Module – 8
Lecture 45: Continuous Functions
(Heine's Definition)

So, in the previous lecture, we have introduced, various types of functions, of real variables and an important result. Namely the Weierstrass theorem, on the bond, of the functions effects, in continuation, we have also discussed the cauchy definition of continuity, of the functions. In this lecture, we will introduce the hennas definitions of continuity, of the function and then, we will establish the equalization, between these two definition of continuity. you can say important note.

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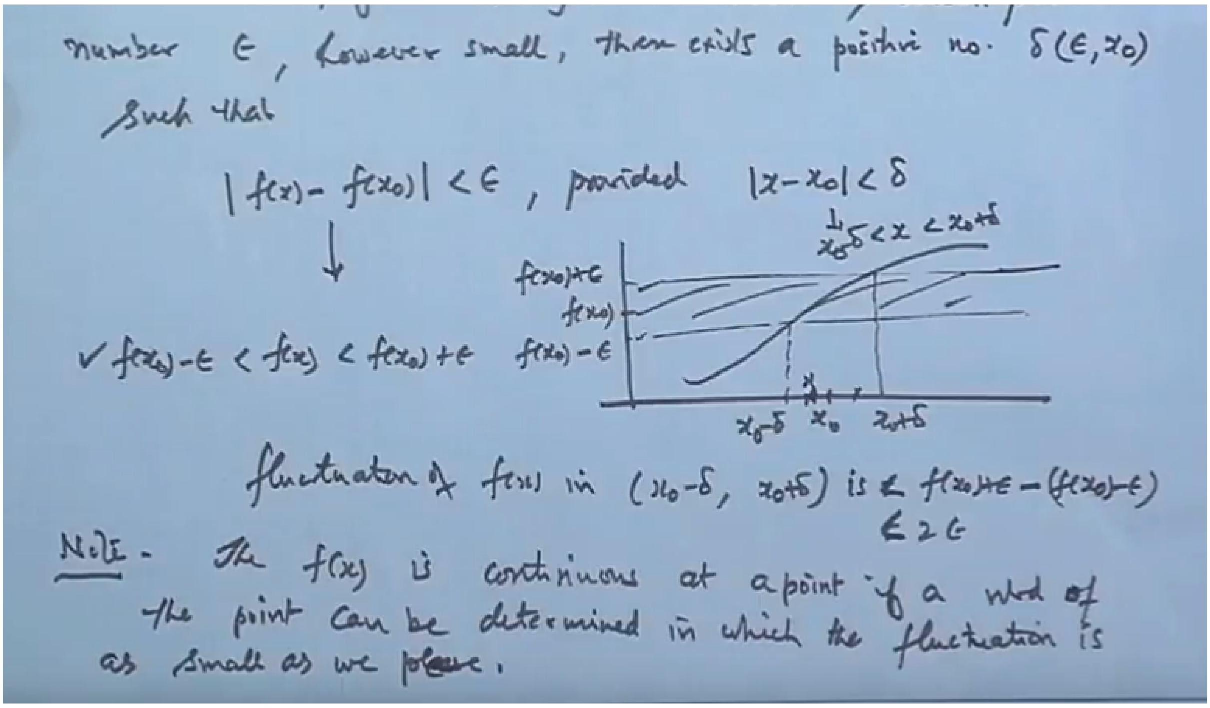

We can say the function FX, the function FX, is continuous at a point, at the point, function FX, is continuous at a point, if a neighbourhood, if a neighbourhood, if a neighbourhood, of the point, never of the point, the neighbourhood of the point, can be determined, can be determined, in which, in which, the fluctuation, the fluctuation, is as small, as we preach, is it more HB please as we please so this is consequence this is considered in the necessary condition because if this does not satisfy the function cannot be continuous function.

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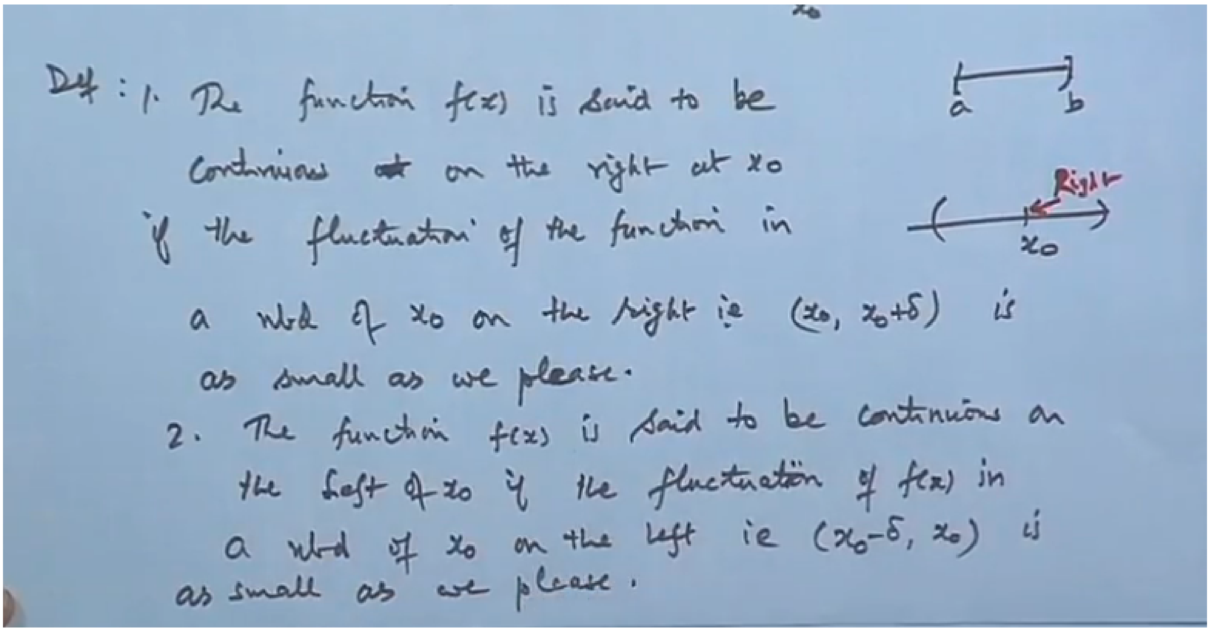

For example suppose I take this graph say this is our graph point X_o is here function f is this now how Sabre is more with this given F sign say with this given F epsilon I take this point is say F X_o so this point is f X_o - Epsilon this point will be F X_o + epsilon and here this is f X_o so for this f epsilon this is the Wits to epsilon for a given epsilon if we get there must exist an interval neighbourhood around the point X_o such that the friction should not exceed by Epsilon but what happen if it suppose I take this neighbourhood okay this is our neighbourhood like this so as soon as you take this point say here this is point here as soon as this point the corresponding image of this will go here it means the fluctuation may exceed by 2 Epsilon so it cannot be made as small as we please and this is because there is no continuity of the curve at this point X_o so such a function will not be considered as a continuous function so that's the important that's why we say this is a case of discontinuity, continuous function case of discontinuity.

Okay? Because of this thing, clear so that's why, it is necessary good now, as a corollary of this, of course, the fruit we further extend this. Suppose a function is given over a closed interval the domain of definition is closed interval and when we say the function is continuous over the closed interval then what we do, we take the open interval first and then at the point a and B, also be test it. But before if the function, may not be defined after B the function main also not be defined so we have to modeify our definition of the continuity at the endpoint of the interval, so we say the function is said to be the function the function is said to be FX is said to be continuous, at continuous, on the right, on the right at X naught, So for this is X_o on the right means this side this side okay on from the right hand side this is right on the right of X_o right on the right at X_o, it X not on a function set we continue that if the fluctuation of the function if the physician, of the function, in a neighbourhood of X_o if the fluctuation of the function in a neighbourhood of X_o, on the right, on the right, that is, that is (X_o, $X_o+\delta$) this is, the neighbourhood on the right of this if the fluctuation of the function in the neighbourhood of X_o, on the right of X_o on the right each edge small as we please then we say the function is continuous at a point X_o, from the right hand side. Similarly the function is said to be continuous on the left the function FX is said to be continuous on the left, left, of X_o, if the fluctuation is the fluctuation of the function in a Neighbourhood in a neighbourhood of X_o, on the left on the left means that is $(X_o - \delta, X_o)$ on the left is Edge moor, as we please so that's the important one okay so now we can define the a function over the continuity in the interval.

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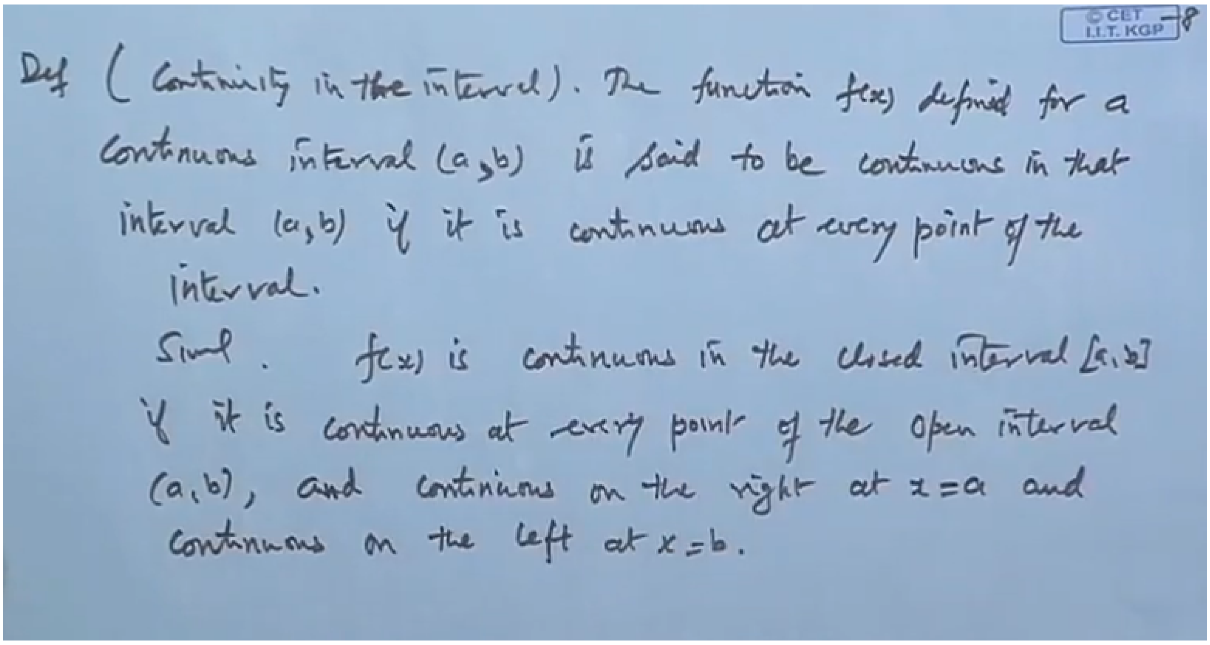

So, we define as a continuity of the function, continuity in the interval, the function FX, function FX, defined for a continuous interval defined for or continuous interval a B continuous interval a B is said to be is said to be continuous is said to be continuous in the interval a B in that interval a B that is at each point if it is continuous if it is continuous if it is continuous at every point at every point of the interval okay if it is a closed similarly we say FX is continuous FX is continuous in the closed interval or over the closed interval a B over the closed interval a B if it is if it is continuous at every point at every point of the open interval of the open interval a B of the open interval a B and continuous on in continuous and continuous on the right at X equal to a and continuous and continuous on the left at X equal to B.

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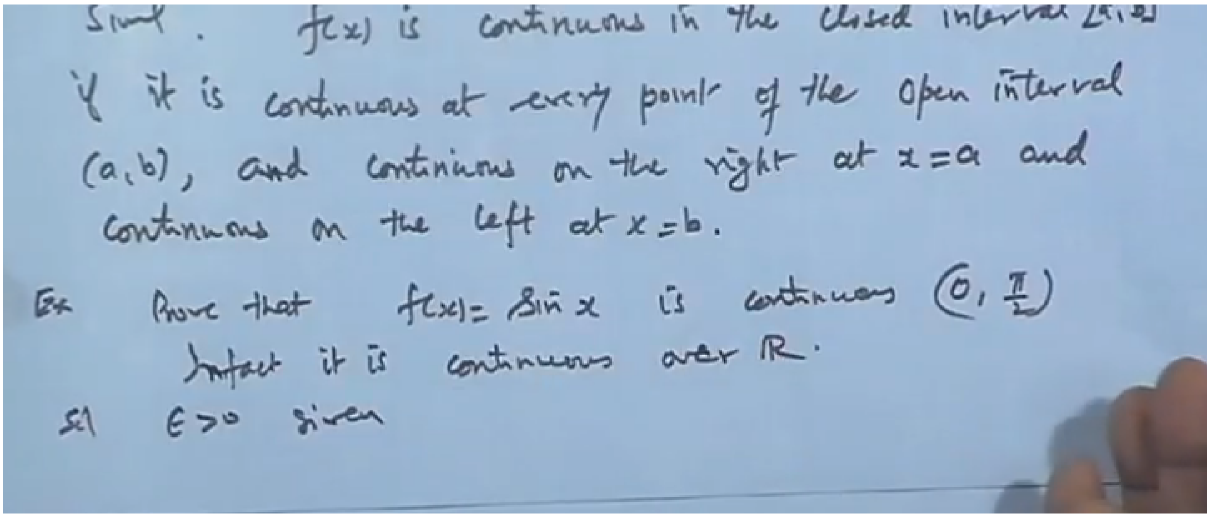

We say the function is continuous okay now let us take an example which is using the epsilon-δ definition suppose put that FX equal to sine X is continuous function it's continuous say over the interval 0 to PI by 2 in fact it is continuous everywhere because it is a periodic function so we can say it ok say 0 to PI by 2 it's continuous in fact it is continuous over the entire real line are over the entire deal I not so the proof is simple suppose epsilon is given greater than zero is given okay we have to fight δ depends on epsilon so that the mode of FX - FX $_o$ less than epsilon is less than epsilon provided X – X not less than δ so consider this

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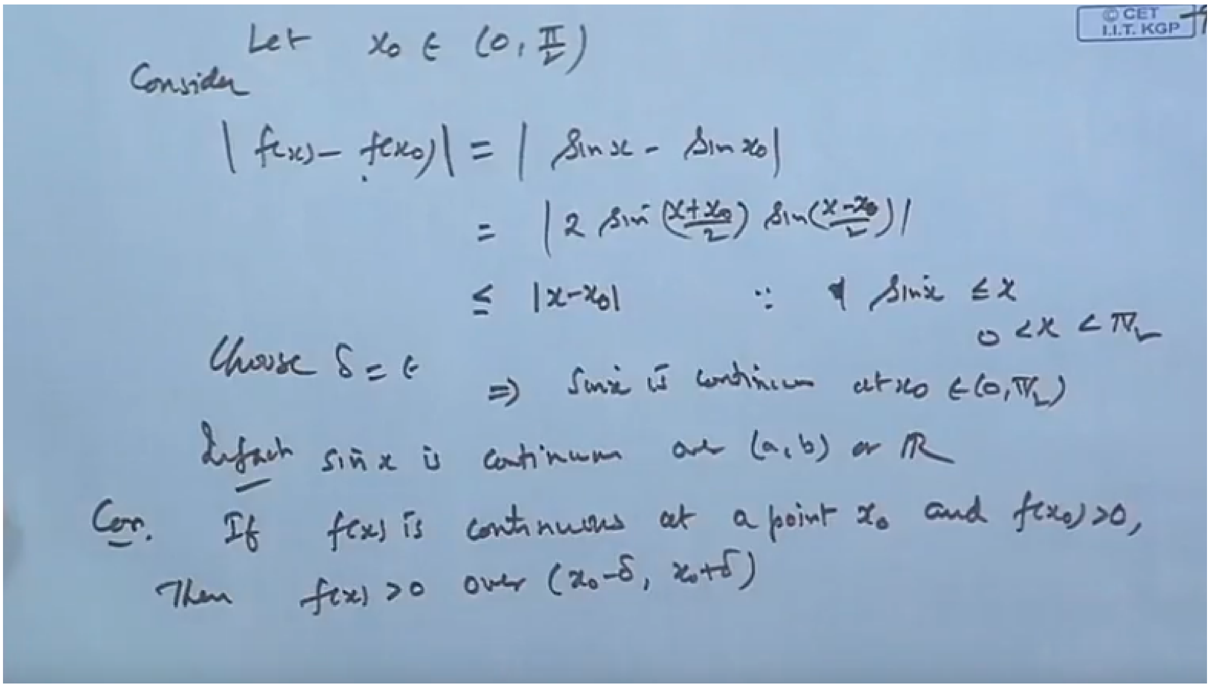

consider mode of FX - FX $_o$ we are let X not be a point in this interval suppose okay or not then what is this this is sine X - sine X not but this is equal to 2 times sine X + X $_o$ by 2 into sine X - X $_o$ by 2 mode of this which is less than equal to now sine of this thing is a bounded

function bounded by 1 so this is less than equal to 1 and this sign will be less than equal to mode $X - X_o$ why because mode of sine because the sine X is less than equal to X for X lying between 0 and PI by 2 sine 0 is your sine 90 is PI by 2 or even 0 to PI it will help ok so any interval you can just say this is in fact this can be extended for any xvlx ok so this is 2 now if I take the X δ so it shows δ equal to Epsilon so if this is less than δ obviously this will less
than Epsilon so this source the sine X is continuous at X_o but X_o is an arbitrary so it is continuous everywhere and even if we stake interval sine X is basically is continuous over any close any open interval a B or in the D line in fact this one can prove easily so this is one of the example which we have I have chuji now currently there is one small result which we call the Connery if FX is continuous if FX is continuous at a point X_o and FX_o is positive then then FX is positive over the entire interval X_o - X $X_o + \delta$ means if the function is continuous at point.

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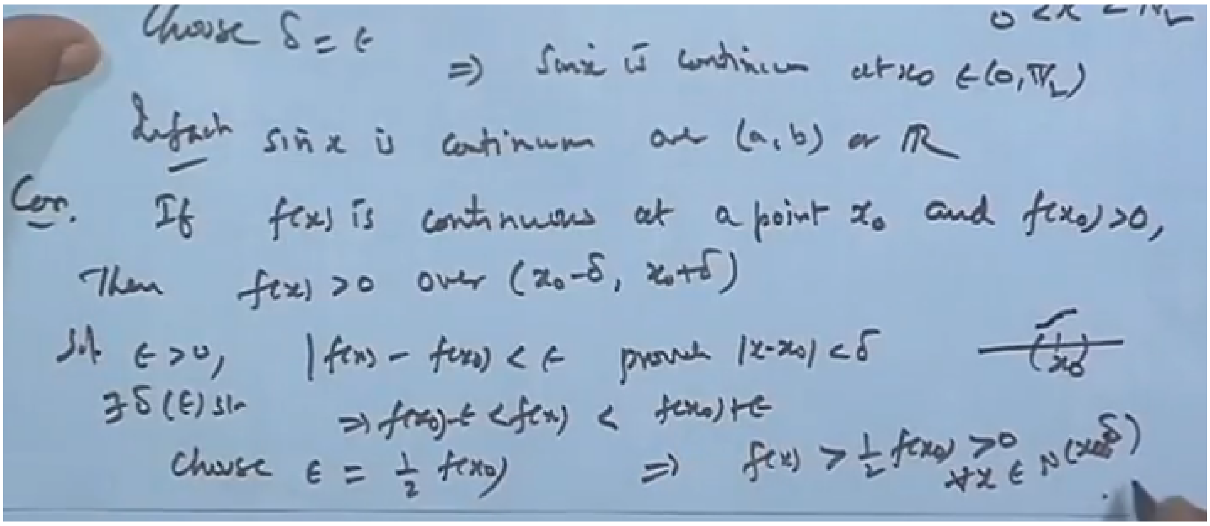

At the point x_o it is positive function it is a positive function graph is up then we can find a neighbourhood where the function will remain positive okay the reason is because it is continuous so this condition is satisfied for a given epsilon greater than 0 there exists a δ depending on epsilon such that this is to provide it mode of $X - X_o$ less than δ so when the X lies between this then what happen this so the FX is lying between FX_o + epsilon greater than FX_o - epsilon so if I choose a channel to be half of FX_o then obviously FX will always be greater than the will be basically half of FX_o because of the left hand side which is positive so this is true for all X belongs to the internet would and neighbourhood of X_o neighbourhood of X_o with a radius δ.

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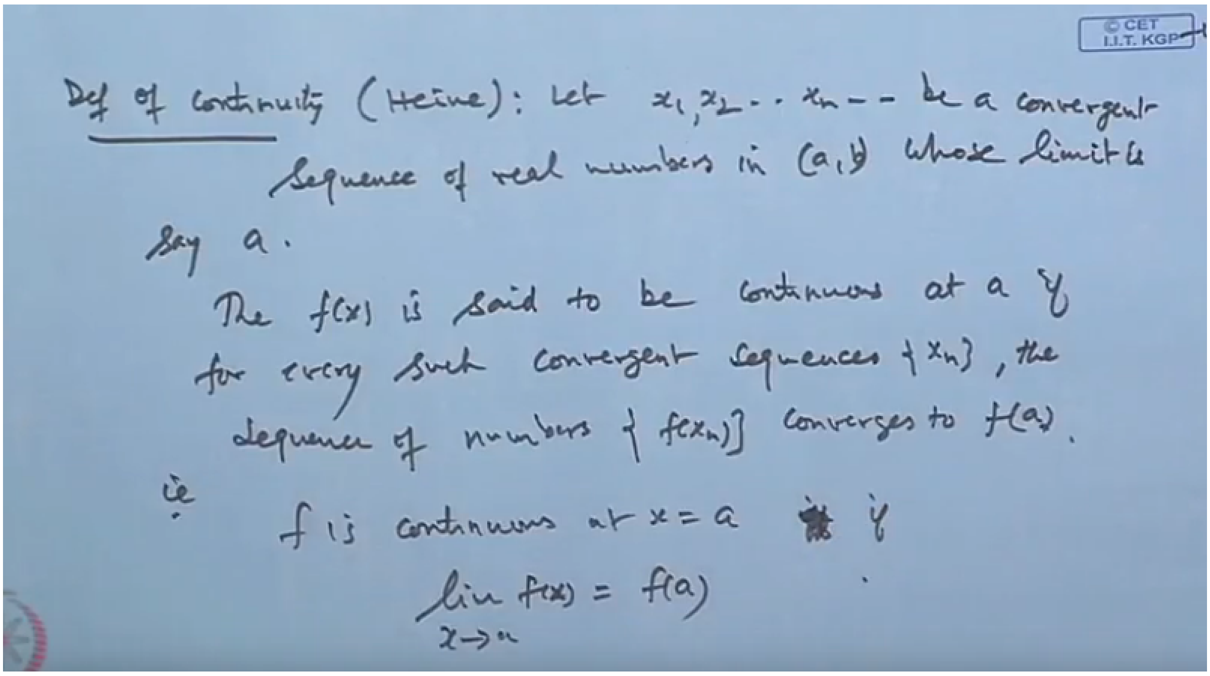

Okay so this disc now Honeywell definition foreign definition of continuity given by Hanny this is given in terms of the sequence so what he said let X 1 X 2 X n X 1 X 2 X n V a convergent sequence convergent sequence of real number set of real numbers all points on the real line in the given interval a B in the given interval whose limit is limit is say a limited say a the function FX by definition of the function FX is said to be continuous is said to be continuous at a point a if for every such sequence for every such convergent sequences sequences xn every such convergent sequence xn the sequence of their functional value sequence of numbers functional means it is numbers now f of X n the serial numbers converges to F of a that is the meaning is f is continuous at a point X equal to say X equal to a if the limit of the function FX when X tends to a exists and coincide with the functional value at this definition is if and only if in fact this is the definition given by this is okay if this condition okay in fact is part if and only if so constantly.

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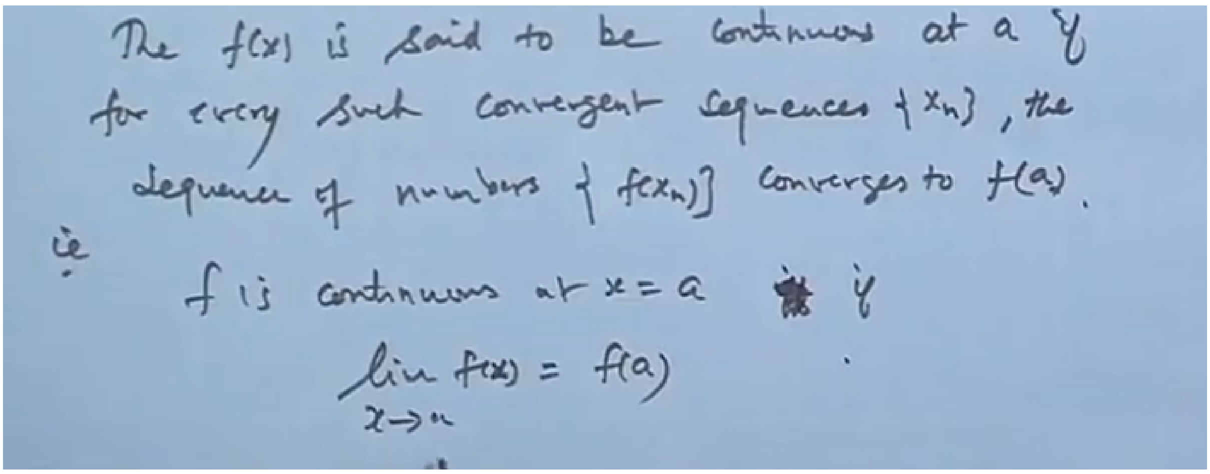

So what he says is that if we take a sequence action which converges to a then f of xn, will go to F, the definition given by the Eny, now these two definitions are basically equivalent definition the equivalence of Cauchy's and Eny's definition of continuity so let us assume the function is continuous assume.

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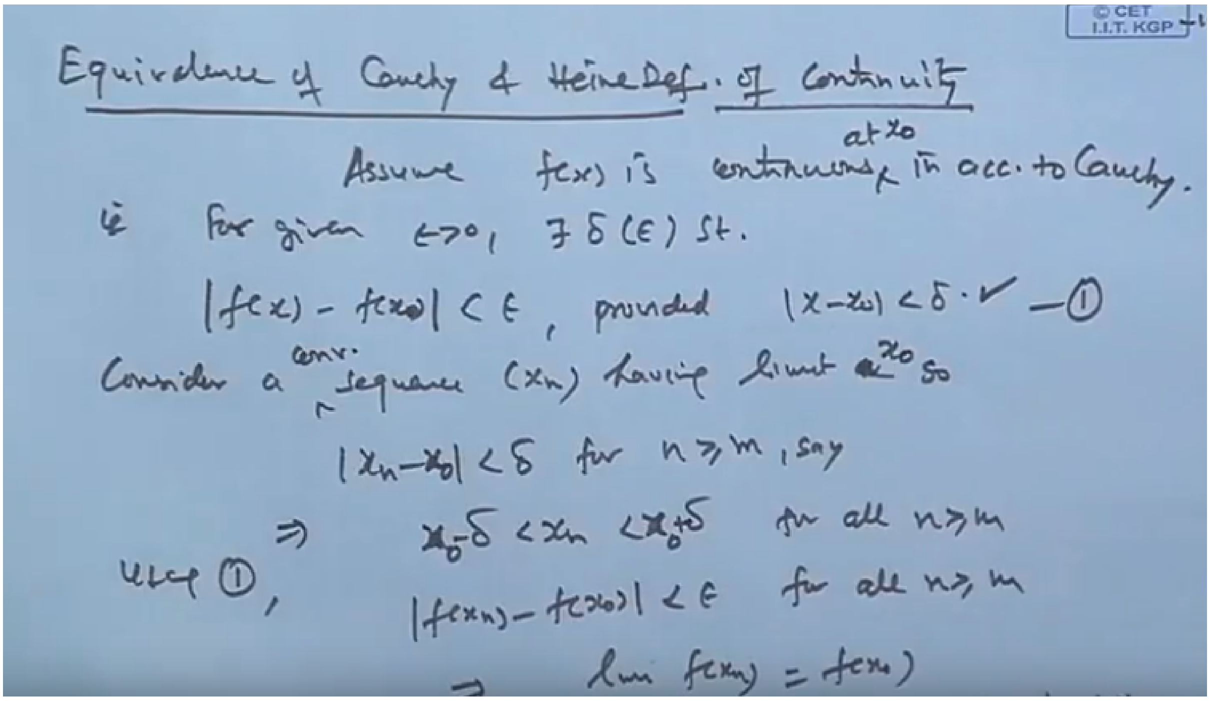

FX is continuous, continuous, in accordance, in according to Cauchy, Cauchy's, it means that is or a given epsilon greater than 0 there exists a δ depending on epsilon such that mode of FX - FX not continuous at X ₒ FX ₒ is less than F epsilon provided mode of X - X ₒ less than δ suppose this is true okay now let us consider a sequence xn a convergent sequence a convergent sequence xn having the limit a means excellence you can converge this to a so by definition so by definition so modeulus of xn - a less than said δ for all N greater than equal to say capital is small M say because limit of xn is a so xn - a will remain less than after a certain stage but if xn lightness this implies that xn lies between a - δ n a + δ for all N greater than equal to n and for such point which lies in this one satisfy this condition so using one what we get is mode of fxn – FX ₒ yes continuous at X ₒ so here X ₒ actually I have taken a limit is X ₒ let us take X ₒ here so X ₒ so here is X ₒ this is also X ₒ because in place of this so FX ₒ this will remain less than epsilon for all N greater than equal to therefore limit of fxn, as n tends to infinity is FX ₒ hence Eny's definition follows hence by Eny's definition f is continuous at conversely, conversely, suppose Eny definition.

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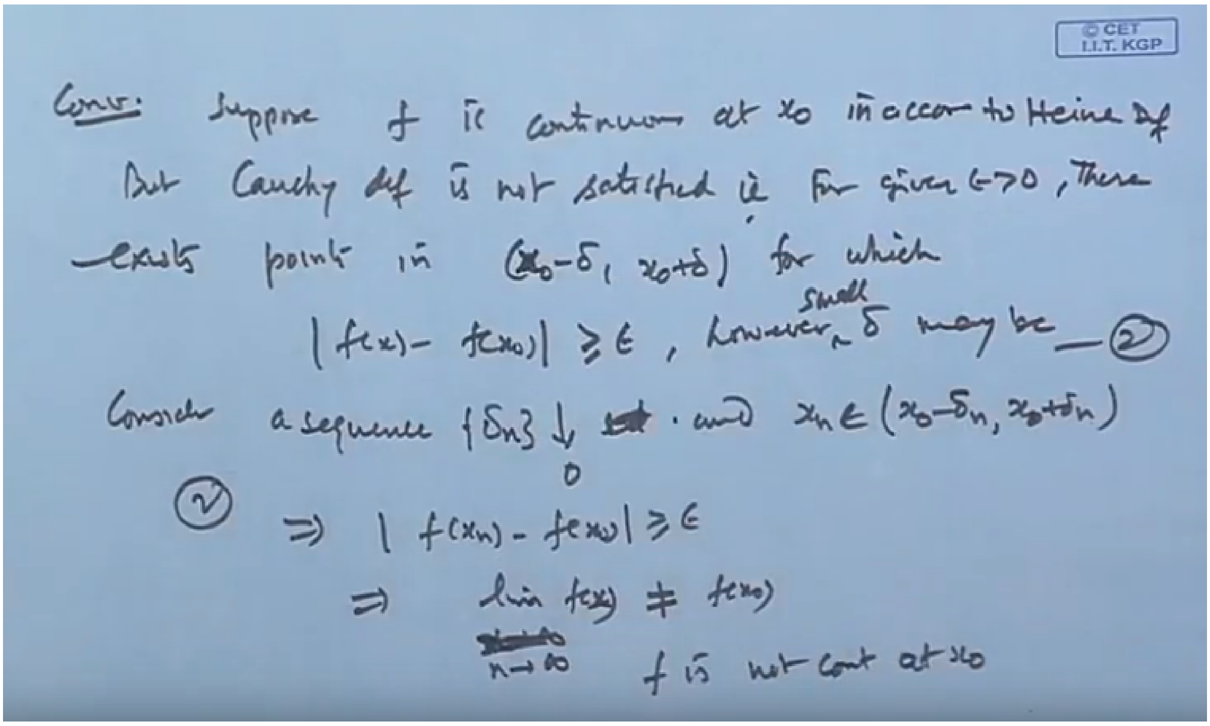

Suppose FX is f is continuous at x $_o$ in accordance to Eny's definition and its definition so for this suppose we say it's continuous in accordance well but, but Cauchy definition is not, Cauchy's definition is not, is not satisfied. That is for a given epsilon greater than 0 for a given epsilon greater than 0 there exist points there exist points in the interval in the interval say X $_o$ - δ, X $_o$ + δ, for which, the fluctuation mode of FX – FX $_o$ exceed by any number F epsilon house ever is more δ maybe, δ may be. So what do you mean by this? This means, if you take a sequence which is in this okay then all this you can so let us take a consider, consider, a sequence a sequence δ n of decreasing nature which decreases to zero, converges to 0, such that secant δ n will converges to 0, and it says that and xn be a point belongs to this X $_o$ - δ n X $_o$ + δ n if we take this point in this sequence of the point in this interval then in then according to the second implies that mode of fxn - FX $_o$ will greater than, equal to f epsilon. This shows the limit of the FX n, as n tends to infinity, as sorry, as X tends to X $_o$, that is n tends to infinity, mean X n, n tends to infinity, is not equal to FX $_o$. So function f is not continuous at X $_o$ all this contradicts the definition of this therefore this assumption that Cauchy definition is not very, satisfied is wrong here for this. Okay, thank you very much. So this was the equivalence of this.