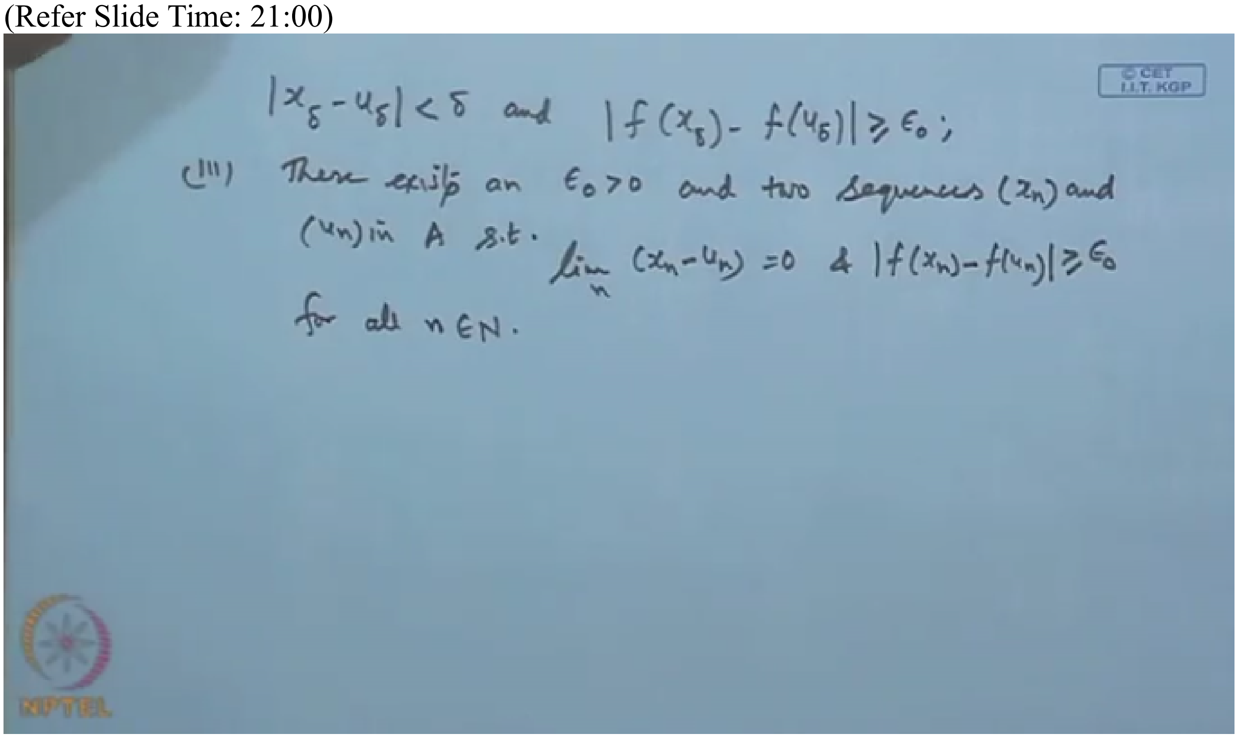

Let us see this, what is that? So uniform continuity criteria says if suppose function is not uniform then by the definition of not uniform means a function is said to be uniform continuous, if at over the set A if for each epsilon there exist a delta which depends only on epsilon not on the delta such that the difference of F(x) – F(u) is can we made less than epsilon provided the point R in the delta neighborhood. So if the function is not uniformly continuous it means this condition will be violated, if we choose the point in the neighborhood of delta the images of this, the fluctuation may not be less than epsilon it can exceed to any number, arbitrary number epsilon naught, so that's why what he says is that if F is not uniformly continuous then there exists an epsilon naught such that whenever the point X and U are in the delta neighborhood the corresponding images exceed that bound epsilon naught greater than it.

Similarly this is in a Cauchy definition, this is Hahn's definition instead of choosing the two arbitrary point if we picked up the two sequence XN/N which are tending to 0, the limiting, both are difference of this is tending to 0 means XN and UN all very close to each other, as N is sufficiently large, then what the corresponding image is not close, it's greater than equal to some positive number epsilon naught, then we say the function F is not uniformly continuous okay, we got this point.

Now this criteria can be applied very directly suppose I applied this function to show G(x) which is 1/X is not uniformly continuous on the set A where X is greater than 0, set of all real number A, so what we do is we have to picked up the two arbitrary sequence, so let XN I choose the 1/N and UN I take to be 1 over N+1 both are in A, and the difference of these two sequences obviously XN-UN this goes to 0 as N tends to infinity, but what is our G(xn) – G(un)? This mod is nothing but what? G(xn) is nothing but N+1 - N that is 1, which does not go to 0, in fact it is greater than equal to any number epsilon naught, therefore G is not uniformly continuous, that's what.
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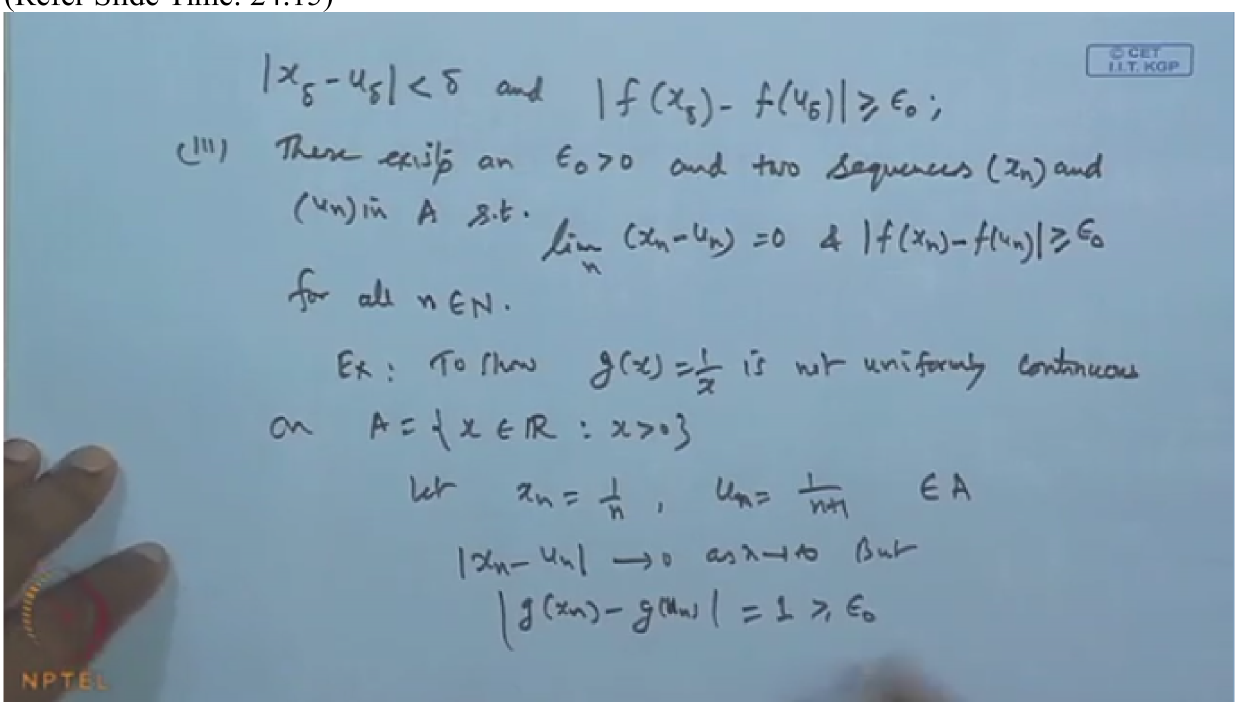

Now uniform continuity, when it is function is defined over a closed interval and if function is continuous then it will be uniformly so this result is known as the uniform continuity theorem, the theorem says let I be a closed bounded interval, and let F which is a mapping from I to R be continuous on I, then F is uniformly continuous on I, so what we said if the function F which is continuous over a closed interval bound, closed and bounded interval then function must be uniformly continuous.

So suppose F is not uniformly continuous on I, so I can use one of the criteria which I listed earlier, I will take that in the form of the sequence, okay, now so what we say? So by the previous result or previous theorems where criterias are there we can choose then there exist a epsilon naught greater than 0 and two sequences XN and UN's in I such that the difference between these two that a limit of this is going to zero means, difference is very, very small say 1/N and but the mod of F(xn) – F(un) this difference exceed by this epsilon naught for all N, so this is why the criteria where non-uniform the continuous criteria, from here we are getting this part, okay.

Now since our, since I is bounded and the sequence XN and UN's both are the sequences belonging to, belong to I, so by Bolzano Weierstrass Theorem every bounded sequence has a convergent subsequence, so there is convergent subsequences say XNK, if there is convergent subsequence, let us take first this part then we can take belongs to this, so there is a convergent subsequence XN of XN that converges to an element Z, to an element say Z belongs to I, Z that converges to Z.
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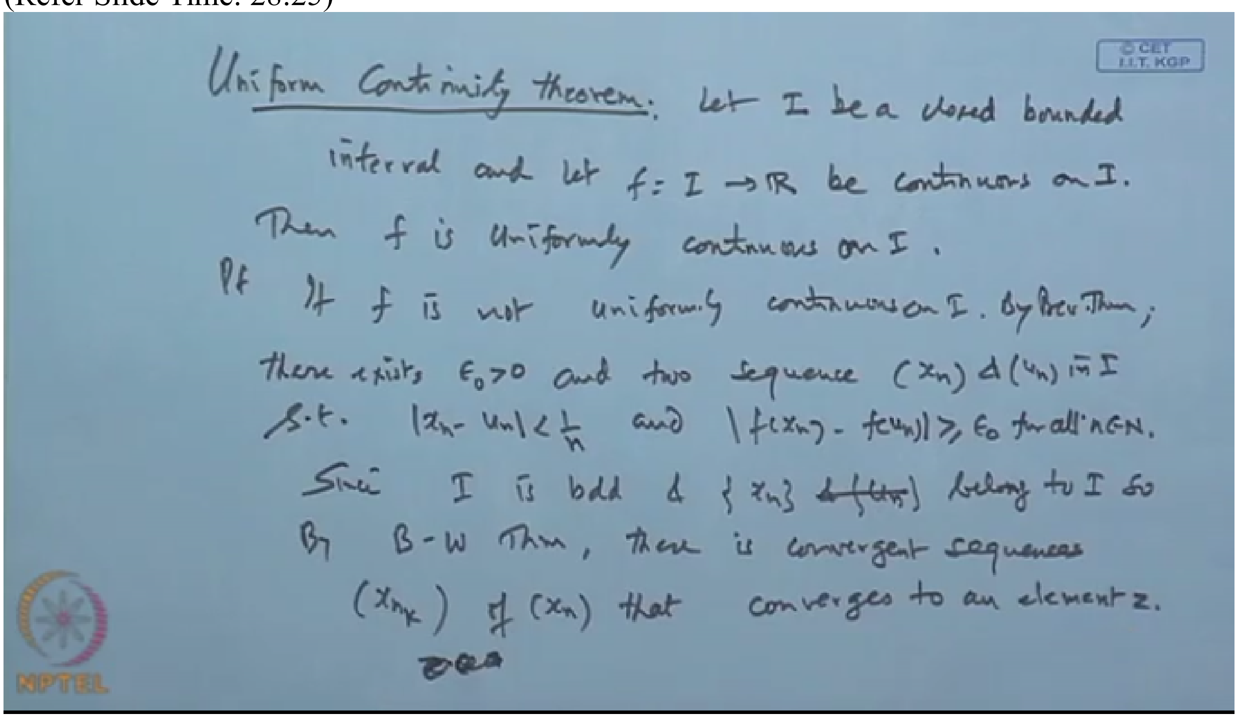

Now we wanted to show Z is a point in I, which follows because I is closed, since I is closed so all the limit point of a sequence in I must be point in there, and also all these sequence XNK lies between the lower and the upper bound of this interval, so by the Sandwich theorem the limit of this sequence XNK must be the point in I, so this implies, since I is closed the limit point Z belongs to I, okay.

Similarly we can say V, similarly we can also show, we also claim that the sequence UN will have a subsequence UNK in I whose limit point belongs to I, limit point belongs to I but the limit point of UNK and XNK will be the same, but limit point of UNK and the limit point of say XNK will be the same, the reason is because if I consider the UNK – Z then this can be written as the UNK - XNK + XNK – Z, now this term is less than equal to K by condition which we have chosen, because both are in A and we are choosing the interval neighborhood is such a way so that this is less than equal to K, and this converges to Z, this is the limit point of this so it goes to 0, so S stands for total tends to zero, therefore both will have the same limit point, once they are having the same limit point therefore F, so F, so Z belongs, so F is continuous.

Now further since F is continuous at Z then both the sequences then F(xnk) and F(unk) must converge to F(z) because XNK goes to Z, so F(xnk) will go to F(z), UNK goes to Z so F(unk) will also go to A, because F is continuous,
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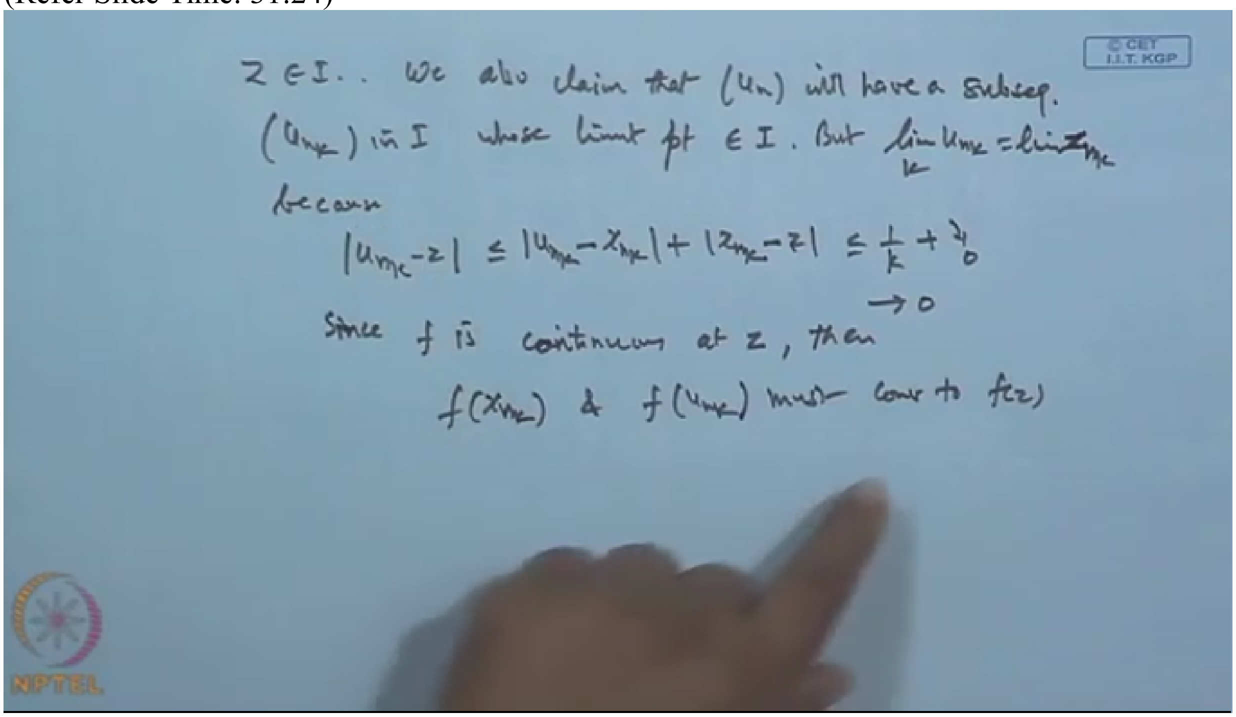

so once they are contain but the given condition is, given hypothesis is that mod of F(xn) – F(un) is greater than equal to epsilon naught, this is given, so they’re contradiction, and contradiction is our, because our wrong assumption that function is not uniformly continuous, so therefore F is uniformly continuous over A,
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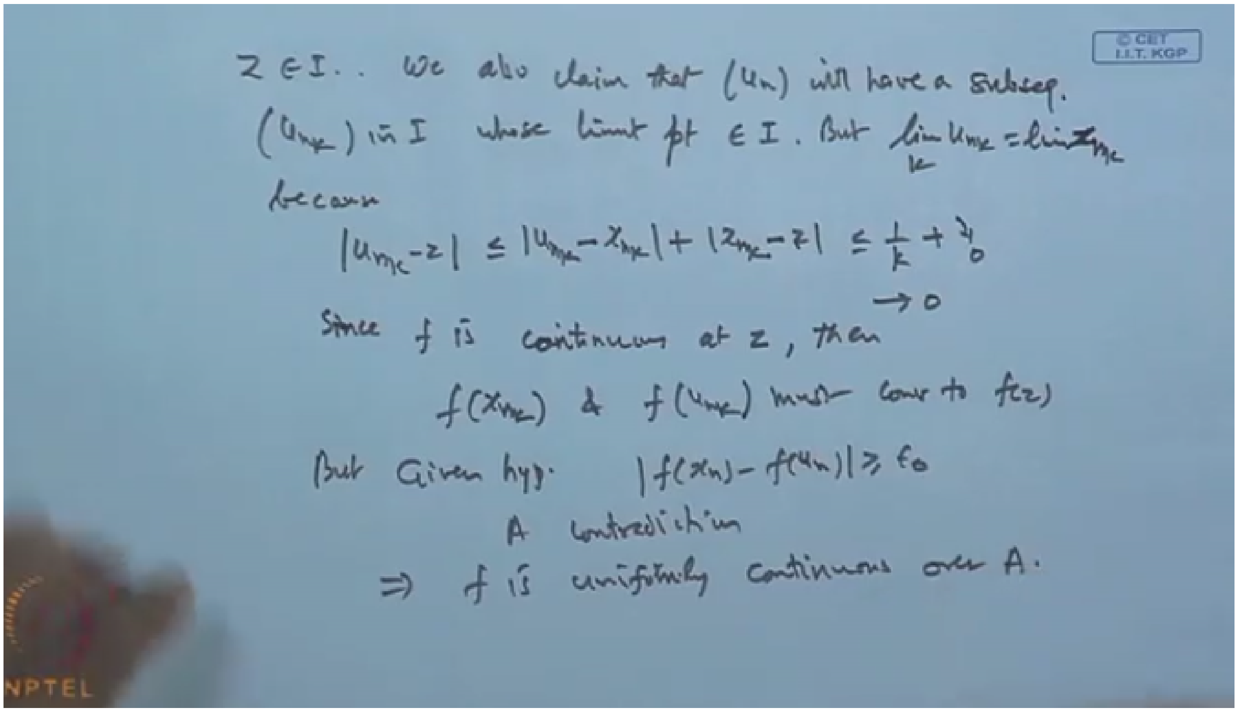

that shows that, is that okay?