Module – 8

Lecture 46: Properties of Continuous Functions

Okay, so in the last lecture, we have defined the continuity, of the function, using the Cauchy and Eny's definition and both are later on shown to be equivalent. And that case we have considered F to be continuous over an interval, but we can extend it we can take an arbiter set a which is subset of all we have that similar way the continuity of the definition of the continuity over the set a, which is subset, over all any point X, C, belongs to their subset can be introduced. Let me just recall that that we say.

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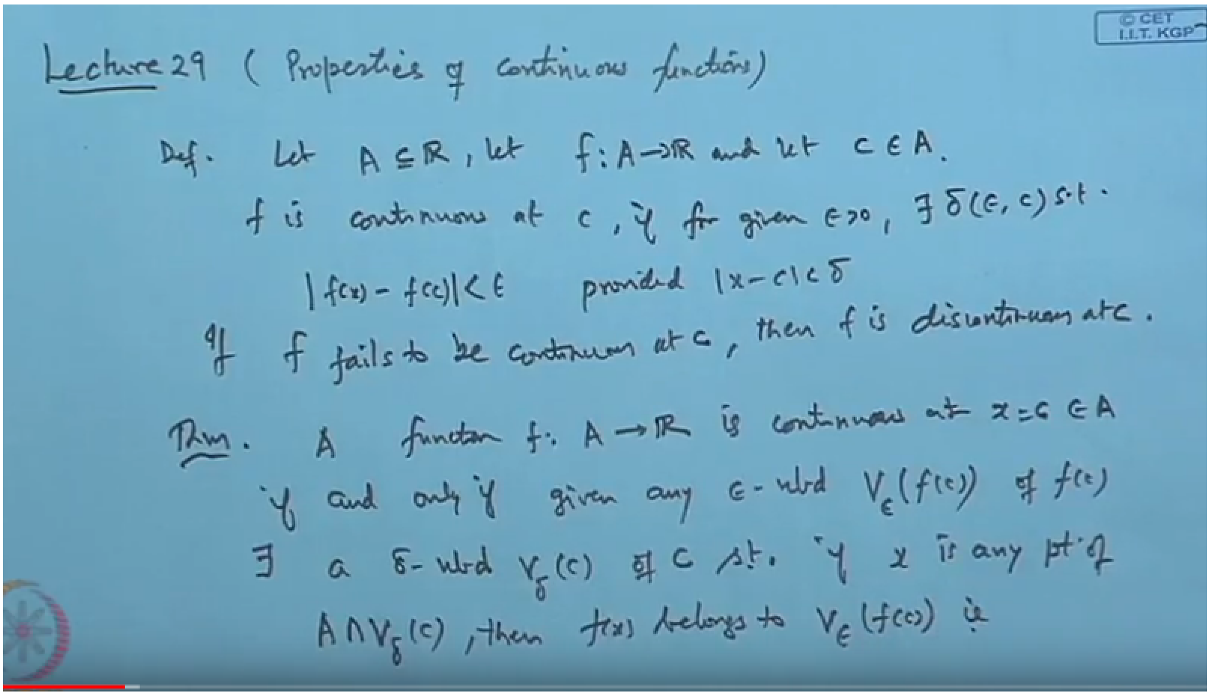

Let a, which is a subset of a non-empty subset or a continuous subset, so for let F is a mapping from a to R and let C be a point in a then we say F is continuous at the point C if for a given epsilon greater than 0 there exists a delta depends on epsilon and the point C such that such that the mode of FX minus FC is less than F epsilon provided mode of X minus C is less than Delta so for if X is any point in this set separately then the correspondingly we get this if F fails to be F else to be continuous at sea when we say means if this condition does not hold then efforts to be continued C then F is said to be discontinuous function f is discontinuous at C and in the neighbourhood form we can say in the neighbourhood form is a function a if from a to R, is continuous at, X equal to C, belongs to a, if and only if, if and only if, given any epsilon neighbourhood ,epsilon neighbourhood, say V F epsilon, depends of FC given any epsilon neighbourhood of FC there exists there exists a delta neighbourhood.

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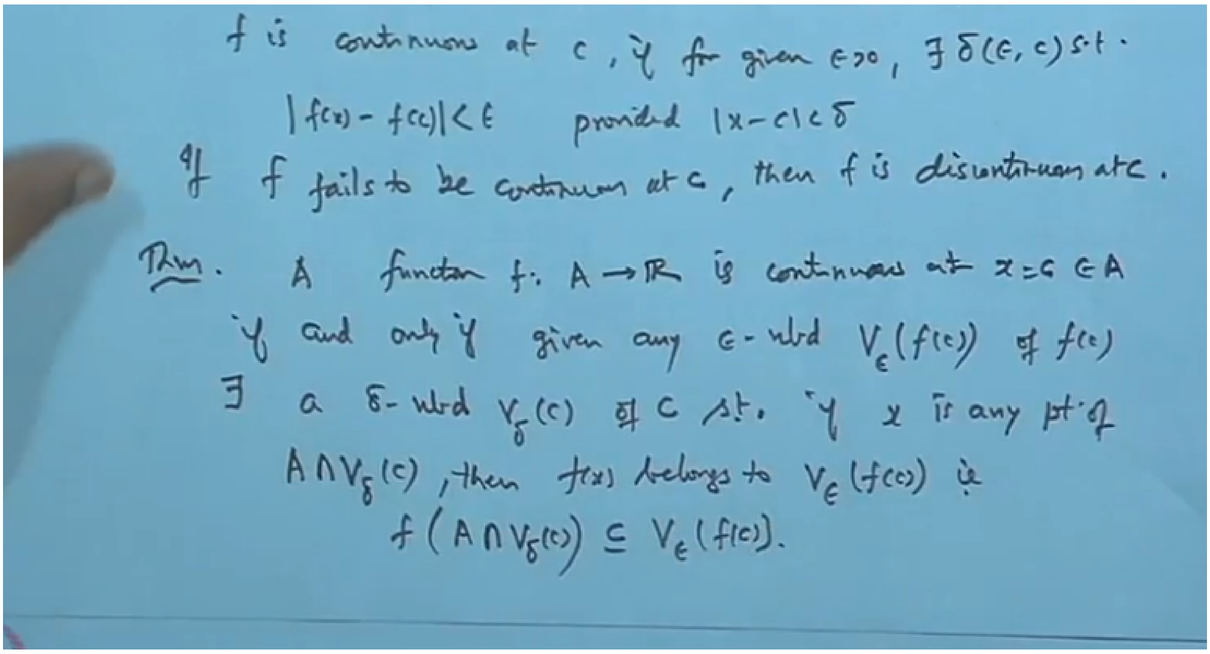

Say, V Delta, C Delta, neighbourhood of C, such that, such that, if X, is any point any point X is any point of the a intersection V Delta C of the common point of this in neighbourhood and then means it belongs to the neighbourhood then FX belongs to, belongs to, the neighbourhood of epsilon, neighbourhood of FC, that is the imas, of this imas, of this interval the Delta neighbourhood of C will be contained in the epsilon neighbourhood of FC.

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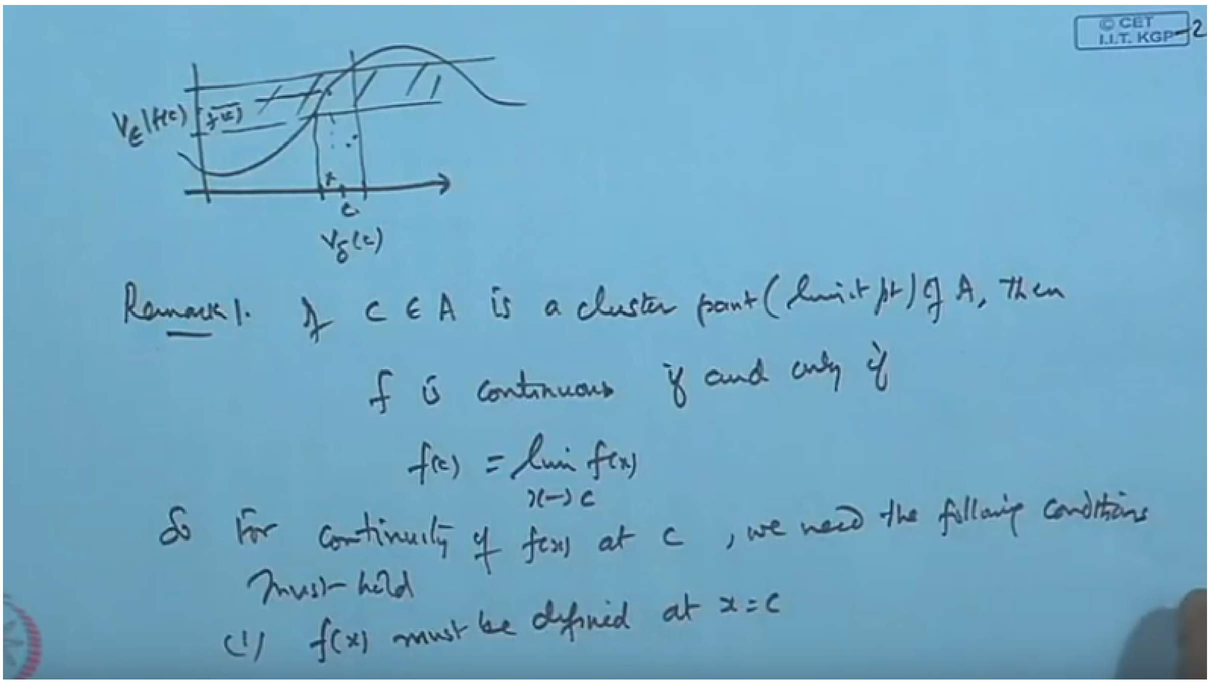

That's what is said and we have seen that graphically, also that if here say this one is there so this is our function and here this is our say Delta so this we get B Delta C here is our F epsilon this is V F epsilon FC C is point somewhere here, here, is FC this is FC we are the C is there so we get given any epsilon neighbourhood of FC there exist a delta neighbourhood this is the Delta neighbourhood B Delta C such that in as of any point in this will fall inside it then it is a continuous function so here C is a point in this neighbourhood now it may or may not be the limit point but if C is a limit point if C belongs to a is a limit point or is a cluster point that is the limit point and cluster point of a then the continuity of definition can be written a then f is continuous if it if and only if the limit of this FX when X approaches to C is F of C and in fact this says if you look that Eny's theorem, Eny, is equivalent definition says the resist sequence x n in the neighbourhood which goes to C then the corresponding F xn will go to F C so basically it is the same as the henna stone so FC it means if a function we wanted to be continuity what the condition requires so for the continuous for the continuity of the function FX at a point C we need the following condition we need the following conditions must hold the first condition.

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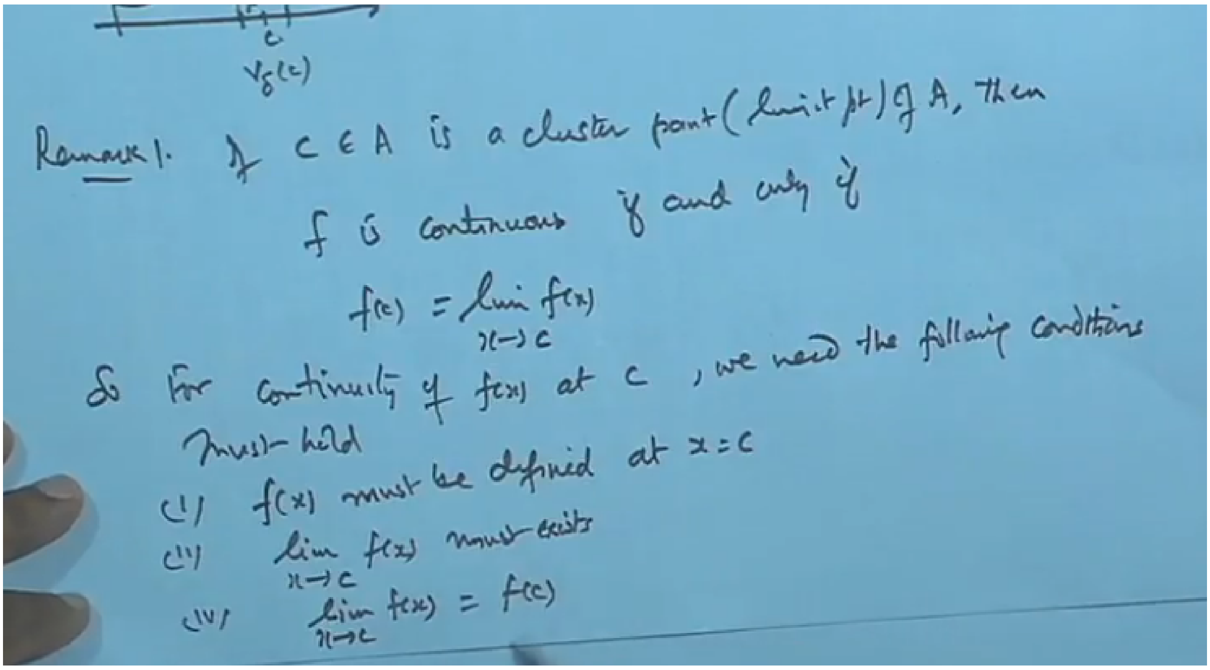

The limit of the function the function f X must redefine at the point FX equal to C where the continuity is testing second port is the limit of this function FX when X approaches 2 C must exist must exist and third condition is both should be identical studies the limit of FX when X tends to C must coincide with the value FX if these three conditions are satisfied then we say the function FX is continuous at a point C so it using this definition it is very easy to confirm now find out, whether the function is continuous or not if suppose in this case any one of the condition fails then we said function is discontinuous. So again there are different types of discontinuities that we will discuss later on that if suppose the function is defined limit exists what both may not be equal then in that case that discontinuity will be the removable discontinuity and so on so forth so that we part we will make up later but using this definition.

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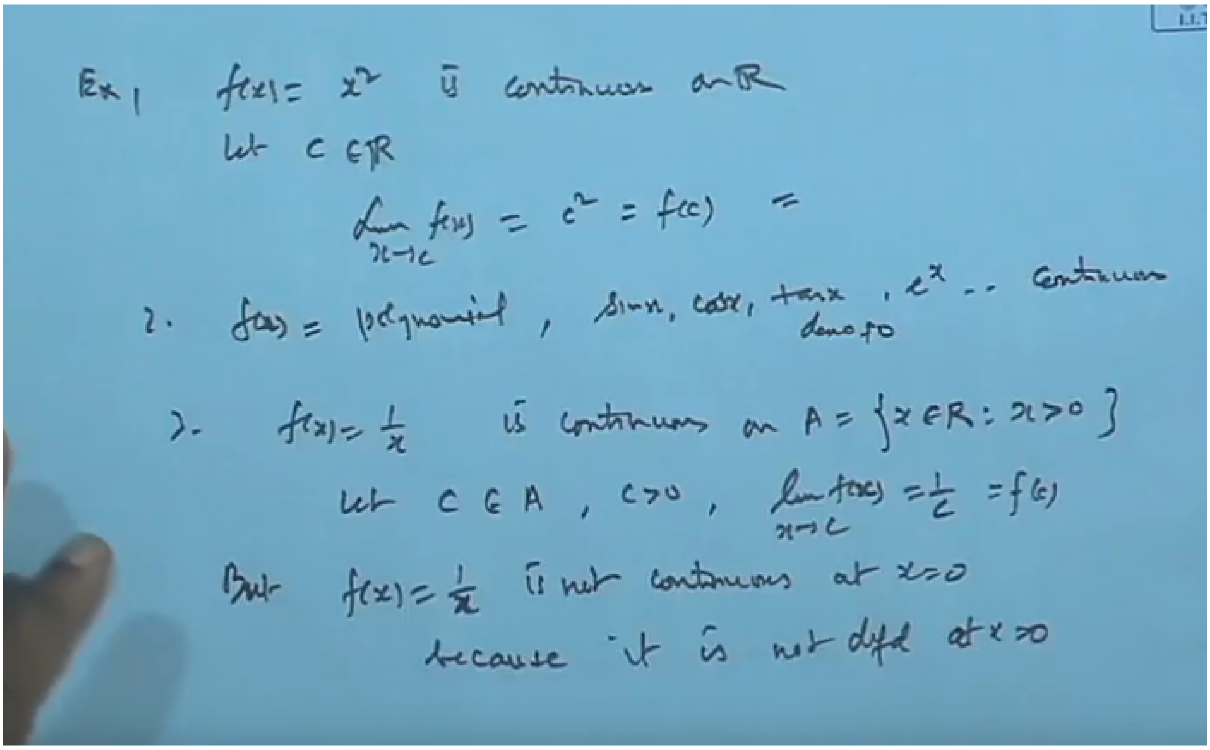

Because so many exams suppose I take FX equal to X square FX equal to X square and B claim is a continuous function on the entire real line R because let to see as any point in R then what is the limit of FX when X tends to C as we have seen already this is C square this is the same as F of C so it is continuity follows similarly the function FX all the polynomials function polynomials they are also continuous from self sine functions cosine X tan X we have the denominator is not zero denominator is different from zero these are all function e to the Y etcetera all continuous functions and follows from this definition easily so we are not touching for we are not going in detail the proof of this but take the limit of this and so much as we have already discussed the limit of fun various functions and that will help in identifying whether the given function is continuous or not now let us take the function FX equal to 1 by X now FX equal to 1 by X this function where X is its continuous over the set continuous on the set of all real line, real number, so ever it is positive because if we take see any point in a so obviously C is greater than 0 so limit of FX when X tends to C is nothing but 1 by C which is the value of the function at C so continuity follows but f of X which is 1 by X is not continuous is not continuous at X equal to 0 because it is not defined at X equal to zero so there is no question of continuity Liza.

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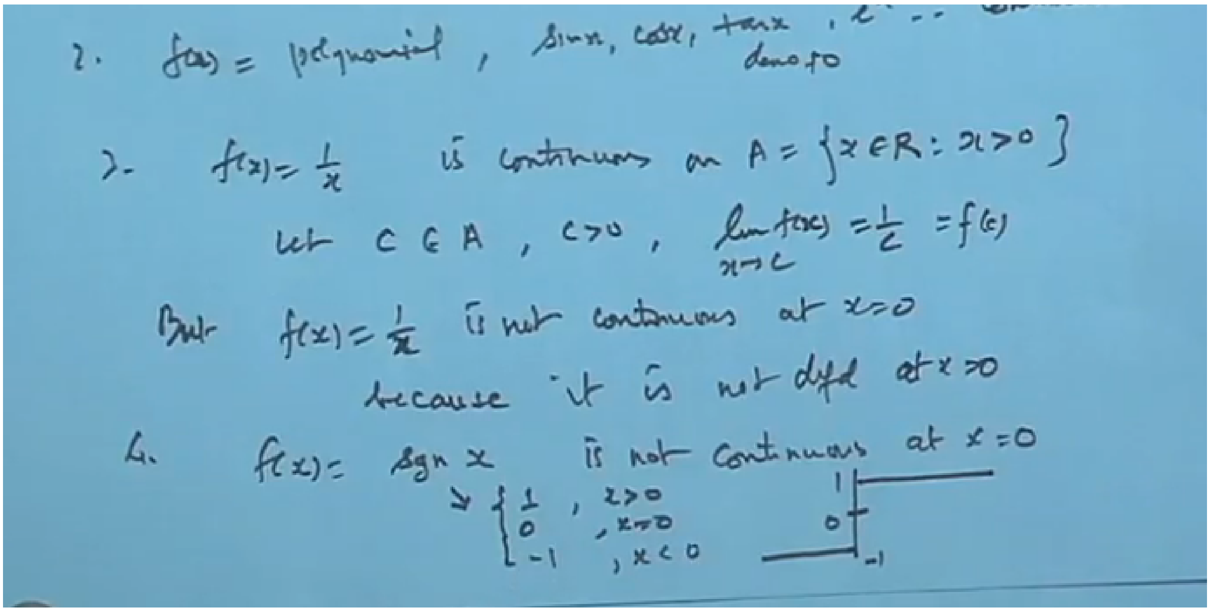

Similarly if we take the function sine X sorry FX equal to Signum of X Signum of X this function is not continuous at 0 at X equal to 0 the reason is because the Signum function is defined like this when X is negative the Signum value is minus 1, 0 it is 0, when X is positive, it is 1, so Signum function this is the function which is defined as 1 if X is positive 0 when x is 0 and equal to minus 1 when X is negative so when you take the limit of this Signum function as X tends to 0 the limit will go to 1 and minus 1 when X is tending to 0 from positive side or when X is tending to 0 from the negative side and therefore limit does not exist, limit does not exist, at X equal to 0. So it is not a continuous function. Another example which is also very interesting example let a be the entire real line and F define f is defined as f is defined as the value is 1 if X is rational and 0 if X is irrational, irrational. Now we claim that this function f is not continuous at any point of our because the region is simple because suppose I take a point C let us take let if C test it at C which is a rational number say okay now every rational number can be approximated by mass of the sequence of rational so there exist a sequence xn, of rational irrational point irrational numbers in our whose limit is this is C but what is the behaviour of what is the value of xn, but xn, when xn, is rational it 0 always for each n therefore the limit of this FX n when n tends to infinity will also be 0 which differs from the limit but limit is C because C is a rational number so the value of this is 1 according to this so it is not continuous at any rational point in size similarly we can show that it is also discontinuous at every irrational point every irrational points in the real line in a similar a function however is modified and made it rational at some point and that function is let us define let a be the set of all real number which are greater than zero positive then define the function define H function H H.

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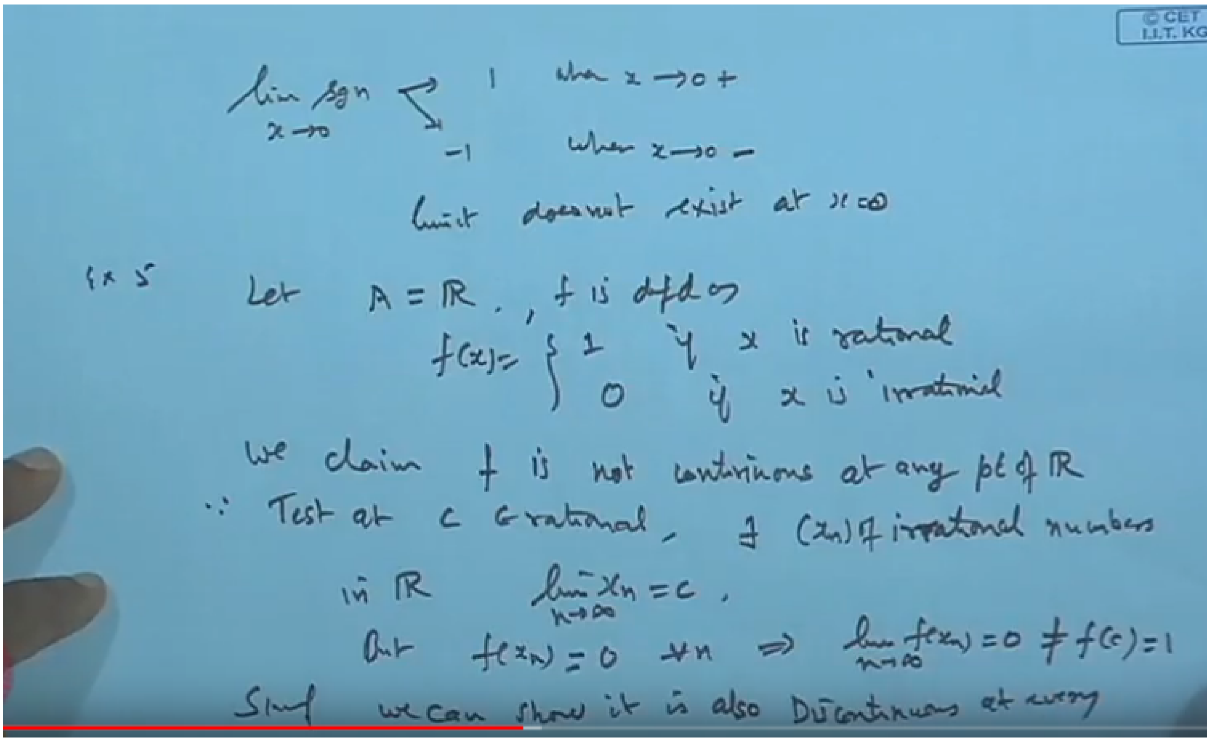

Follows define H on a H follows so H of X equal to say 0 if X is greater than 0 a rational number is a irrational number if this is irrational number we are taking H of X to be 0 and equal to 1 by n if X is a rational number M by n which is in the lowest form M by n which is in the lowest form and natural minor and the lowest form is that means M n is 1 the divisor the greatest common divisor of this is one where so let us define the X if it is rational number is of the form M by n may be right but by M and if X is irrational then we will write to be 0 now we claim that H this function H is basically is discontinuous at every at every rational number in a in a but it is continuous at every irrational number in a so let's see the proof for it the solution is suppose I take first a rational number let us take a be a take a which is greater than one zero as a rational points in a, so we can find a sequence so there is a sequence xn of irrational numbers irrational points which converges to a. Okay? But what is the value of affection HS x HX n HX n is defined to be 0 for every rational point so along the 0 it’s this sequence H of xn will be 0 but what is the H of a H of a is given to be what if it is rational number then it will be some say 1 by N or whatever maybe which is positive this is positive so here the limit of H s H xn as n tends to infinity will come out to be 0 because for all points the values are 0 so limit will be 0 and here the limit value is coming to be non zero therefore this limit of HX n, over N is differs from the value of the function at a point a so when a rational it is not a continuous function, so H is not continuous at any rational numbers in let's see the irrational point so suppose B is an irrational point V belongs to a is a irrational point is an irrational number it is irrational so when V Innis nollette choose f epsilon greater than zero so for this given epsilon we can identify a integer so a natural number so there exists a natural number n such that one by n naught can be made less than epsilon which is possible by our our material properties so this point now let us consider an interval

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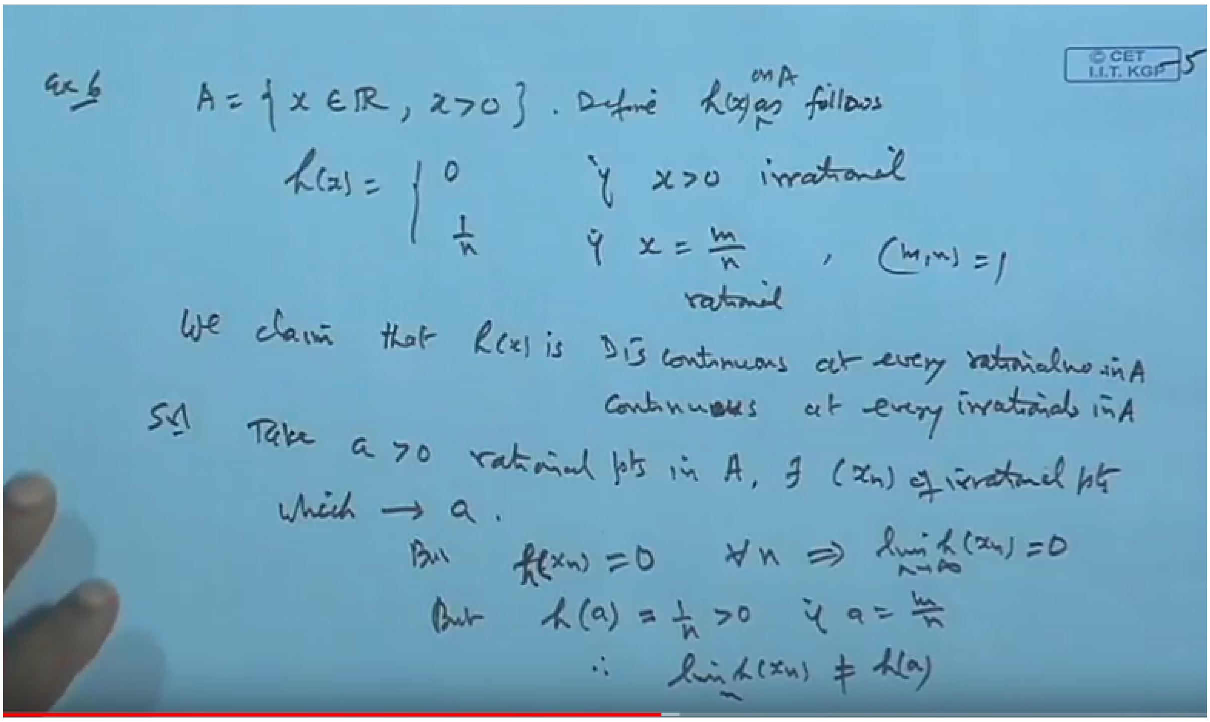

The interval if I take the interval B minus 1 B plus 1 now in this interval we can get the rational point whose denominator is greater than and not as well as less than n not okay so what we claim is that the national with nominal say less than n naught there are all only B whose neighbourhood or choose an interval, interval, B minus 1 B plus 1 and choose an interval then there are obviously there are only a finite only find a finite numbers of rational finite only finite number of rationals of Reznor's fine animal is a big denominator, denominator, less than n naught because obviously when this denominator less than n naught it will get up then n naught this will follow this will follow and be minus 1 here we will get somewhere number so it will approach to worse this number okay so we say any arbitrary number we can get hence we neighbourhood a neighbourhood V minus Delta V play excuse it so we can get the neighbourhood B minus Delta B plus Delta which contains which contains which obtains no rational number with no rational numbers with numbers with denominator less than the denominator less than n naught less than n naught means this neighbourhood we can find V minus Delta V Plus Delta we are all this point X n whatever the number the denominators will be having the denominator Gator than n okay so it will satisfy this condition that's what we wanted to have that neighbourhood. Okay? So once you have this neighbourhood then clearly then if X, then for any X, we satisfy this condition X minus B is less than Delta we are the X belongs to a we have mode of HX minus H of B but H or B is 0 because b is irrational and the function is defined in listener to be 0, so this is H of X H of X means X is of the form say 1 by n so this is 1 by n and this will be less than 1 by n naught 1 by say n not even by n HX X is equal to 1 by n so which is less than equal to 1 by n not so this will remain less than Epsilon help therefore follow epsilon neighbourhood of this we can find that is a delta such that for all X lying between these the function HX minus is be less than F epsilon so this shows the H is continuous at a rational point or irrational points.

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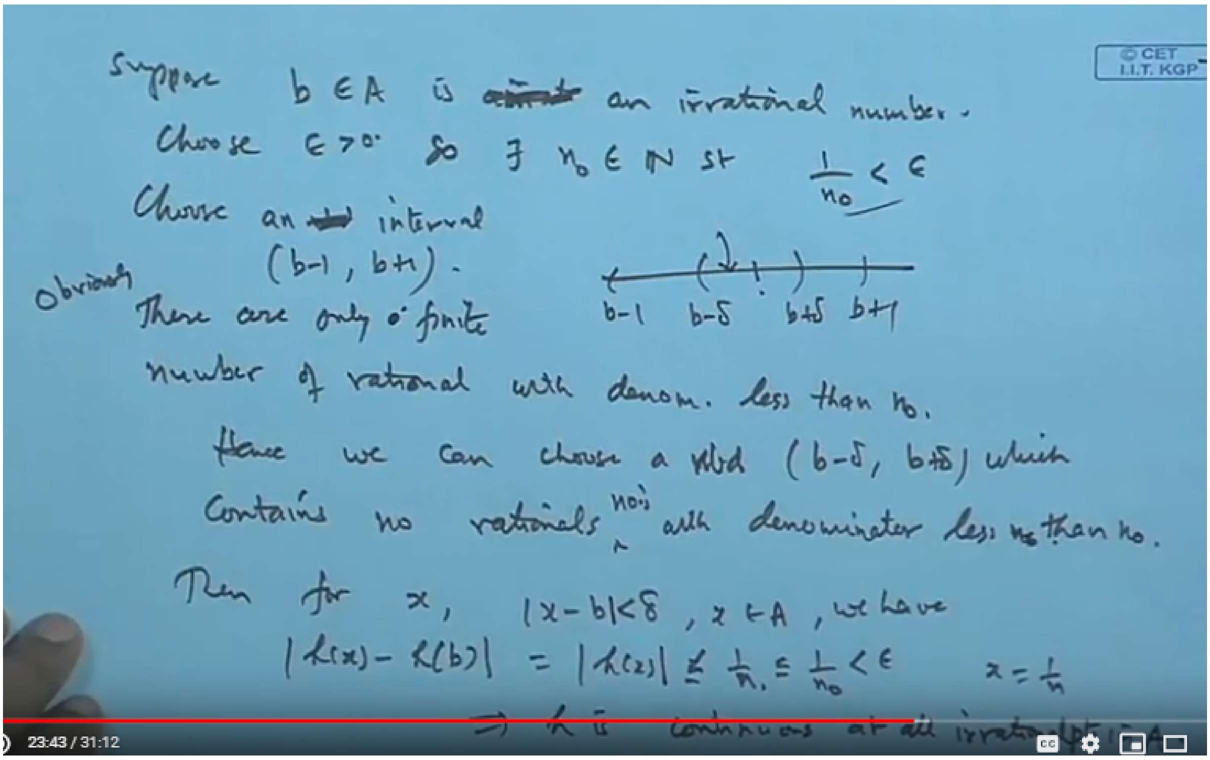

That's now, one more thing, which we have a remark. We have seen the continuity of the function in terms of the limit if suppose the limit does not exist but the function is well defined everywhere except the limiting point then the two possibility are there either the limit will exist function is defined everywhere except the point where the continuity is required so the possibility is when we take the limit of the function X when X tends to that point limit may or may not exist if it exists then we can redefine the function in such a way so that the new function soft n becomes continuous so that is the limit sometimes suppose, suppose, a function a suppose a function of from a to R, is not continuous, is not continuous, because the value of the function at that point is not continuous at a point C because the value of the function FC is not defined is not defined.

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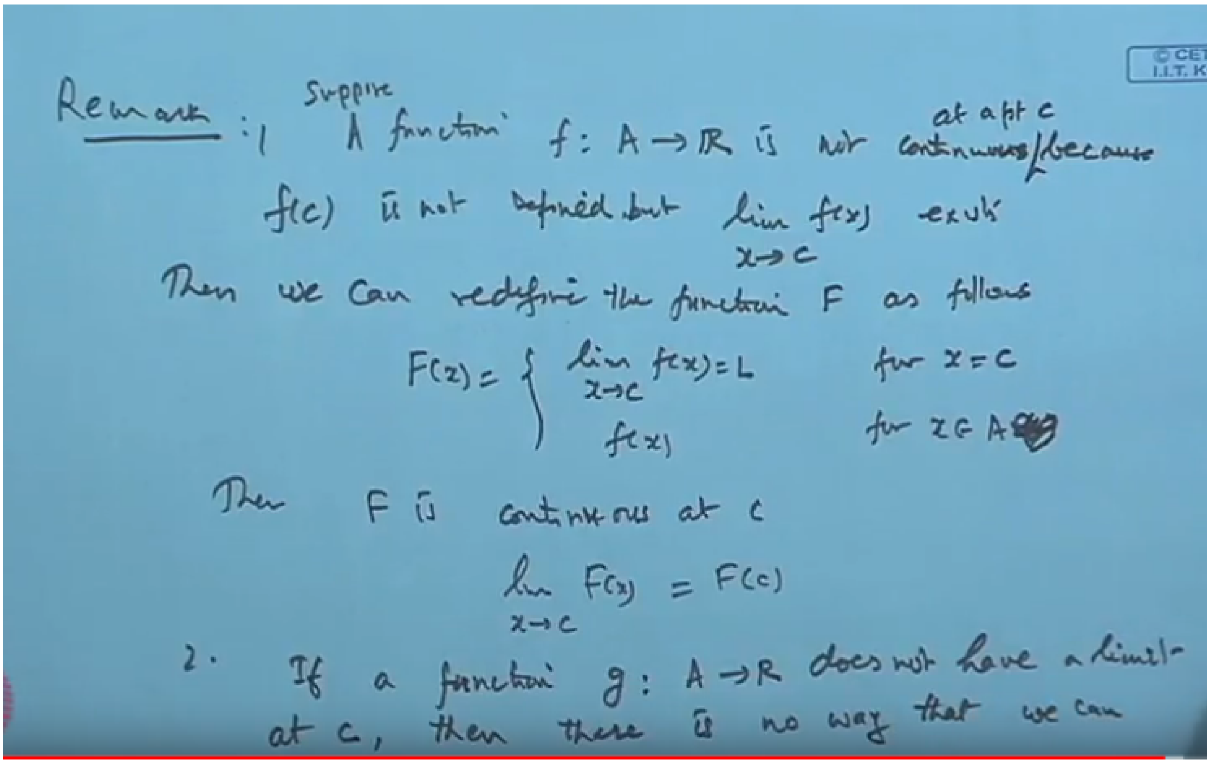

Then the two possibility then but the limit of this function limit of say FX limit of FX when X tends to that C exists then we can we can redefine the function function capital F s follows capital FX is equal to the limit of FX when X tends to C C for X equal to C we are defined this and f of X for X belongs to a which is not seen different from C minus C ok - see that point the function is not defined in frequency okay so this so that is equal to suppose a now we claim this F is continuous is continuous is continuous at C because F is already given to be continuous function accept it and is continuous accept it at the Point C so when you take any point X so the function FX is not defined only at C and rest of the point is defined so when you take that limiting value of the function FX this is comes out to the value of the function f at the point C so limit of this FX when X tends to C is equal to f of C and that is equal to the limit FX exist so this function is continuous it means the original function f from the original function we can remove the continuity we can remove the point of discontinuity and made it to be continuous function and this way we call at that point to be a removable discontinuity ok this but if if the limit of the function if a function does not have a limit see if a function if a function G from a to R does not does not have a limit does not have a limit as she does not have the limit as C then there is no way there is no way that we can do obtain we can obtain there is no way.

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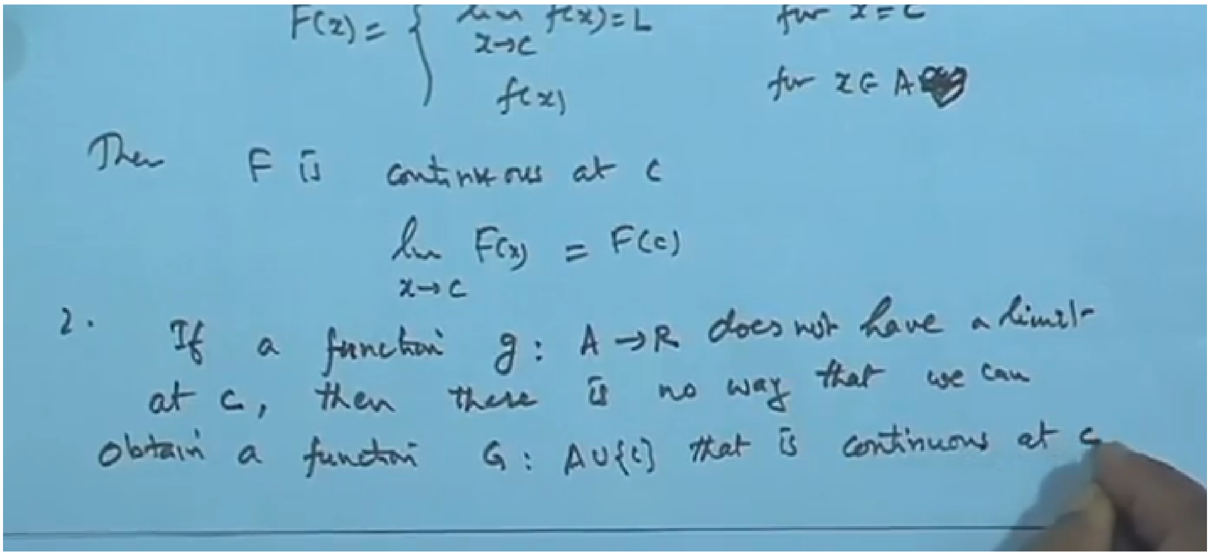

That we can obtain a function G, a union C, here we write, on a union C, this is what left, to R. Okay? To R, please check it. Okay? To, R. So A union C, can that is continuous, that is continuous at C,

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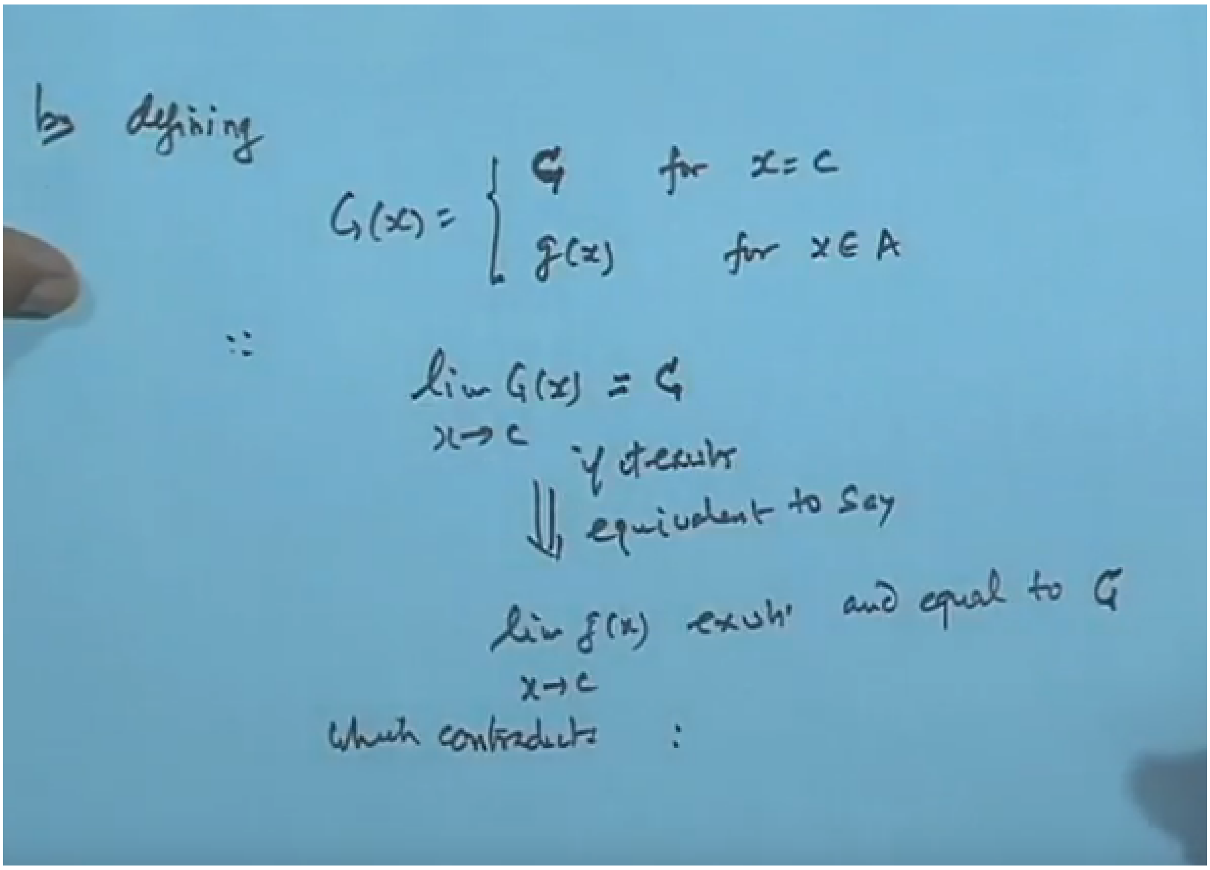

by defining, by defining, G X, as equal to C, for X is equal to say capital C, I am just saying capital C. Okay? For X equal to C and G X for X belongs to a X belongs to the reason is because, if we assume, this pair, then what happens? Because the reason is this because what is the limit of GX when X tends to C, this is given to be, if it exists if it exists then it is given to be the same but this limit when X tends to C it is equivalent to which is equivalent to say the limit of this GX, when X tends to C exists, the if, this limit exists, it means when X is different from C the GX is defined like this is small G so then you are choosing the limit of GX when X approach you have you have for all X, we have the first from C the G will be replaced by small G so limit of this and if this limit exists this will exist which will contradicts must also say and equal to C and equal to capital C, but this is not given, which is a contradiction, which is which contradicts. Therefore we cannot define, redefine the function. So that it becomes continuous that's what.