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Course

On

Introductory Course in Real Analysis

By

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Lecture 49: Boundness Theorem and Max-Min Theorem

So this is in continuation of our previous lecture, we wanted to discuss the various properties of the continuous functions. And as a consequence of that we see that if function is continuous over a closed bounded intervals then we have some results which is known as the Boundedness Theorem, max or minimum theorem, and the Bolzano's theorem. And this gives you the also a criteria to find out an approximate solution for the root of the function, that a root location with the help of this, okay.

So let's see first what is the bounded, before going for the Boundedness theorem I will revise, recall the definition of a bounded function, bounded function, we define like this,
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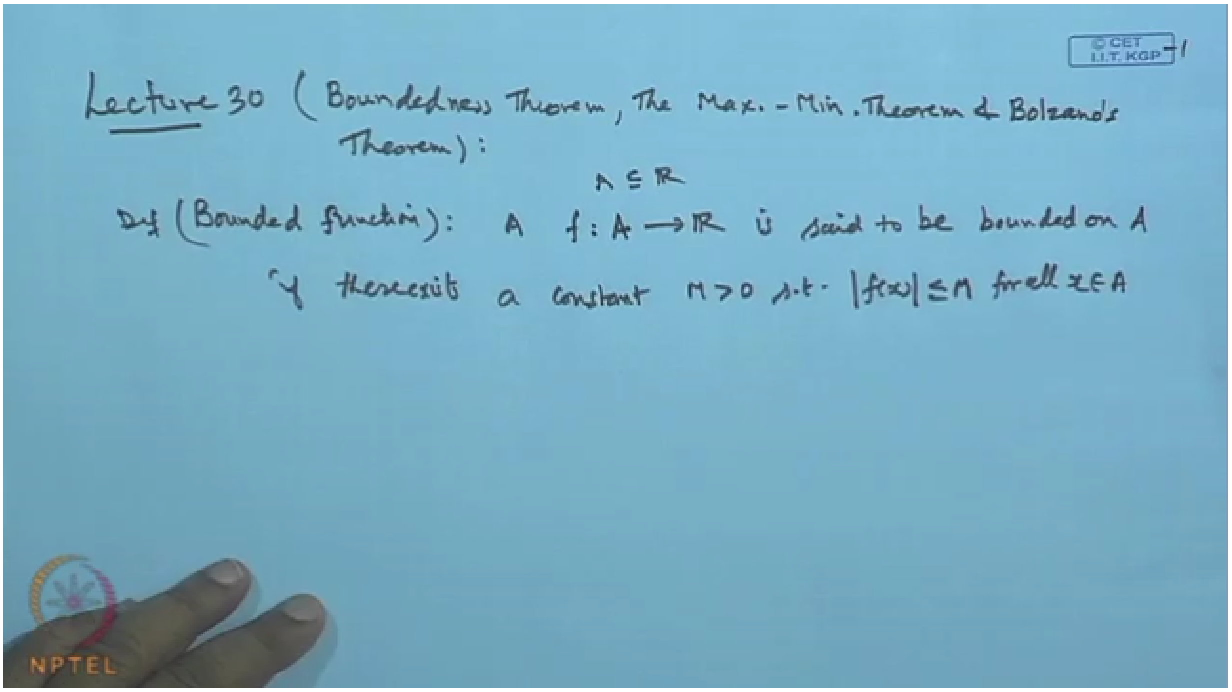

a function F from a set A to R, where A is a non-subset of R, non-empty subset of R is said to be bounded on A if there exists a constant M greater than 0 such that the mod of F(x) is less than or equal to M for all X belongs to N, it means as function is said to be bounded if the corresponding run set is a bounded set.

So when we say that F is not bounded it means that is, so a function F from A to R is not bounded, it means we are unable to get such an M for this equity horse, or we can say that if given any M then a function F is said to be unbounded, then if given any M greater than 0 there exists a point X, bit depends on this bound M, XM point M belongs to A such that the value of the function at these points will be greater than the given number M, so whatever the number you choose you can always find a corresponding a point in it for which the functional value will exceed that number M, then we say F is unbounded or is not bounded on the set M, okay, so this is we already discussed.

We wanted now the result the theorem which is known as the Boundedness theorem, the theorem states says let I be a closed bounded interval, and let F be a function from this closed bounded interval to R be a continuous function, be continuous on I, then this theorem says F is bounded on I, so every continuous function on a closed bounded interval will be a bounded function that's what it says.
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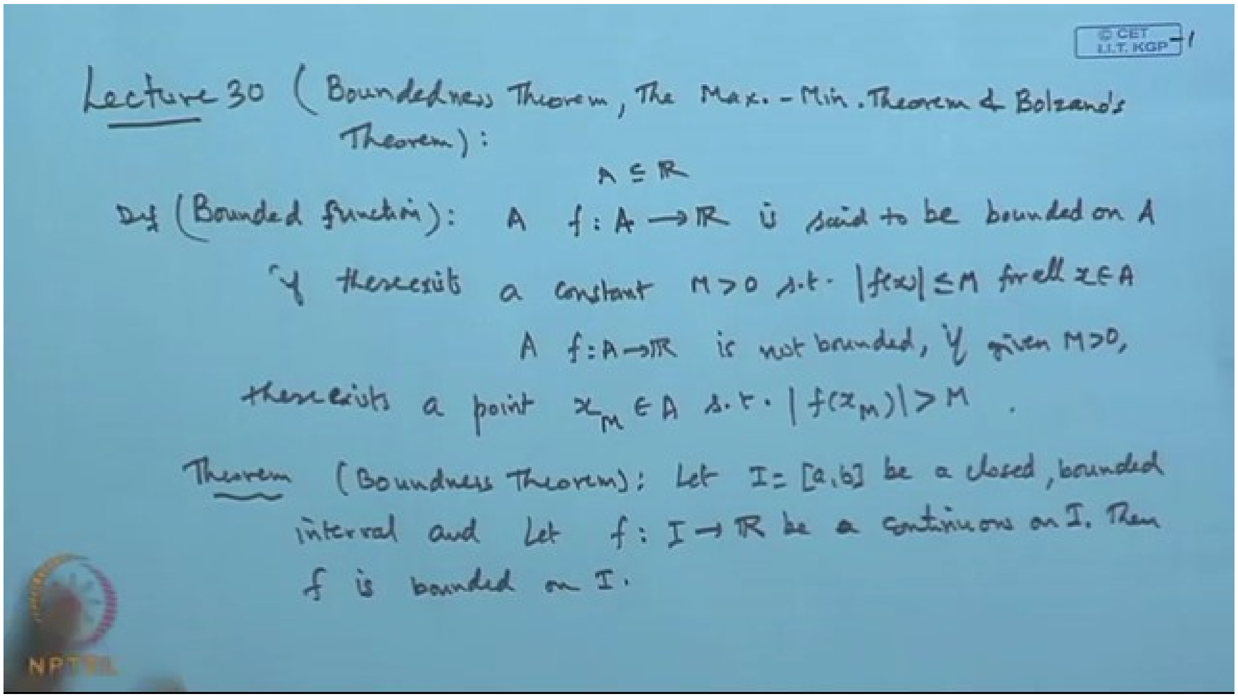

Proof, we will prove by contradiction, suppose F is not bounded on I, suppose F is not bounded on I, so by definition we cannot find an M such that the mod of F(x) is less than equal to M, so we can get then for any N belongs to natural number there is a number XN in the set A such that the value of the function, absolute value of this functional value F(xn) will exceed by N, so corresponding to 1 we get 1 point X1 in A so that F(x1) is greater than 1, 2 we get F(x2) so we get a sequence of the points in A which will satisfy this condition F(xn) is greater than A, okay.

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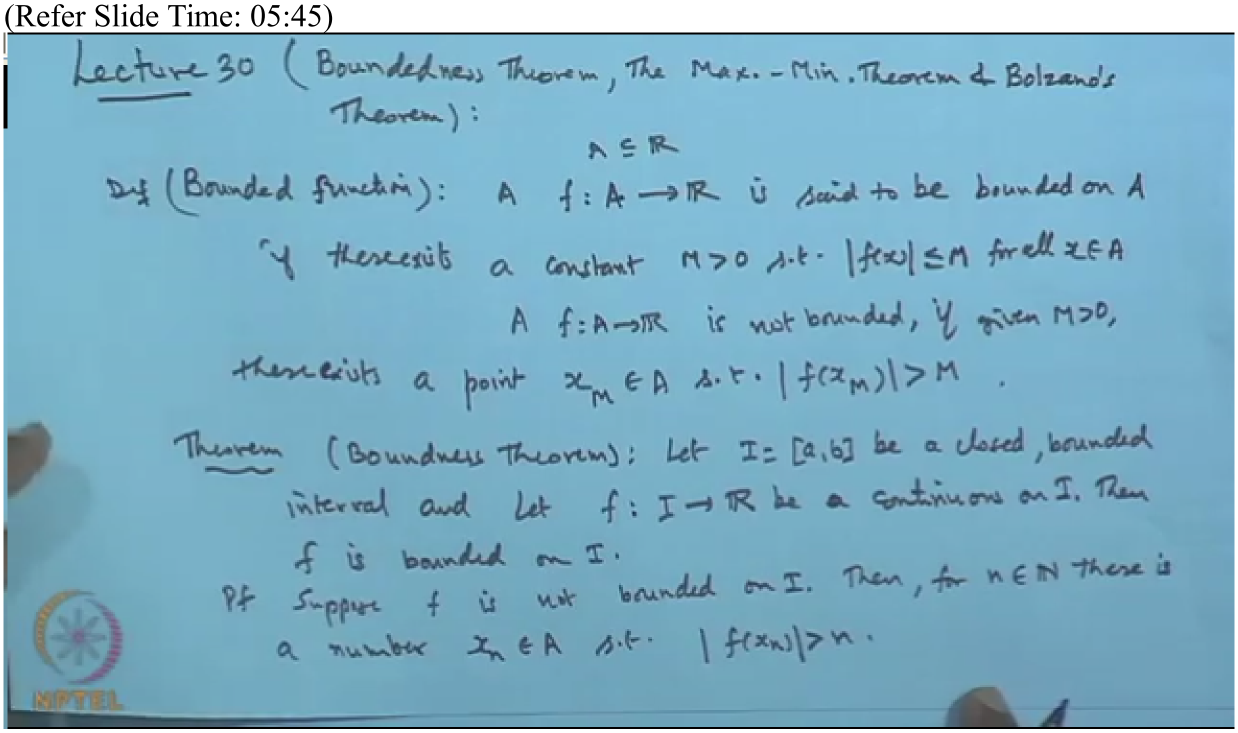

But since I is bounded, since I which is giving to be the closed bounded interval is a bounded set, and all these sequences XN which you are getting satisfying the condition say 1, satisfying 1 lies in I, lies in I, because these are all the sequences belonging to I, and I is a bounded set, so the sequence XN is a bounded sequence, because XN lies between A and B so that all the terms of the sequence have a lower bound say, upper bound say B, it is a bounded sequence and we know by Bolzano–Weierstrass theorem every bounded sequence has a convergent sequence, so use the Bolzano–Weierstrass theorem, the Bolzano–Weierstrass theorem, so use Bolzano–Weierstrass theorem this will give, which gives a convergent subsequence XNK of, say XNK of this XN, convergence subsequence of XN F again that converges to the point X converges to a number X, to a number say X, because by definition XN is a bounded sequence,

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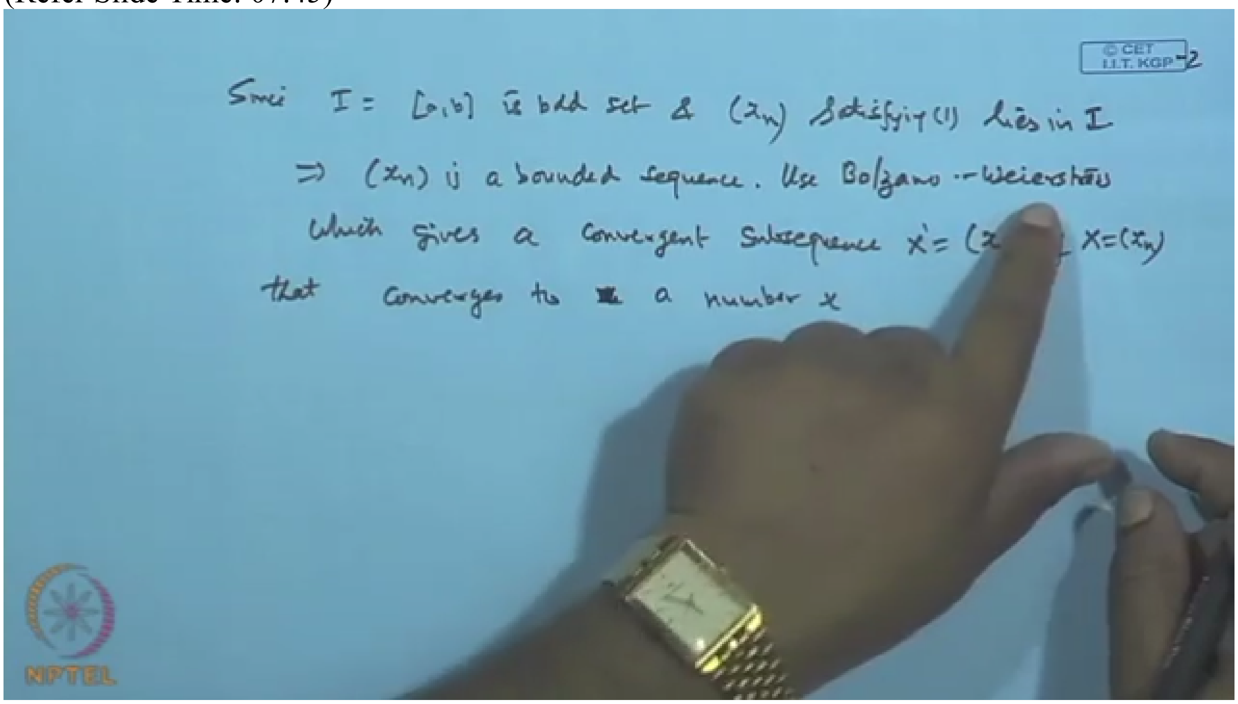

so Bolzano–Weierstrass theorem says by Bolzano–Weierstrass theorem we can get a subsequence which is convergent and converges to a number X.

Now this X obviously belongs to I, why? Because since X is, XNK all these terms of the sequence XNK lies between A and B, this is a closed interval, I is closed interval, all the terms of the sequence lies between this, so the limit of this sequence XN cannot exceed between these, will always lie between these two bond, so since this is there, where I is a closed bounded interval, so the limit point of this sequence XNK, over K this limit point obviously belongs to I, okay. But this limit point is X, so that X belongs to I you know, so what we get is that a sequence XNK has a subsequence which is convergent and the limit point belongs to I.

Now F is given to be continuous, since F is given a continuous function over the interval I, and X is one of the point inside the I, so this implies F is continuous at X, so by Hahn–Banach theorem therefore this implies limit of XNK, when K tends to infinity is X will give you, will give F(xnk) limit of this as K tends to infinity is nothing, but what? F(x), F(x) okay, because by the convergence part, so what it shows? This implies the sequence F(xnk) this sequence is a convergent sequence, is a convergent sequence okay, and convergent sequence is always bounded, so it is bounded, but that gives a contradiction to our result,

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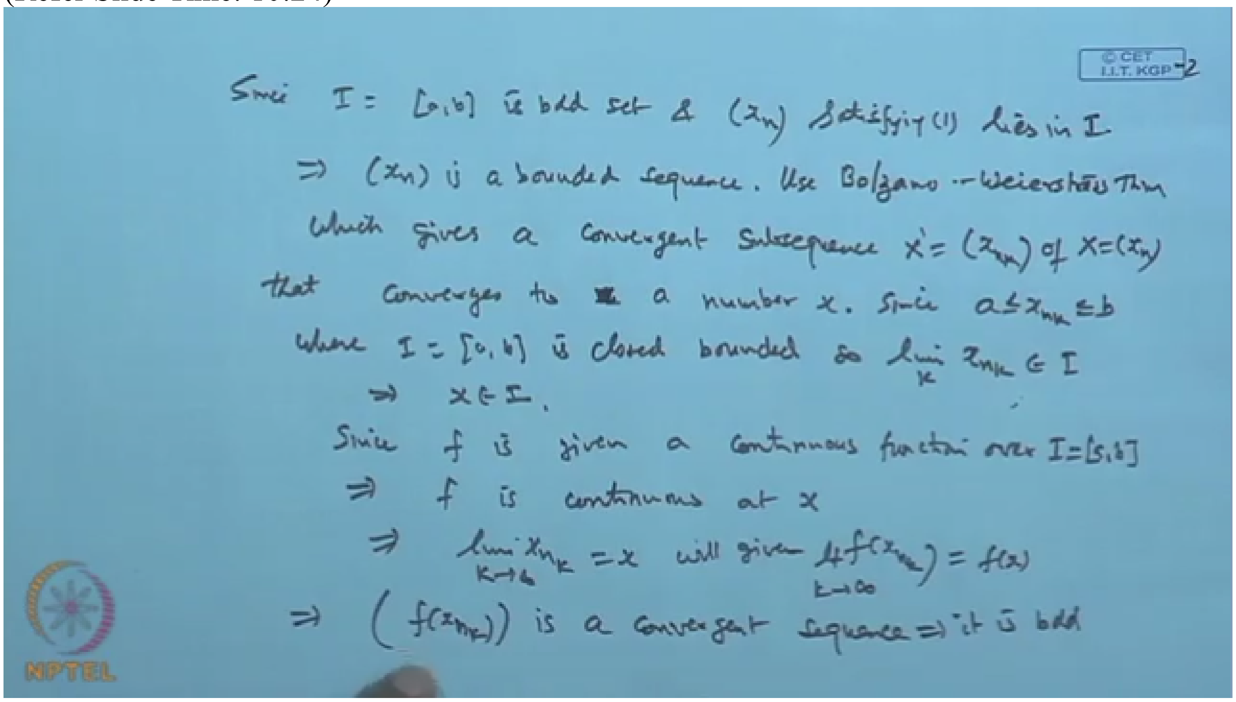

because this sequence F(xn) is greater than N, so from here from 1 but from 1 what we get? We get mod of F(xnk) is greater than N of K which is greater than K of course, greater than K and this is true for all K belongs to N, all K belongs to N, so this continuous function is not bounded on the closed bounded interval,

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so this shows that function F, this over this sequence, so supposition that function is not bounded gives a contradiction, so this gives a contradiction of our second part, but from bigger this which contradicts 2, because here this shows this is unbounded, well we have already shown it is bounded, so it gives a contradiction, in contradiction is because of our assumption that our function F is not bounded on I, so this shows that function F is bounded on I, this implies F is bounded on I, that is closed interval, so this proves the result which is known as the Boundedness theorem.

Now in the Boundedness theorem we have assumed these conditions, what are the condition in the Boundedness theorem is, first condition is the interval on which the function is defined must be closed and bounded, so this is one of the conditions which we have taken.

Second condition which we have seen the function must be a continuous function on this, so interval is closed bounded, and function is continuous, then only we can say F is a bounded function only. If any one of the condition is relaxed that is if we take I to be a simply bounded interval not closed, or simply closed not bounded or function is not continuous then our conclusion that F is bounded on I cannot be drawn, in fact we will get a contradiction, we will get the example we get this function is unbounded when we relaxed anyone of the condition.

So let's see the examples, note the conditions of Boundedness theorem, the hypothesis of Boundedness theorem is each conditions, each condition is needed to justify or to get the function F to get a continuous function to be bounded on I, it be relaxed any one of the condition then we get the conclusion fails, the conclusion of the theorem fails, if any one of the hypothesis or anyone, if anyone of the conditions of Boundedness theorem is relaxed, for example suppose I take the function say function F(x) is suppose X, the interval I I'm choosing I as an interval is 0 to infinity, okay, and now this function is continuous function, it is a continuous function on this interval I, but I is not that is the function is closed but is not bounded, it is unbounded well though it is closed, because all the limits point of between 0, 1 are inside it, so I is a closed interval but it's not bounded, okay, then what happens? The function is continuous throughout, but the bond for this function but function F(x) = X is an unbounded function, because as X increases the value of the F(x) keeps on increasing and interval is up to infinity, so it is without bond so it is unbounded therefore the conclusion fail.
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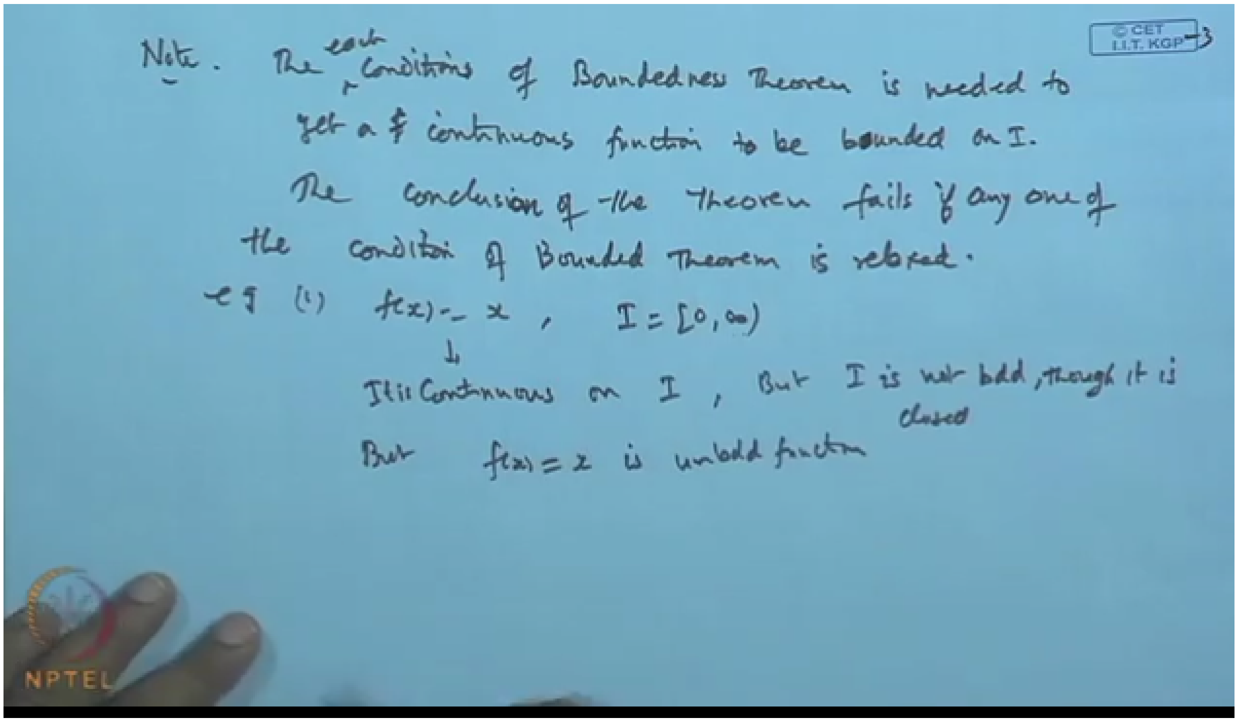

Second if suppose I take the same function G(x) let's say 1/X this function and the interval I choose, I to be the interval say 0, 1, now this interval is not closed but bounded, function F(x) is continuous over the interval 0, 1, because 0 is not included but we have seen this function is also unbounded, is unbounded as because as X tends to 0 the function G(x) will go to infinity unbounded,

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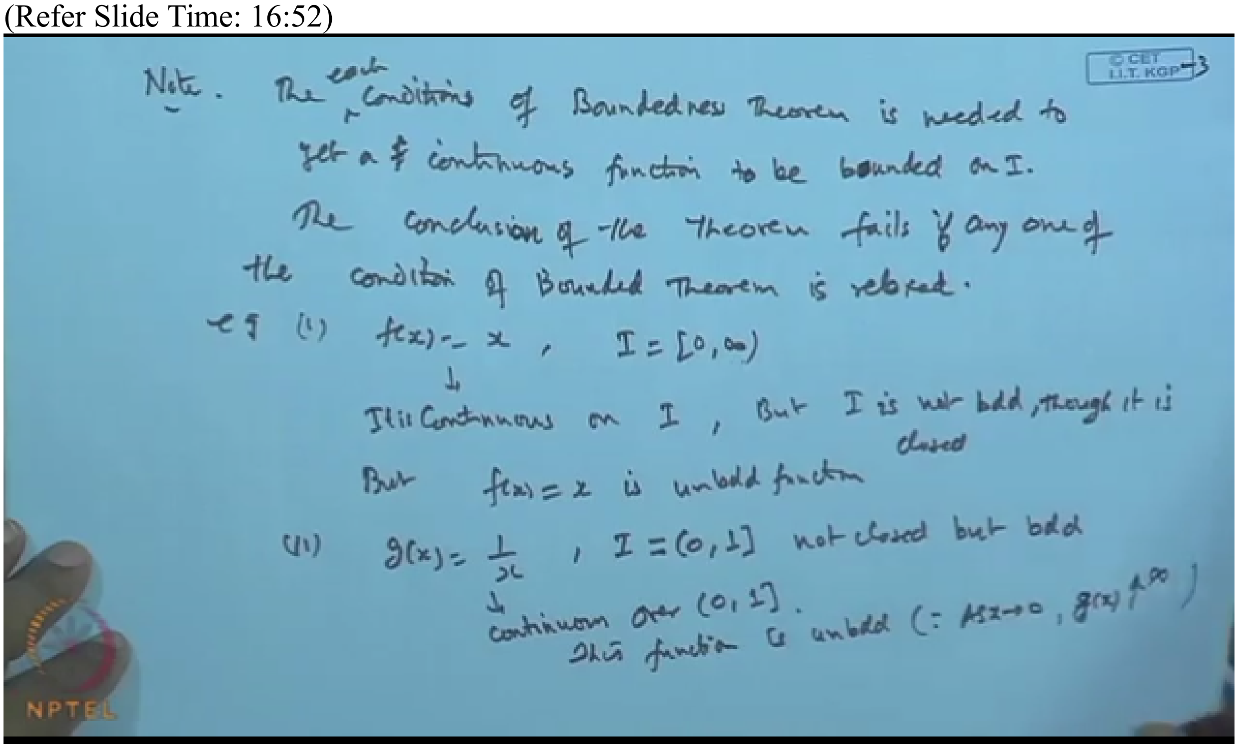

so again the relaxing the condition is again not going to help that conclusion fails, then if I take the interval I as a closed interval which is a closed and bounded, okay, but now I am taking the function H(x) as 1/X, if X belongs to the interval say 0, 1, and 0, 1 if X is 0, I define this function, the function is continuous over the 0, 1 interval okay, but it is discontinuous at the point 0, clearly H is discontinuous at X = 0, so again the conditions is not satisfied and obviously clearly H is unbounded, when X approaches to 0 this is not a bounded function okay, 1/X for this 0, 1,

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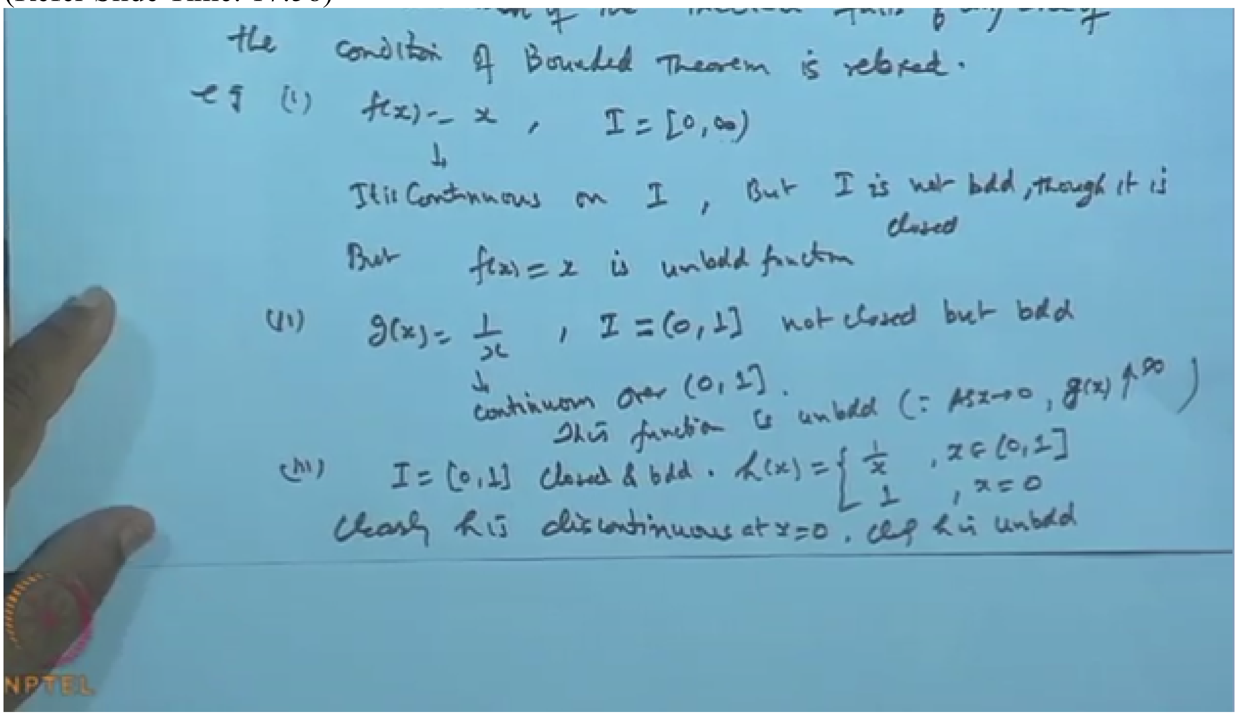

and 0 when it's discontinuous and unbounded on C, okay, so this will be there, okay. So we can say that in Boundedness theorem is going to only be applicable when all the three conditions are taken in consideration.

The next result which one to a maximum minimum theorem, the maximum minimum theorem, the theorem says let I be a closed bounded interval, and let F which is mapping from I to R be continuous on I, then F has an absolute maximum and an absolute minimum on I, this results.

Now we have already discuss the maximum, absolute maximum and absolute minimum in the last lecture, so proof we go, what is the absolute maximum mean? That if suppose a function is from A to R,

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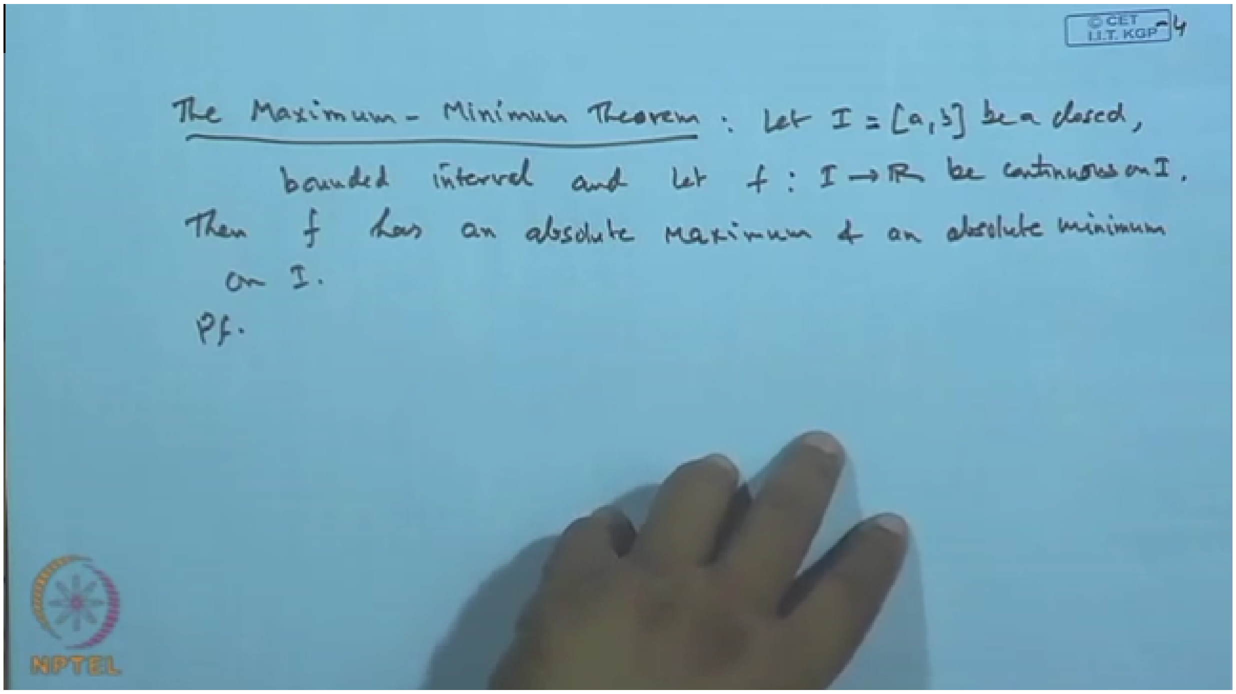

then F has a absolute maximum at a point, if there exists some point X star, such that F(x) star is greater than F(x) for all X, and minimum when F(x) in lower star is less than equal to F(x), so that is the way we have introduced the maximum of this function.

We will discuss after this proof, so suppose what we want is suppose A be a closed bounded interval, and function is given to be continuous then it will attains this maximum value as well as minimum value over the interval I, that is there exist some point we have the absolute maximum will be the attained, absolute minimum will be attained, so let's see the problem.

So consider the set F(I) the set of those values F(x) where X belongs to I, means consider the range set of function F which is defined over I, now this then set the values of F(I) is clearly is bounded subset of R, and this follows from the previous, from Boundedness theorem, because Boundedness theorem we have seen if the I is a closed bounded interval, F is a continuous function then F(I), then F is a bounded on I, that is the range set will be a bounded set on I, so it is a bounded subsets of all, this much be,

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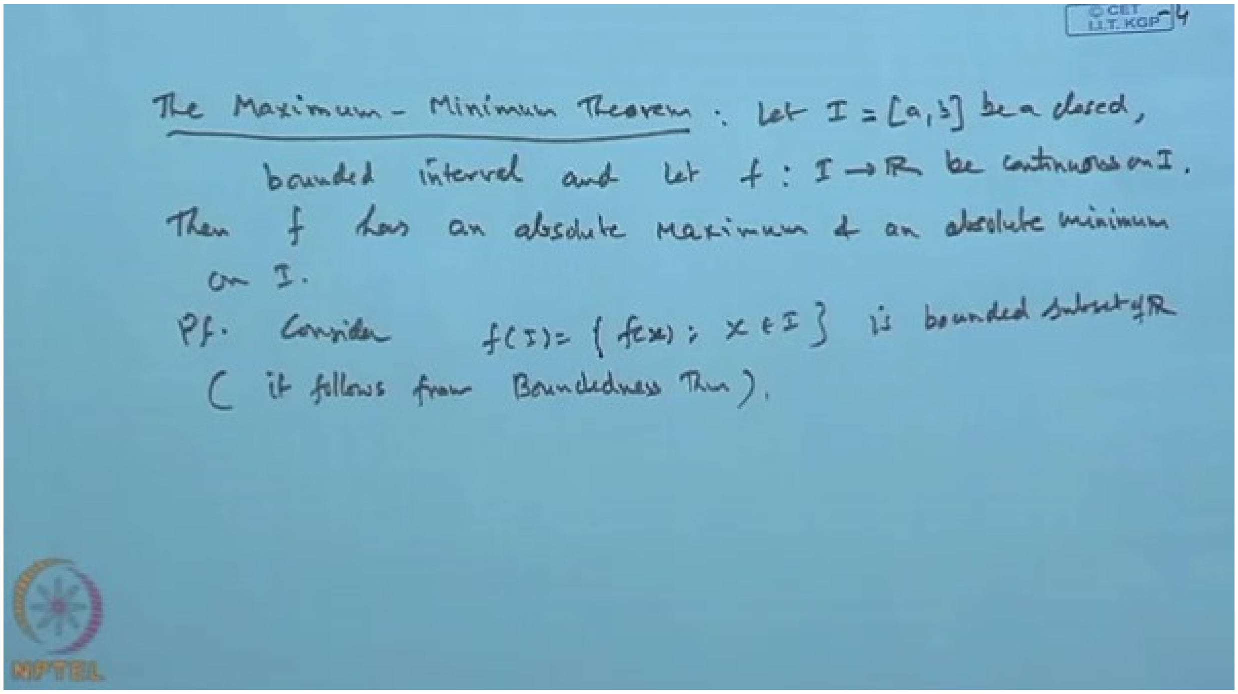

so once it is bounded subset we can talk about the upper bound and lower bound and supremum of least upper bound and greatest lower bound.

So let us suppose S star be the upper bound supremum value of the function F(I), and S lower star be the infimum value of the function F(I) that is least upper bound of the function F over the interval is suppose S star.

Now what we want to, this S star and small s star exist, it means there exist some function what we wanted to prove, required to prove is that there exist, there exist points X upper star and X lower star in I such that the S star will coincide with the F(x star) and S lower star will coincide with F (x star) this we wanted to show. So first we will prove for this side and the other will follow in a similar way, okay.

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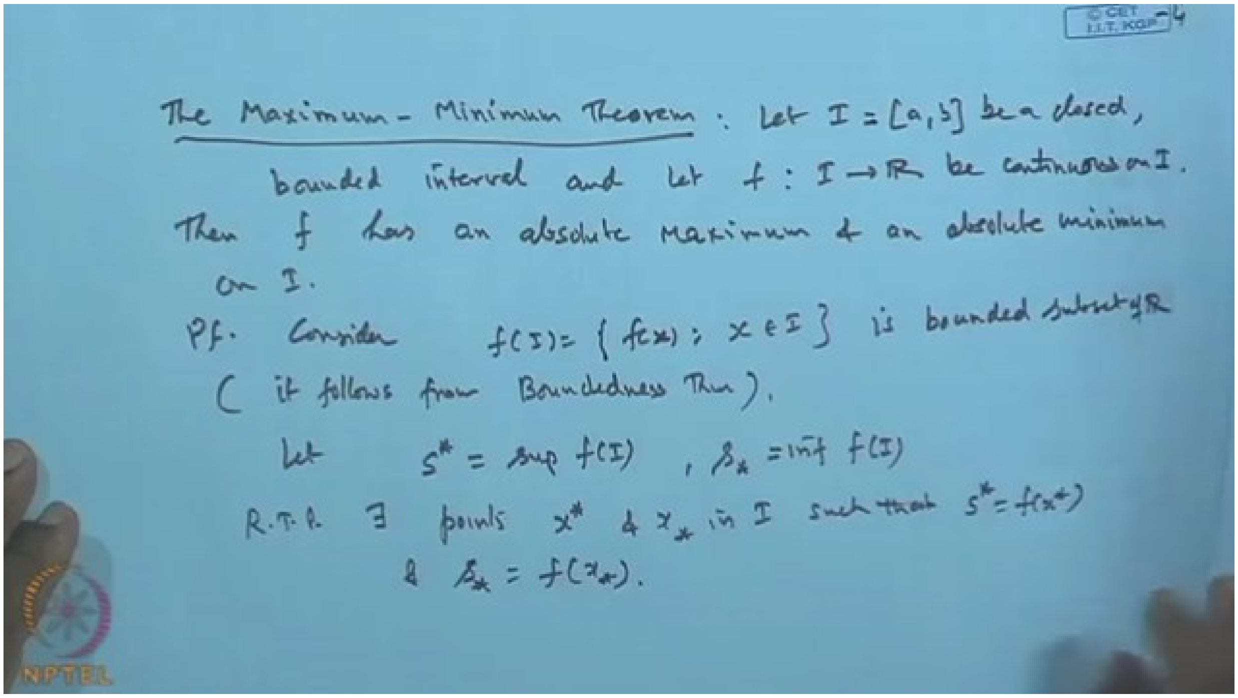

So let us assume, so first to show that there exists an X star in I such that S star which is the supremum value of F(I) exist and equal to the value at this moment, there exists this we wanted to show, okay.

Now since our X star, now since S star is the least upper bound of the function F(I), of the function F(x) for X belongs to I, when X belongs to I the least upper bound of this function is S star, so if I choose a number slightly lower than this it cannot behave as a least upper bound so the number then so the number S star -1/N this number is a slightly lower then, so it's not an upper bound for it, so is not an upper bound for this set or of the set F(I), it's not a bound, therefore they'll exist so we can check consequently there exists a number XN in I such that X star upper -1/N is less than S star, this S star -1/N less than F(xn) which is less than or equal to S tallest, S upper star, for all N belongs to capital N, because this is our upper bound, so when you take a number slightly lower than this then we can find a some number XN in I so that the functional value of XN will exceed by this number, and obviously it will remain less than equal to this because it is the least upper bound for this, okay.

Now this sequence number XN, but these sequence is XN satisfying 1, 1 are lying in the interval I which is say A, B which is a closed and bounded interval, closed unbounded, so by Bolzano Weierstrass theorem there is a subsequence say X days elements are X and say R belonging to of X, of course of X, X is a sequence XN of X that converges to some number X upper star say, okay.

Now we wanted, in fact this number which we have got it this must be a point of A, B, this we wanted to show first, so how to show is, since all the elements of this all again the elements of I,

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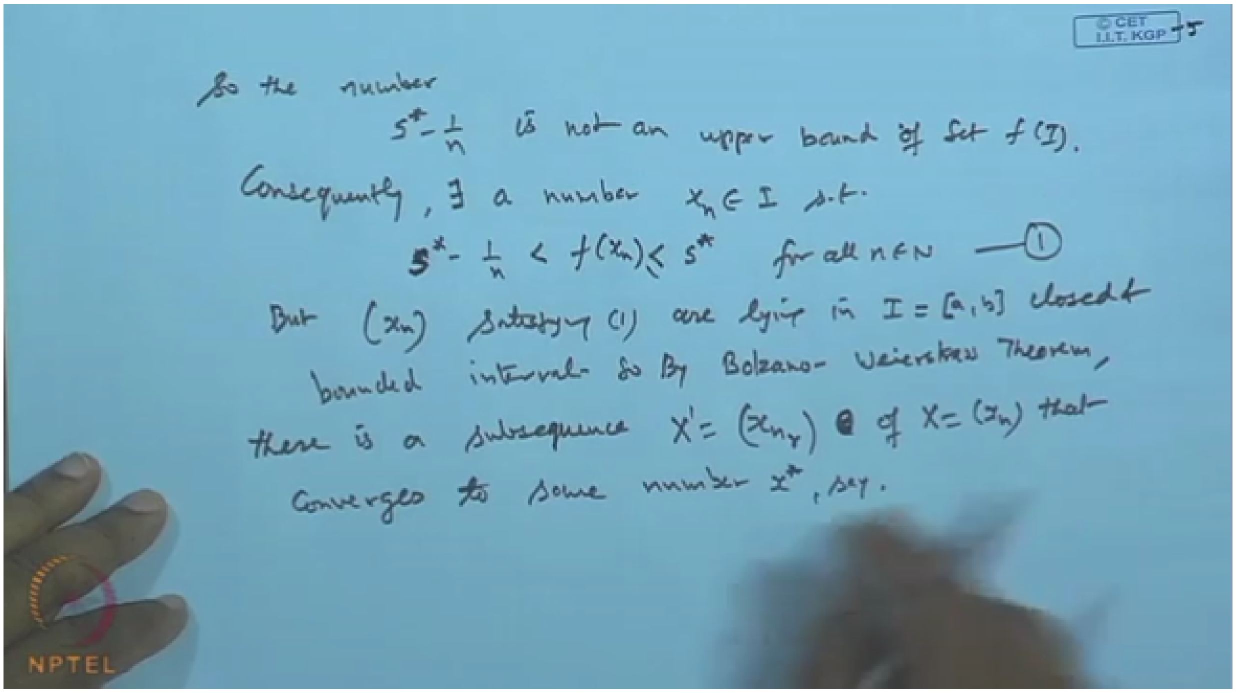

so since the elements of X dash, that is XNR these are, they belongs to I which is closed and bounded interval, so just like a previous thing we can say, so just like previous theorem we have seen that if the sequence of the point belongs to a closed and bounded open interval then limit point will also belongs to it, and since it is closed so limit point well, so it follows from this that the limit point X star is also a point in I, okay.

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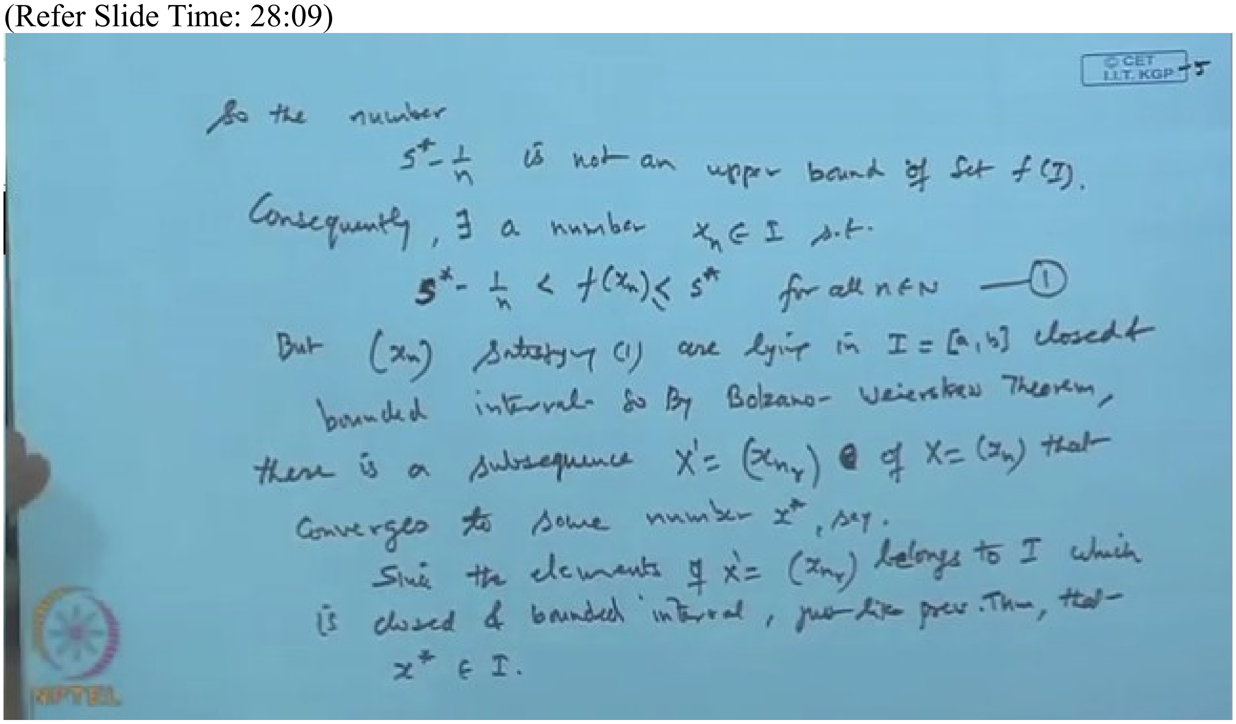

Therefore F is continuous at I, therefore F is continuous at this point also because it is continuous throughout over the I, so once it is continuous so apply the definition Hahn–Banach definition, Hahn theorem says if a sequence converges to X star then F(xn) will also converge XR will also converge to F(x star), so by theorem limit of F(xnr) when R tends to infinity coincide with F(x star) because it's continuity, but by the first one, use the first one, from first what we get is S star -NR is less than F(xnr) which is less than or equal to S star, is it not? So this is true for all R belongs to N.

Now let R tends to infinity, so this limit is S star, this is S star, so by Squeeze theorem, the limit of this function F(xnr) as R tends to infinity will be equal to S star, but this limit is nothing but what? F(x star), so this implies there exist an X star belonging to I such that the supremum, because this is the supremum value of the function F(I) such that this F(x star) is the supremum of F(i) exist, and that proves the existence of X star. Similarly we can show for that there exists a X lower star in I such that X lower star is the infimum of F(I)
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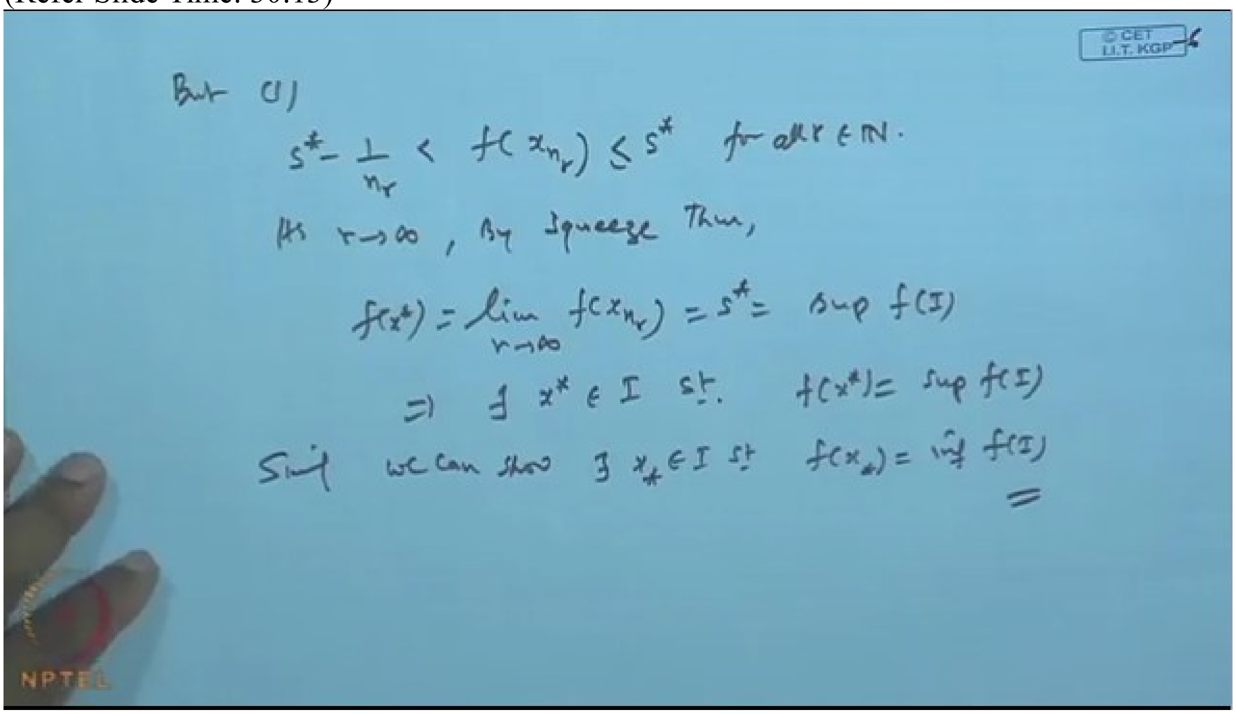

and this completes the proof for this much.