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Course

On

Introductory Course in Real Analysis

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Lecture 50: Location of Root and Bolzano's Theorem

Okay, so today we will discuss next result is also in testing that source the location of the roots, location of roots in fact this also known as the bijection method, it will be used in the bijection method for this known as that, this algorithm is known as the bijection method, so we are not touching okay location.

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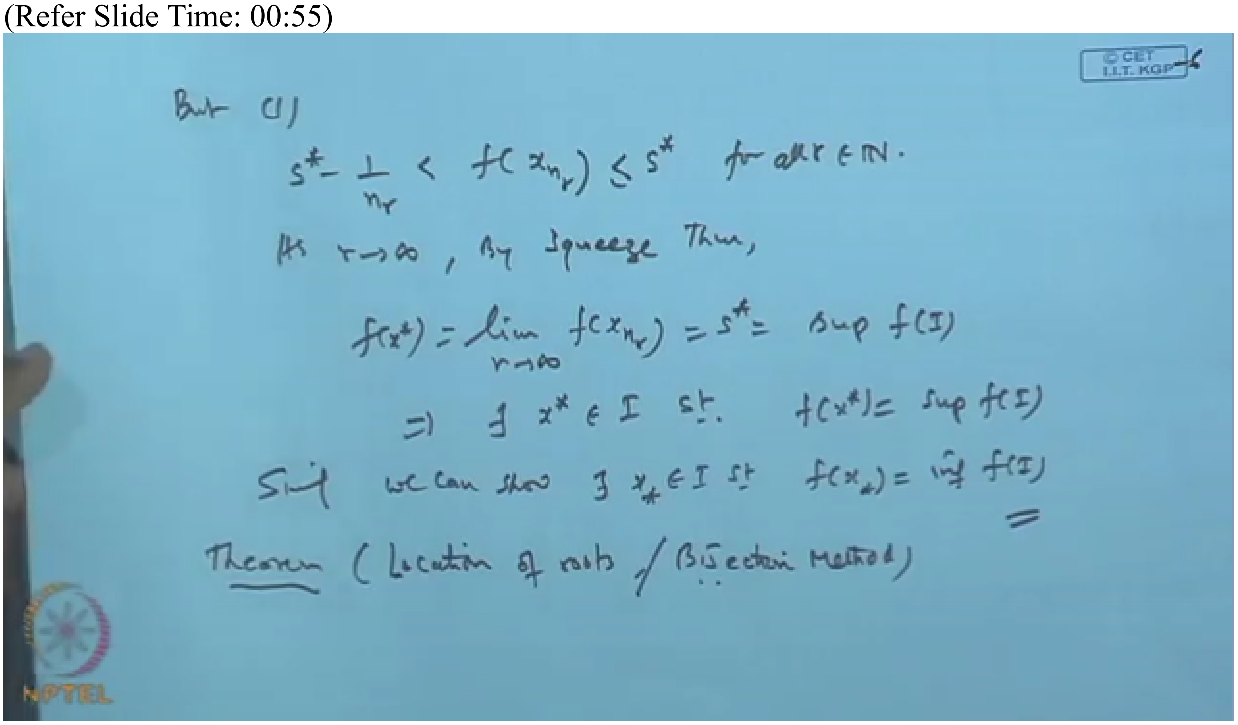

What this result says is let I be a closed and bounded interval of R, and let F is a mapping from I to R be a continuous on I, now if at the point A is suppose negative, at the point B it is positive, or if at the point of A it is positive, and at the point of B it is negative, that is at the corner point if the function attains the different sign,

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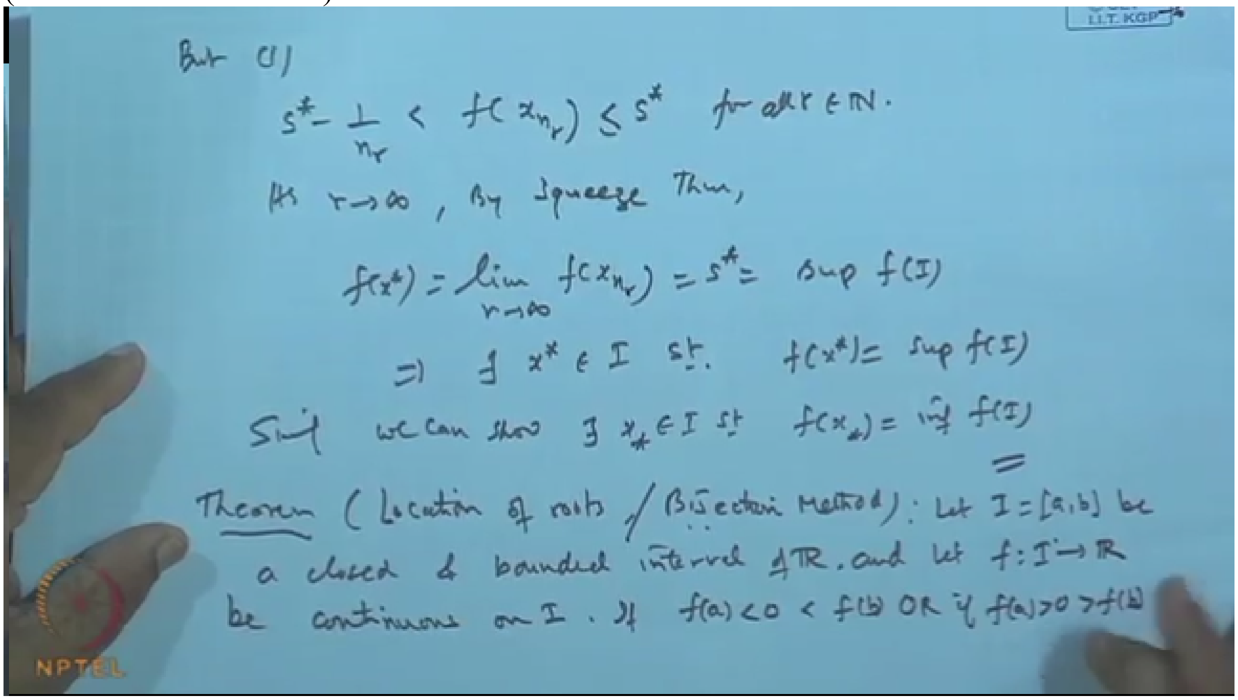

then there exist on number C, there exists a number C belongs to the interval A, B such that the value of the function at the point C will be 0, so this source that we can identify the root of the function, if a function is defined over the closed interval A, B which has an alternate sign that is

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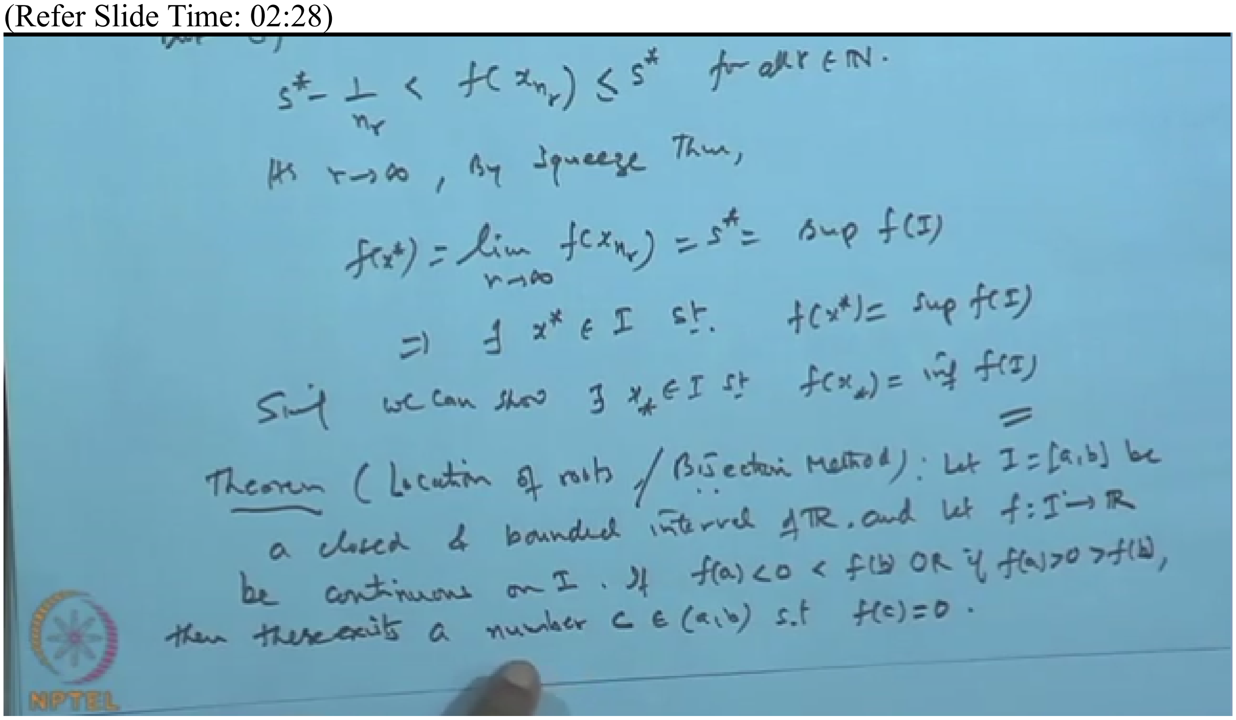

at the point A if negative, at the point B is known, so there will be at some point C where the N function is continuous, so obviously then the function is continuous there is a continuous graph, so when the function is negative it means the part of the graph is below the X-axis, and part of the graph is above the X-axis, so obviously because of the continuity of the curve, the curve definitely cross the X-axis so that point where it crosses will be the point C, where F(c) will be 0, and that's the shows the location.

Now proof of this is in fact we will generate the sequence of successive bijection just like, so let us suppose I1 is an interval say A1, B1, okay, and assume let us first assume that F(a) is negative, and F(b) is positive, okay, now I1 is the interval where A1 is suppose A, and B1 is supposed B, okay, and let P1 is the middle point of A1+B1/2.

Now if our F(p1) is 0, then result follows, is it not? Suppose it is not, suppose F(P1) is not equal to 0, then either F(P1) will be negative or F(P1) will be positive, if F(P1) is negative, if F(P1) is negative then in that case we take the A2, A2 is our P1, B2 is our B1, and in case if this is positive then take the A2 as our A1, where the B2 is our P1, okay, and consider the interval A2, B2, consider the interval A2, B2 like this, okay this is fine. And then one of the case you will be open so one of the interval A2, B2 even that, then find the point V2 which is again the interval half of this, that is basically A2+B2/2, so basically this length when you are taking this A2/2 then test the functional value as F2, if it is 0 then result follows. If not then again either F(P2) will be positive or F(P2) will be negative, so again continue the same process as above, okay.
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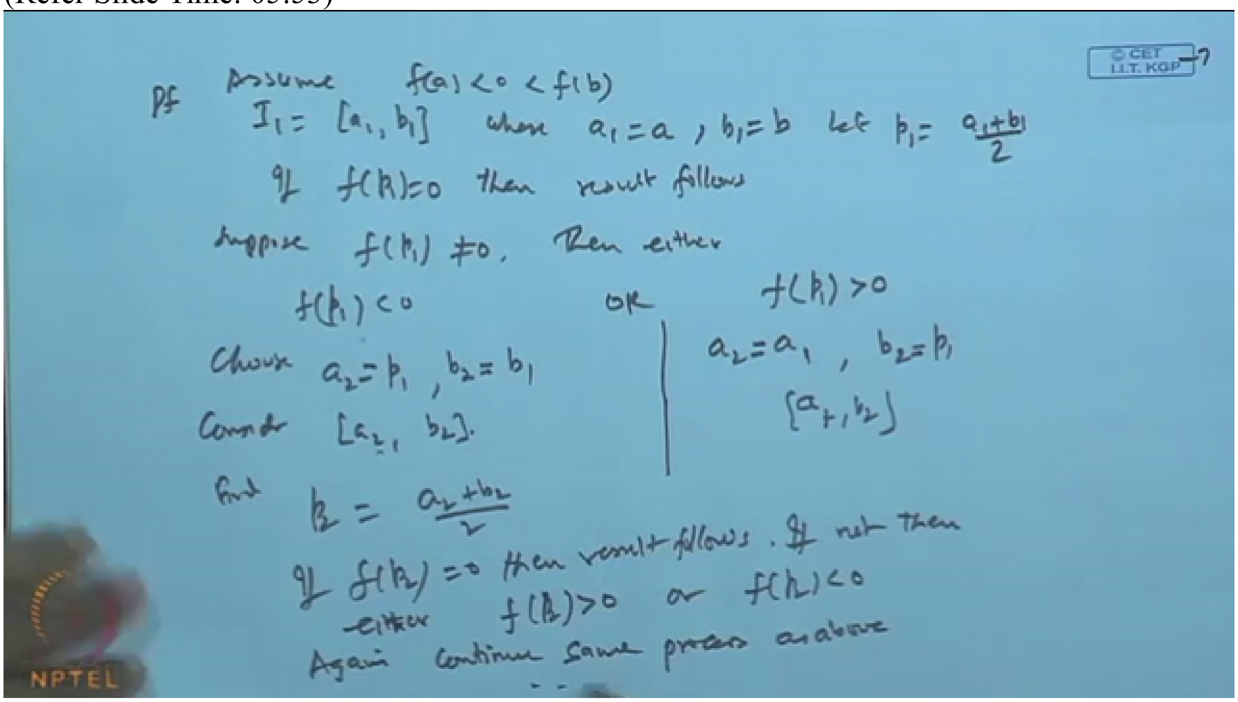

So suppose we are getting after, at the nth stage what we get? So suppose we get the sequence of nested closed intervals AN, BN with length such that for every N belongs to capital N, we have the value of F(n) is negative, and value of the function at the point BN is positive, so this is the sequence of nested intervals, say A1, B1, then maybe once you divide here is A2, B2 like this further divide and like this, so we get this nested, sequence of the nested intervals we are getting or maybe sometimes here or there that also possibility may be like this also that instead of this bigger this or maybe this and so on like this, so we get a sequence of the nested intervals which covers, which is contained totally in the previous one, and length of this, with the length will be BN-AN and that is equal to B-A over 2 to the power N-1, this will be the length of the interval AN, BN, length of I, IN which is AN, BN, this one, okay.

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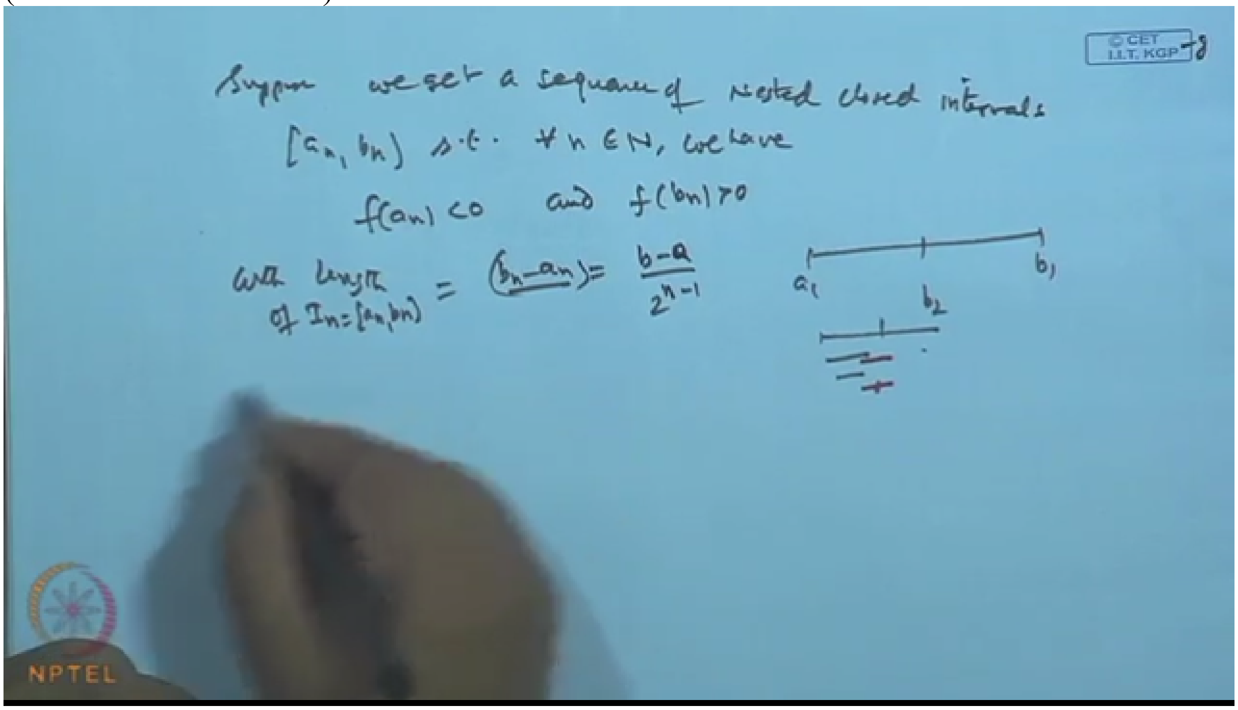

Now let us see, here we get the sequence AN which is less than equal to, okay, AN which is less than equal to BN, next interval, okay, so what we get it here is, so here we get a sequence of the nested interval AN, BN say IN such that I1 covers I2 covers I3 and so on, and the finite intersection of IN, when N = 1 to R is nonempty, okay, so there will be, so by the result which we have nested interval property, by nested interval property there exist a point C, there exist a point C that belongs to IN, belonging to IN, IN for all N, this is nested interval property, okay, so since for all IN, okay.

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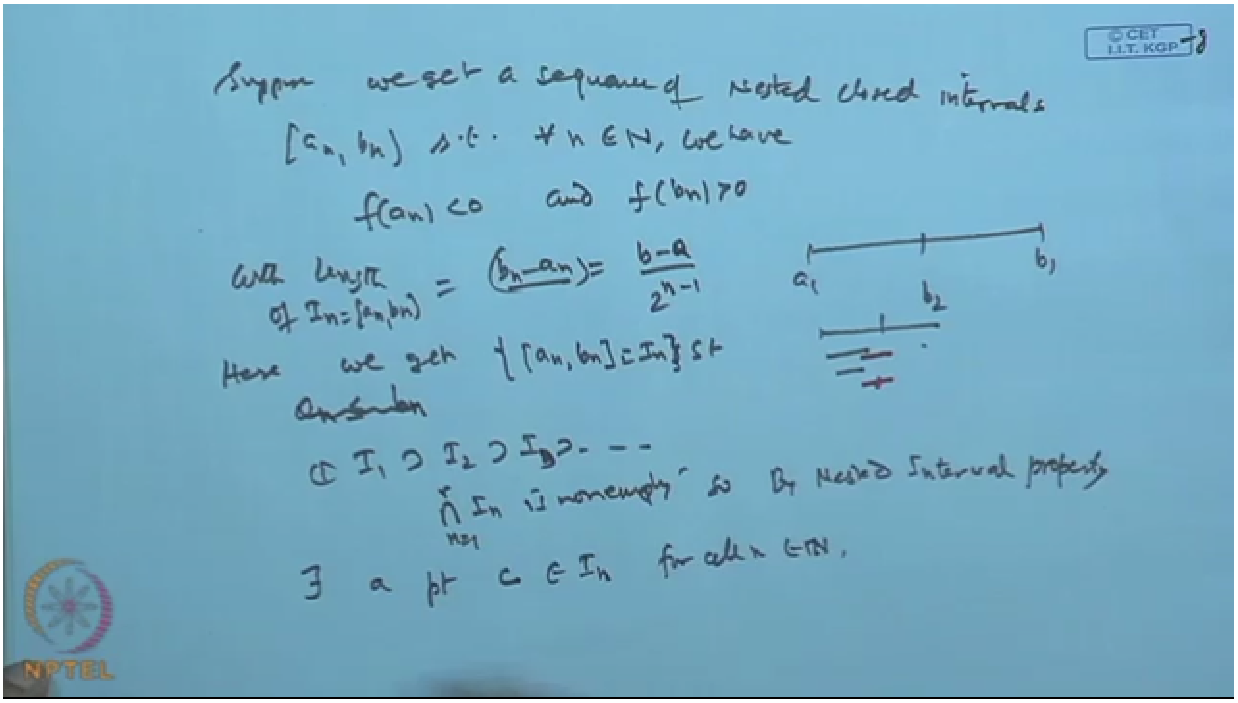

Now since C lies between AN and BN for all N belongs to N, for all N belongs to N we have that 0 less than equal to C-AN which is less than equal to BN-AN which is equal to B-A over 2N-1 and 0 less than equal to BN-C which is less than equal to BN-AN, and which is equal to B-A over 2N-1, this is true, so when N tends to infinity this is tending to 0, this is tending to 0, so this shows limit of AN is C, limit of BN is C,

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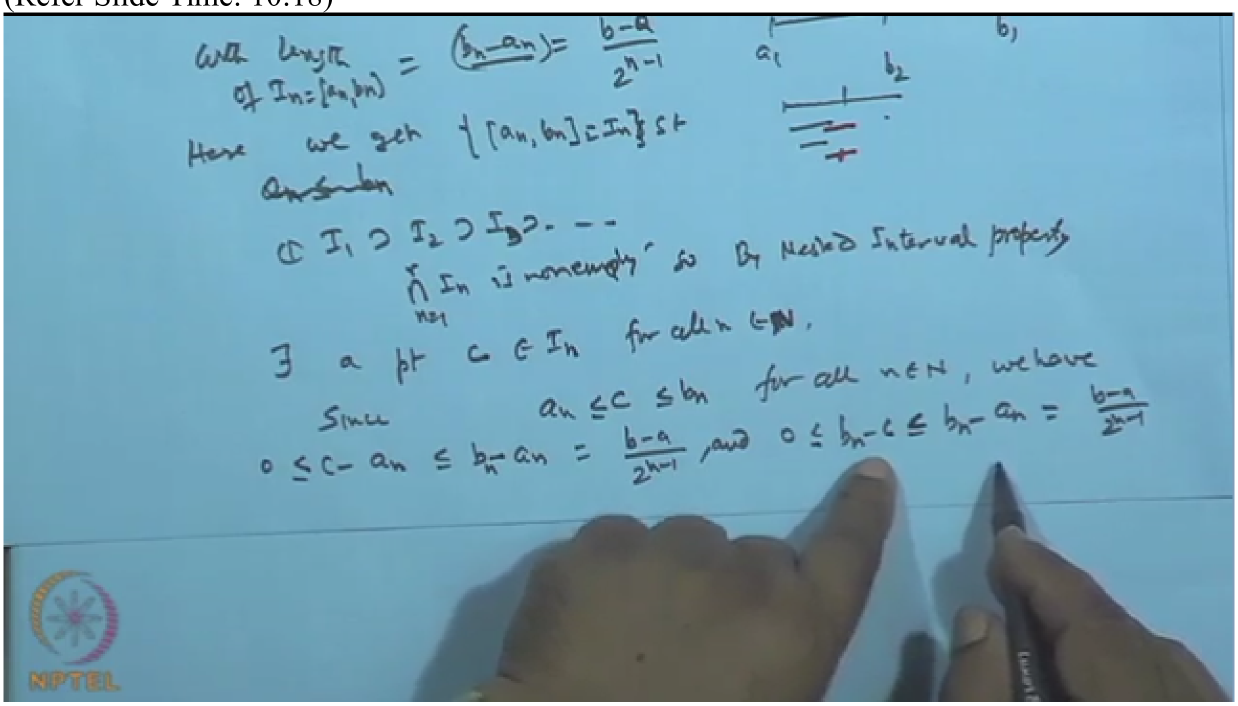

so this implies that as N tends to infinity limit of AN is C which is the same as the limit of BN, okay, but F is given to be continuous, what F is continuous, so continuous at this point, continuous on I, so continuous at C also which is in I, therefore limit of this sequence F(AN) as

N tends to infinity is nothing but the value of the function at the point C, which is limit of F(BN), so this source, okay.

Now further F(AN) is always be negative for each N, and F(BN) will always be positive for each N, so therefore the limit of this sequence F(AN) which is equal to F(c) will be less than or equal to 0, and from here the limit of F(BN) when N tends to infinity which is also F(c) will be greater than equal to 0, so when you take this two together we get F(c) = 0, and that proves the root, the source C is the root of F, that is there exist as C where the function will be 0, so if alternate positive negative then we can get this thing,

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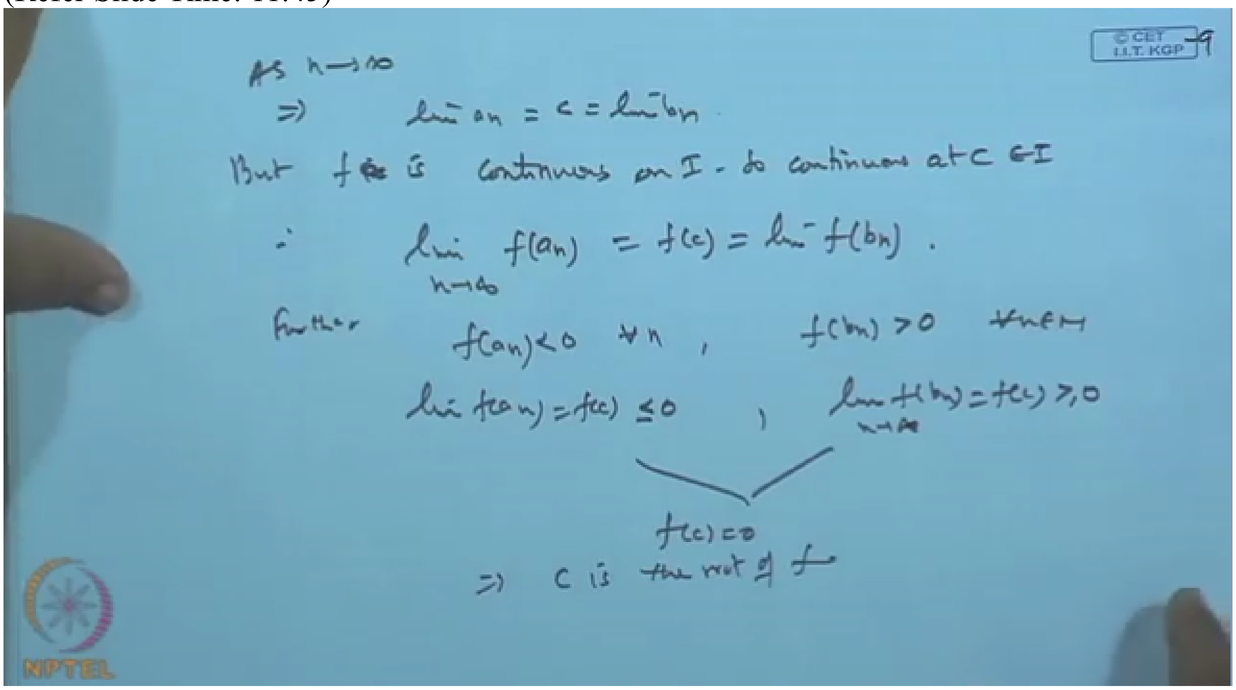

so that's where it testing that, it is basically used by numerical methods, and in numerical to get the approximate root for the function F(x) = 0.

Next result which we have the Bolzano, Bolzano intermediate theorem, what this theorem says let I be an interval and let F which is a mapping from I to R is a continuous function, be continuous on I, F be continuous on I. Now if A and B, if A, B belongs to I and if K is any real number satisfying the condition, satisfying or satisfies the F(a) is less than K which is less than F(b) means in between F(A) and F(B) we are choosing a real number K, then there exists a point C in I between A and B such that the value of the function at the point C is K this is known as the intermediate theorem, means if F is a continuous function then it will attain all its values in between the maximum and minimum value, so in fact here we are not getting mixed, here we are not discussing about the maximum, what we are saying we are taking two particulars of the function F(A) and F(B) and they are distinct.

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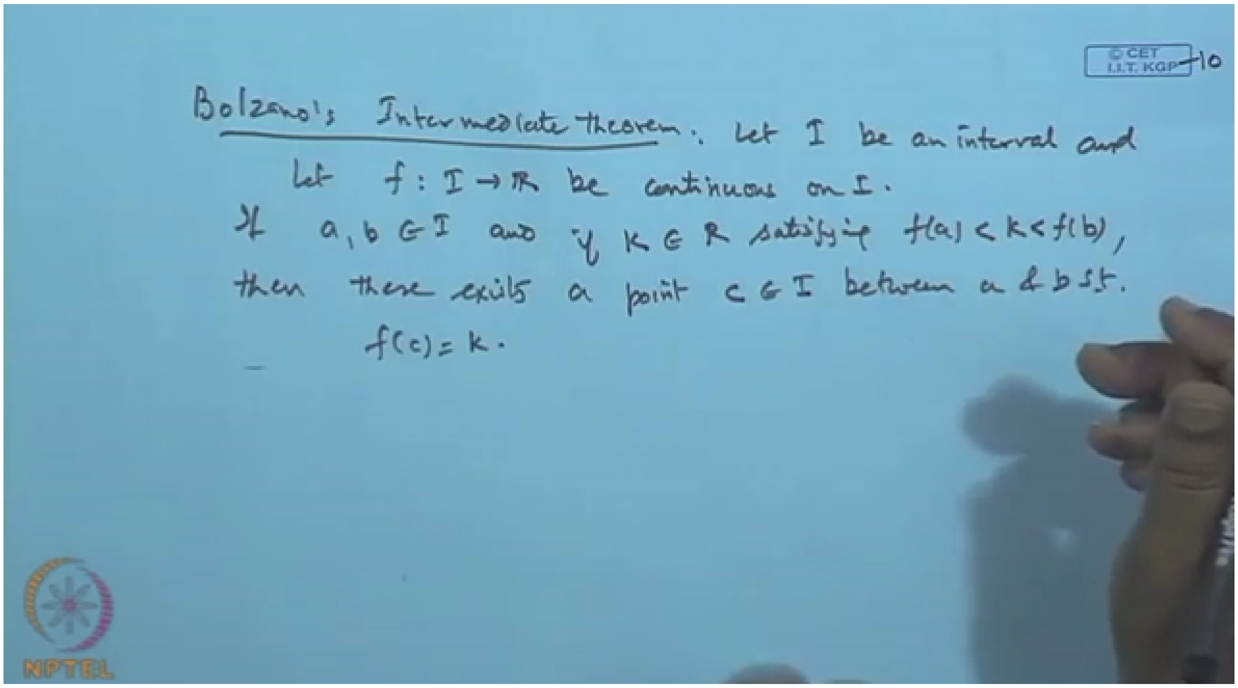

So if we picked up any number in between F(A) and F(B), and F is continuous so there will be at some point C available where this number will be attained by the function at some point, so that is known as the intermediate zone, so proof of this is like that, suppose that A is less than B, let's take this one first, okay, and let G is defined G(x) is choosing X, F(x)-K, so if we look this function then clearly at the point A, G(a) is F(A)-K, F(A)-K is negative and G(b) is positive, so this function G is a value negative at the point A, positive at the point B, G is continuous function because F is continuous, K is constant, so additional, subtraction of the continuous function is continuous, so G is continuous over the interval A, B therefore by the intermediate theorem, by the location of the root which we have proved earlier there will be at some point in between A, B where the derivative, where the function of G will be 0,

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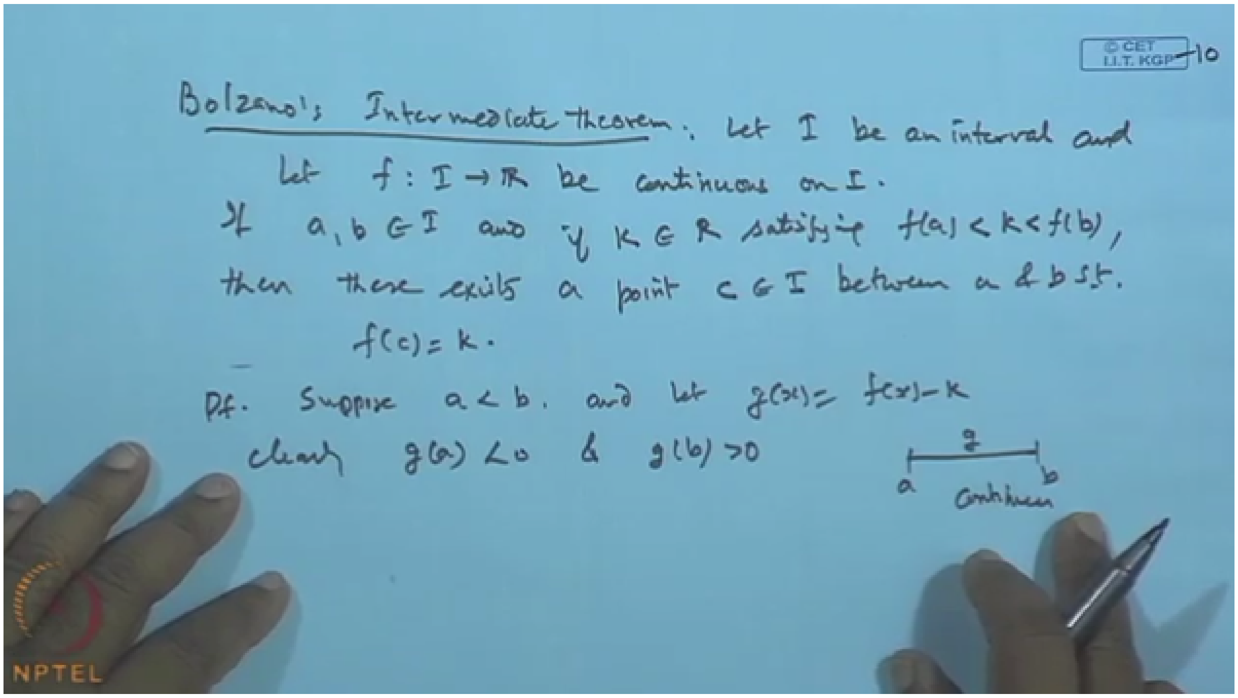
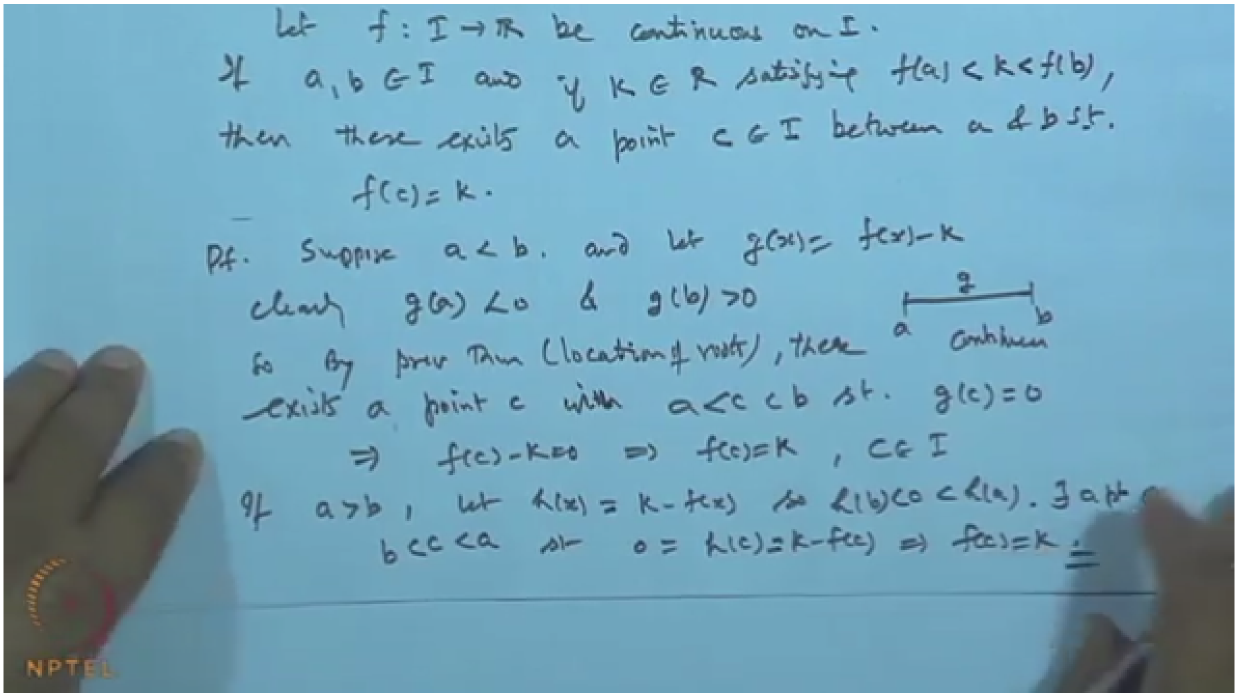


so by previous theorem that is location of roots, there exists a point C such that with C lying between A and B, such that the value of this G(c) is 0, but what is G(c)? But G(c) is nothing but the F(c)-K0, so this implies F(c) = K, so there will be a point C available in I where the function will be attained, and the cable value is attained by the function at this point.

Similarly if we take suppose A is greater than B, second if A is greater than B, then what happens? You consider the function H(x) instead of this, we say K-F(x), okay, so that H(b) will be negative, and H(a) will be positive, so again there exists a point against a C lying between B and A such that the value of H will be 0 at this point that is K-FC, so this implies F(c) = K and that’s proved.



Now we can extend this result, and because this is for any value lying between the two unequal values of F, so if we replace this F, and capital F(a) and capital F(b) by its minimum value or the maximum value that is infimum of this and then supreme of this, if we choose K in between infimum and supremum then also we can get the sum point value, so as a corollary to this we can say let I be a closed bounded interval, and let F is a mapping from I to R be continuous on I.

Now if K is in R, any number satisfying the infimum of F(i) that is infimum of F(i) is less than equal to K which is less than equal to supremum of F(i) that is the minimum value and maximum value, so K lies between minimum and maximum value then they are exist a number C in I such that the value of the function at the point C will be K, means this K will be attained by the function at some point, okay.

Proof follows just from the maximum-minimum theorem and above this previous , Bolzano intermediate theorem, so it follows from use maximum-minimum theorem and location theorem of location of roots theorem, so what we see here there exist a point C star that is so there exist a point say C star, C upper star and C lower star okay in I such that the infimum of this will be F(c) lower star, infimum will attain because I is a closed and bounded interval, F is continuous function, so infimum will be attained, and there exist a point C lower star, we have the F(c) lower star and the infimum value, and then this is less than equal to K, which is less than equal to F(c star) this is the same as the supremum of F(i).

Now conclusion follows from the Bolzano, so from Bolzano Intermediate theorem, we have we get a point C belongs to I such that the value of the function at the point C is K follow, and that's proves the result which is,
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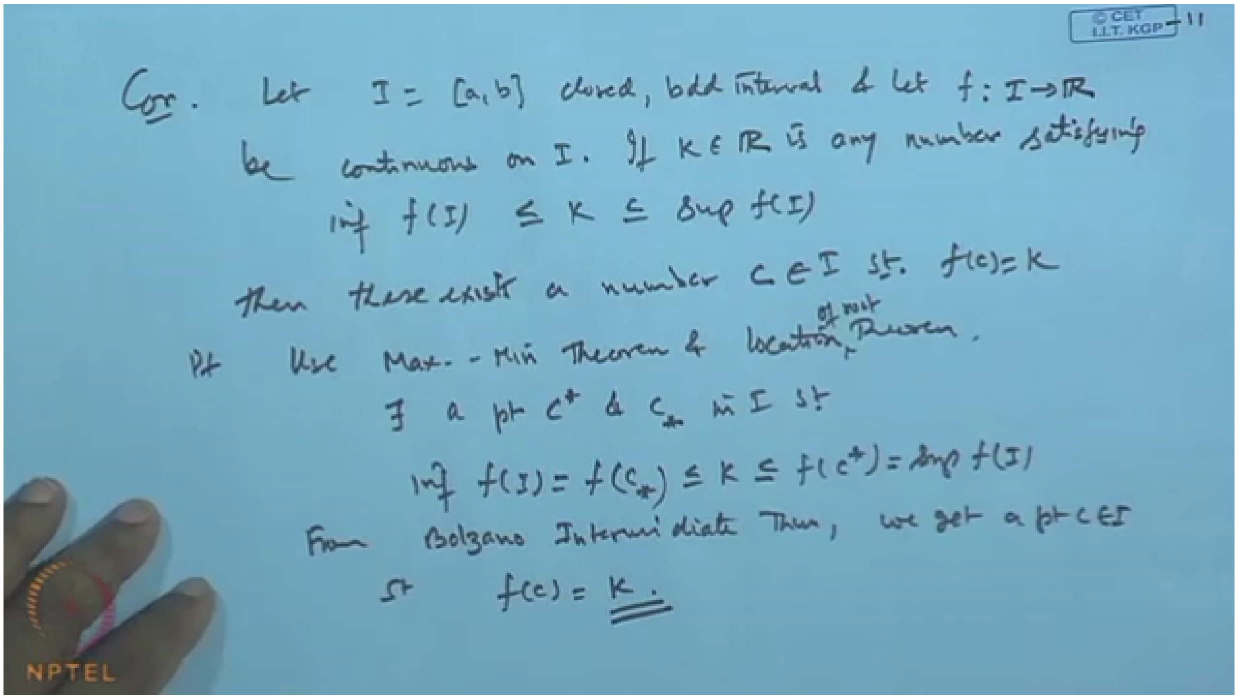

now we get one more results which is also true in case of the continuous function, the transfer of a interval, if F is a continuous function then it will transform the closed set, close interval to the closed interval, but if interval is not closed then the image of the interval other than the closed interval that is open interval or semi closed interval need not remain to open or semi-closed that is the nature of the closed interval is only retained by a continuous function, but if the interval is not open and closed then it's natural may change depending on the function, so we get this result first for this, first of all let I be a closed interval, I be a closed bounded interval, closed bounded interval, and let F is a mapping from I to R be continuous on I, then the set F(I) which is the set of F(x) such that X belongs to I is a closed bounded interval.

So proof is just like, let suppose M is the infimum value of F(I), and capital M be the supremum value of F(I), suppose this M, okay, then we know that from maximum-minimum theorem the MN belongs to MN, now by maximum-minimum theorem this m and capital M belongs to I, because they exist and there will be a point where it belongs to I, okay, so therefore the every value of F(I) moreover the functional value will lie between the interval m and M, because its maximum value and minimum value only it will be there, so we can get the maximum value and minimum value, all the values lie in between this, okay, so if K is any point conversely, if K is any point belonging to this interval then they'll exist a point C, K is any value in between m and capital M then they'll exist a point C in I such that the value of the function at the C is coinciding with K, it means K is an element of F(I), so we conclude that any value in between F, M is also contained in this therefore we can say m capital M this closed interval is contained in F(I), so combine these two we get F(I) is nothing but m capital M is a closed bounded interval, that is image of the closed bounded interval under the continuous function is closed and bounded, okay.

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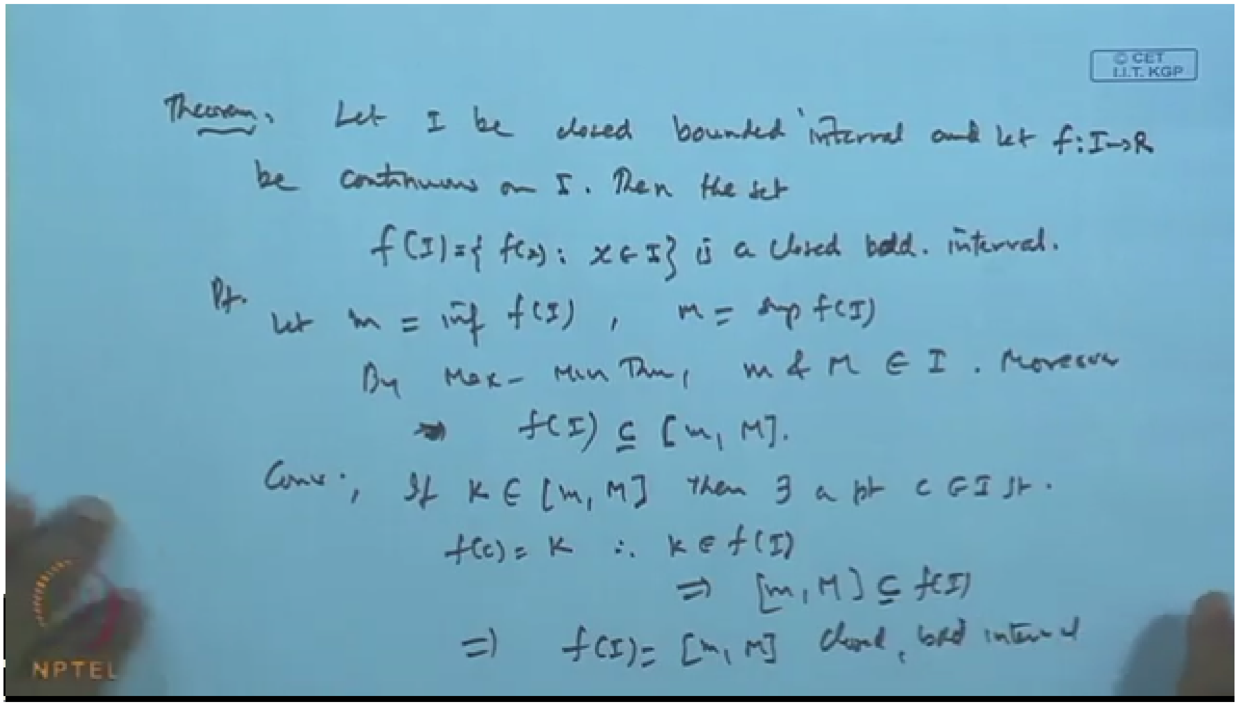

Let's see the note, the image of if F is continuous and I is an open interval, then image of F(i) need not be open, for example if we take the say function F(x) is 1 over X square + 1 and I is -1 to 1, then F(I) you can see just 1/2, 1 this is not open, closed at one point, not open okay

Similarly if we get, if F is say semi-closed interval then also suppose I2 is our interval say 0 infinity, a semi-closed interval and F(x) is the same as X square + 1, we see F(I2), that F(I2) comes out to be a semi-closed interval, but open at this point, so what we say I2 which is not closed interval, okay, so if it is not that big,

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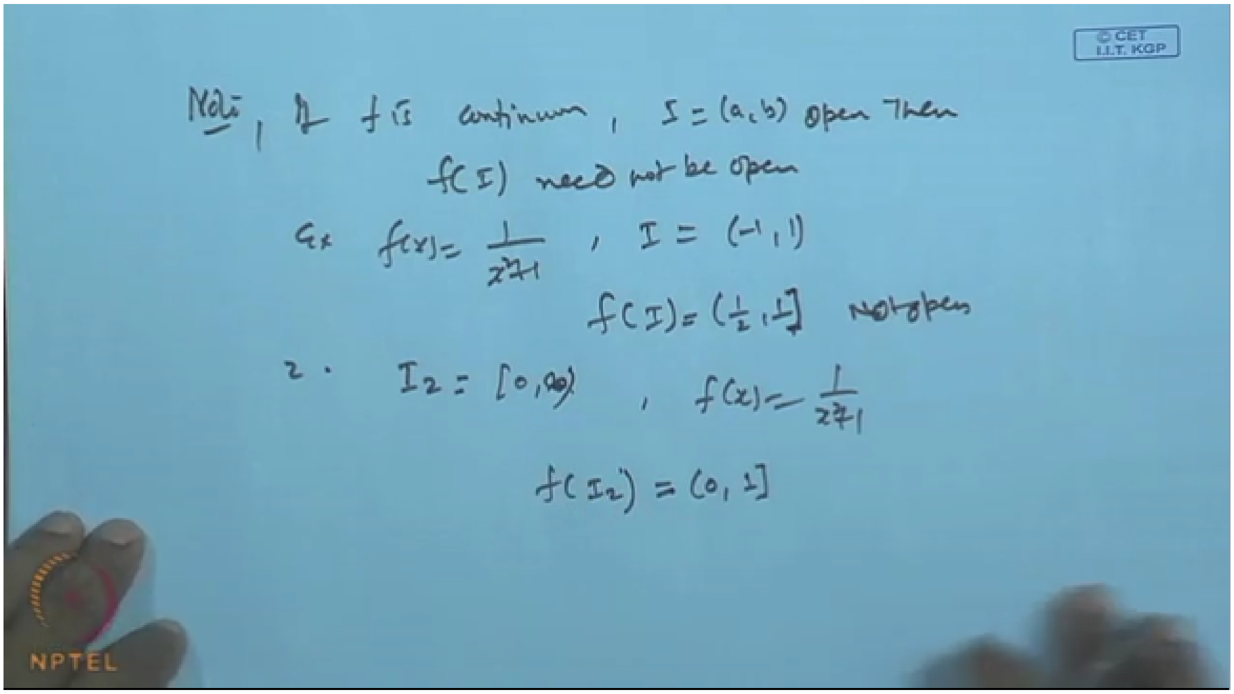

so this shows that similarly third if you take F(x) = sine X and if we choose the interval say -pi to pi then image of this interval F(i) will be the closed interval -1 to 1,

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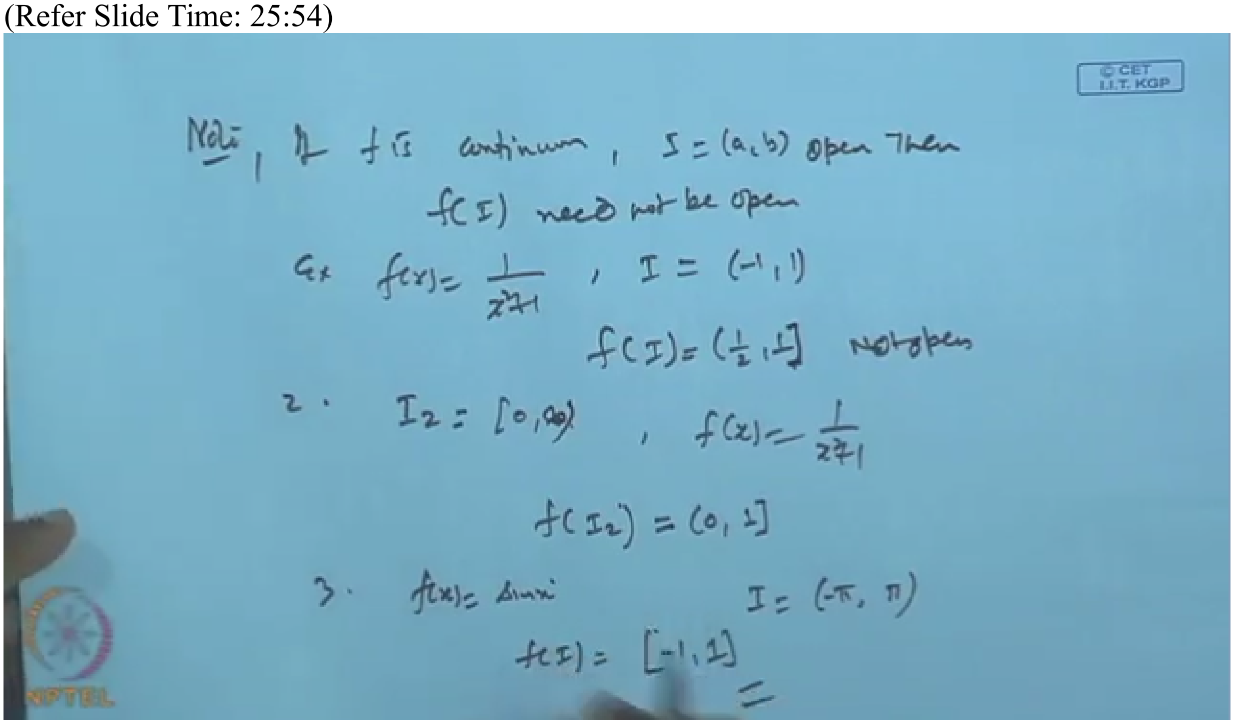

so this shows that only the closed intervals under the continuous function remains closed image, otherwise if the interval is not closed the image of that interval and continuous function need not be the same nature. Thank you very much.