Module 8

Lecture - 44

Lecture 44: Continuous Functions

(Cauchy's Definition)

So, we will discuss the today, the Continuity of Functions. We have already seen, how to define the functions. It is basically a mapping, a route, from a set a, to set B. Such that, each element of a, we can have one, unique element in B. Then, functions is well defined, if there are more than one values of a, then we say, it is not well-defined, of course, function. However, when corresponding to each value X, we have one value, then it is a single valued function. And for each X, if there is more than one value of the imass, more than one imas, then we says, it is a multi valued function. Say for example, FX, is the root X, there for each real x, positive real X, we have two values. So it is a multi valued functions. So we will include, all sorts of the functions, whether it is a single valued function or multi valued function. And when we say, FX is a real valued function, it means, that values of the FX, lies in the real M.

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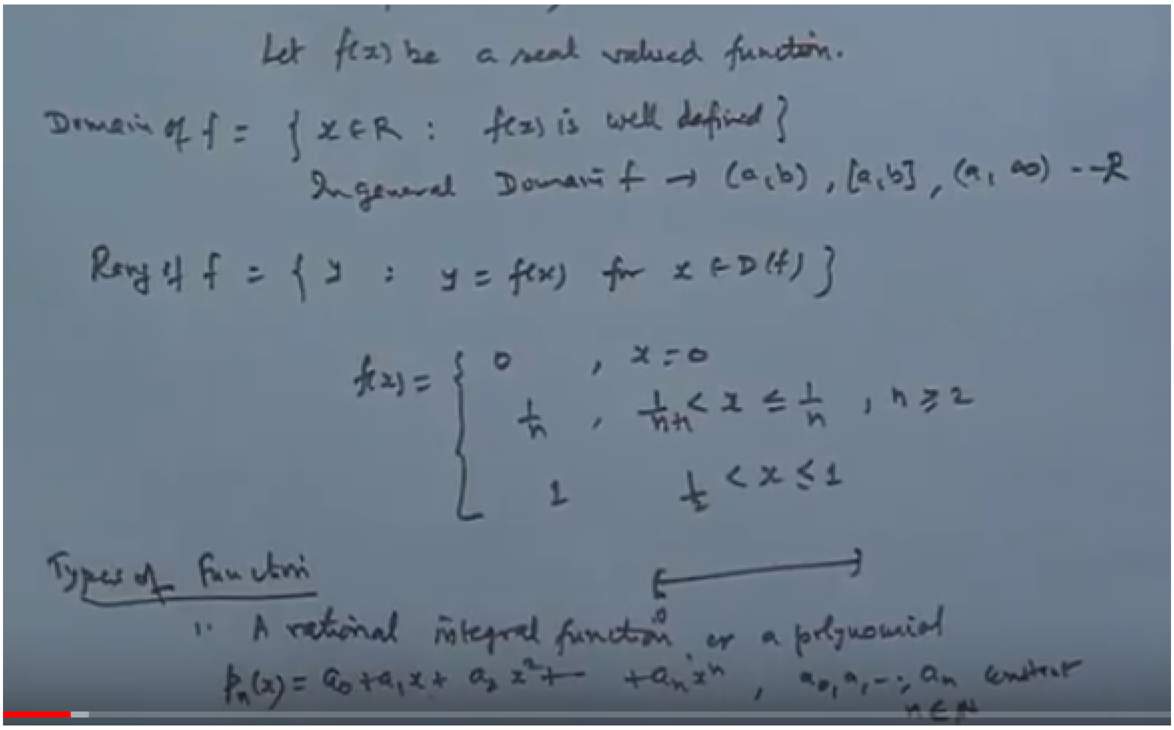

So let FX, be a real valued function, that by the domain of F, we mean, domain of F, we mean, the set of those real numbers, are, set of those real number X belongs to R, such that, the FX is well defined, well defined, at this. Means FX, should not be infinity or minus infinity or the functional at X and it must be, a finite value, well defined. Normally the domain of F, in general, the domain of F, normally we consider, either a open interval or may be a closed interval or may be a semi closed interval, say like this or even the entire real R. So it depends on the function, where it is defined, we are not. And the set of those values, of F, is the range of F, range of F, means, those y's, such that, there exists some X, such that F X equal to Y, for X belongs to the domain of F, okay. So all the imas is set, is the range of the F, so this much we have. Now the domain of F or the function, when we say the function is defined over a dominant F, then it is not necessary, that function, the domain will be a continuously defined like, a b. It may be, in the break up form, may be the function may not be defined, same function may not behave, may not be defined, over the entire domain, it may be, if several functions, can be take up, over the domain of the F. For example, if we say FX is say 0, when x is 0, say FX, is equal to, lying between, say 1 by, n plus 1, less than X, less than equal to 1 by n, where n is, an integer ,greater than equal to 2 say, then the value of this is defined as the one functional, FX is defined by 1 by n, so in the interval when it is lying between, 1 into, 1 ½, less than X, less than equal to 1, the function is defined suppose 1. So this is, also a way of defining, the function, means a function, may be take up of this type of also, where the interval, the domain of definition, is from 0 to 1. But it is not the same function, by means of the same function; we are not defining over the interval. What we are doing, what we have here is, over these several parts of this interval 0 1, this is 0 1 interval, we have a different form of the F, at different points, all even if different subintervals like this, so this means. Now there are different types of the functions, which we come across, the like the rational, types of real valued function or types of functions, which we will come across. The first, may be, a rational integral function, a rational integral function, all polynomial, we can also say it is a or a polynomial, which is of the form, like, a naught, plus a 1 X, a 2, X square, plus a n, X to the power n, well the coefficients, a naught 8, a naught, a 1, a 2, a n, these are all constants, real constants. Okay? And N is a positive integer, positive integer, belongs to n. So this is also a functional and we denote by PN X, a polynomial, of degree n.

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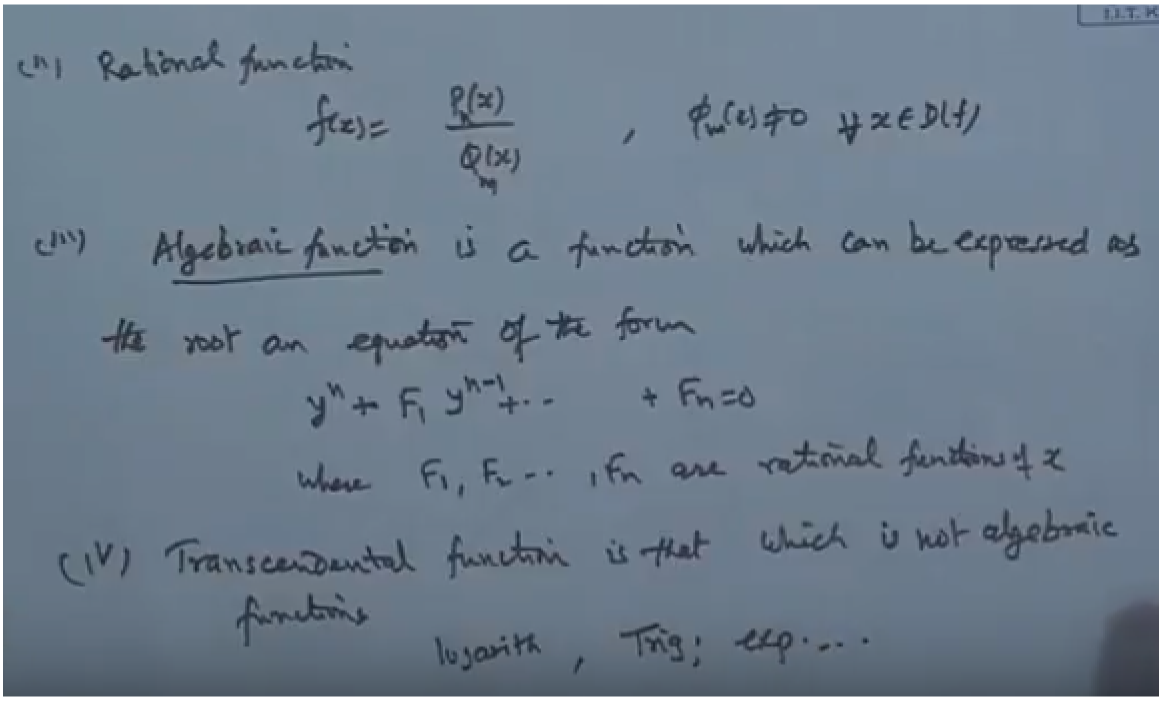

another type of the function, which we come across, about the rational functions, rational functions. A function FX, which is of the form, which is of the form, P X by qx, where px is also a polynomial, say of degree n, QX is a polynomial of degree say, N. But Phi of M X, is not 0. For any X, belongs to the domain of F, for every, for every X, belongs to F, Phi M X, is not 0. Then such a function, we call it as a, rational function. And then, a third type of the function, which we can approach, as algebraic function. Algebraic function, is a function, is a function, which is or which can, be which can be, expressed as, as, as the root of, root of, an equation, of the form, say, Y to the power n, plus, f1, Y to the power, n minus 1, plus f2, y to the power, n minus 2, plus FN, equal to 0. Where, this f1, f2, FN, these are rational functions of X. Where these, F, ,F 1, F 2, F n, are rational functions, functions, of X. So the roots of this equation will be a function. Because these are all rational functions, coefficients are rational function, so the roots will be a function and that. Function we call it as a algebraic functions. Then transcendental functions, transcendental function, are those functions, which are not algebraic. Is the, is that function, which is not algebraic function, which is not algebraic function. So, transcendental functions are that. Now in this case, we have seen so many examples, like logarithmic functions, logarithmic function, and then trigonometry functions, like trigonometry functions and then exponential functions, all these, new function, will come across, for this. So we will discuss in general first, the continuity of these type functions. And then we will see that this functions which are smooth continuous and also the some uniform continuity, will also test it will be more useful to develop further the x. Okay, so let's come to now before going for this application say definition for continuity we will also look we require the bounds for it.

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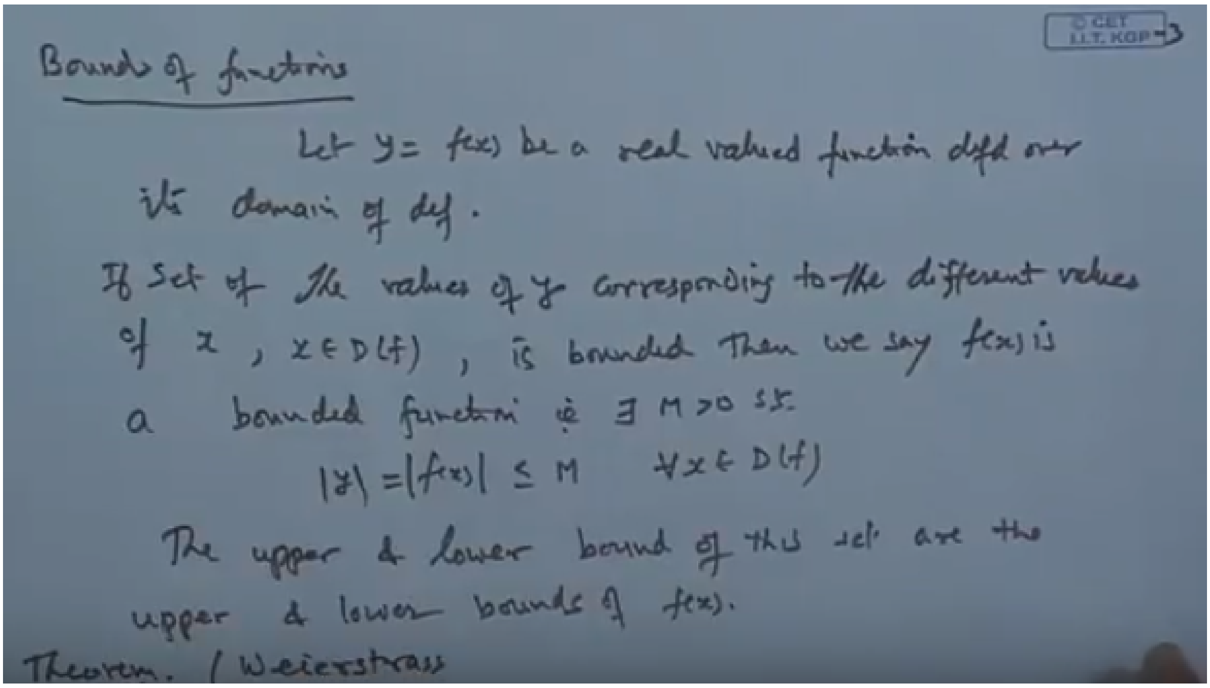

so let Y or the bounds bound of the functions the bounds of functions in fact this is when we say the function FX is bounded when we do we say so let F by is equal to FX be a real valued function define over its domain its domain of definition whatever whether it interval or maybe a union of the disjoint intervals and like this interval the set the values of by the values of by corresponding to corresponding to the different values of X X so this will form a set the values of y corresponding to the   different values of X so if the set of set of the values of Y corresponding to different values of X if it is bounded if the set of values of Y corresponding to different values of X where the X belongs to the domain of F is bounded then we say the function FX is a bounded function, it means that is mod of FX is less than equal to some constant that is there exist some ham greater than zero such that this is true for all X belongs to the domain of F. What is FX? FX is by basically the value of F at the point X do trade able by so if the set of all set of values of FX if this set is a bounded set then we say the function f is a bounded function and in the similar way the upper and lower bound of this set the upper and lower bound of this set upper and lower bound of this sets are the upper and lower bound upper and lower bound bounds of the function FX that is 1 the upper bound of FX then find out the or imass and then among these imass find the upper bound so that upper bound will be the upper bound for FX similarly the lower bound for FX like this if this upper bound is finite fine otherwise a this may be unbounded set if that upper bound we are not able to get m to be finite okay.

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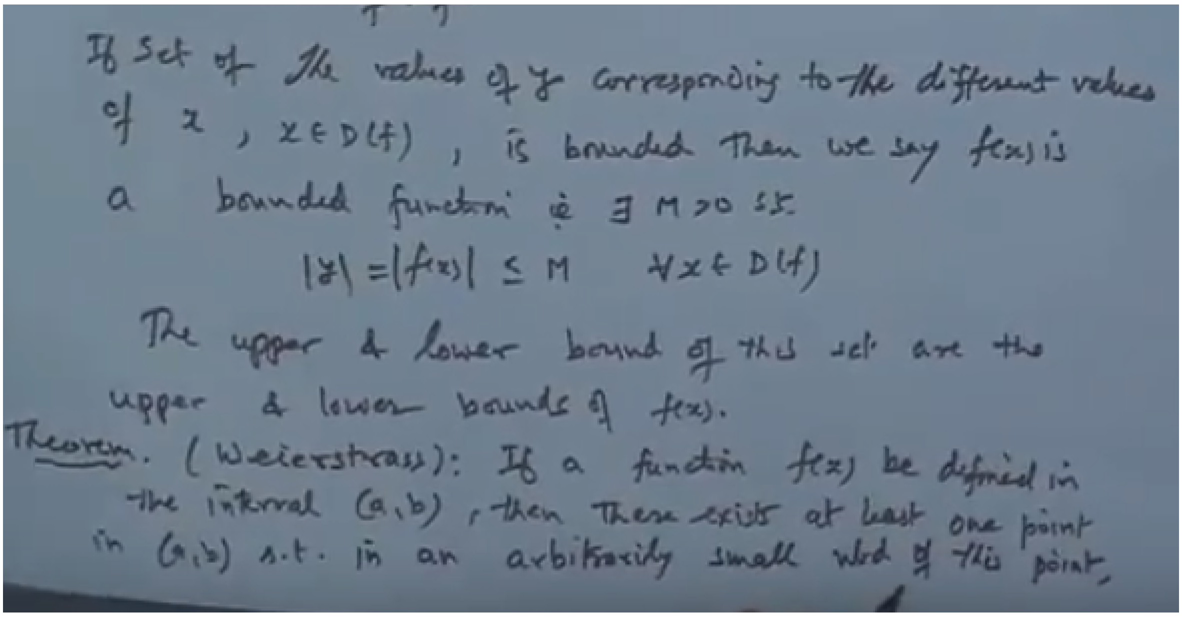

So one more results which we require here that is a theorem given by Weierstrass theorem of says if a function FX, we defined in the interval a B of interval a B, then there exists, then there exists, at least there exists at least one point, at least one point, in the interval a B in the interval a B, such that, says that in an arbitrary small in an arbitrarily small neighbourhood of this point, of this point, of this point, at this point

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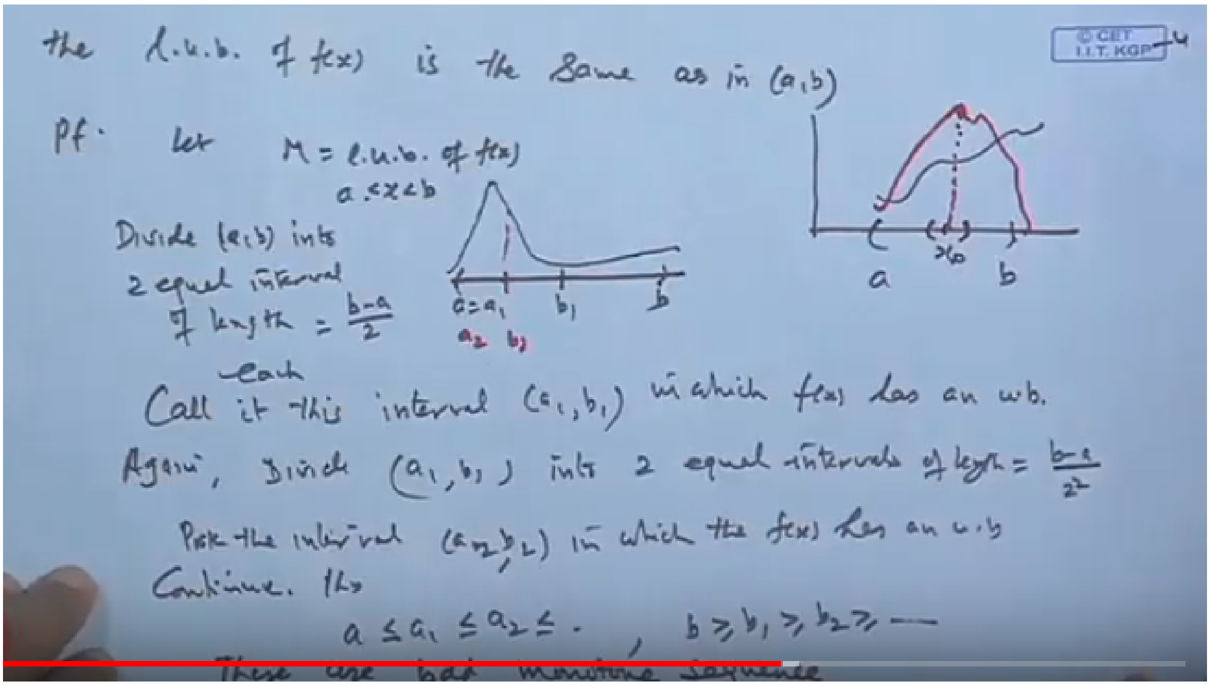

the least upper bound of FX, the least upper bound, of FX, this double bond of FX, is the same as as in a B, so what it says is if suppose the function we define this is a function suppose this is interval and the function is defined everywhere at this everywhere it is the function is well defined then it just a point say X naught, in the interval a B such that in an very alpha T is small neighbourhood, if I choose then this function will have the same upper bound as it is in over the entire thing. Say if I choose this function suppose this is only function suppose I take this function this function suppose I did. Okay? Now you see the upper bound maximum value is attained at this point. Okay? Now we can find it very small a point X naught so that in an arbitrary small neighbourhood the upper bound of the function f a is the same as the upper bound of the function over the entire range a V that's what is the short set the proof is obviously the function is, throughout well-defined function the proof is very simple. Suppose the upper bound of this is, suppose M. Okay? Then what we do? Let us find that, by section of this interval, such that, we get this, let M be the upper bound of this okay so let M be the upper bound of FX over the interval X, belongs to a B. Okay? X belongs to a B, this is an upper bound, for this, say least upper bound, we can say is this if it is close interval then we can say say least upper bound of this on the close interval let it be it's okay no problem okay now this interval is given we divide this interval into two parts of equal length now we pick up that interval in which that least upper bound occurs okay suppose we have this say this curve suppose like this suppose this curve is so what we do if we divide this in a to equal intervals and then we pick up the interval a1 b1 which is divided a B interval into two equal intervals of length B minus a by 2 each of length this each each of this length so let pick call that interval pick up that interval in which the least upper bound is there so obviously here a1 b1 in this interval this upper bound lies okay so this even women call it this call it this interval a 1v1 in which the function FX has an upper bound, a perfect college this now this a1 may consult with a or demon may coincide with B depending on the function now further again divide a 1 B 1 interval into two equal intervals of length equal to B minus a by 2 square and then pick up this called that interval pick up the pick up the interval say a 2v2 in which the function FX has an upper bound so you are further dividing this into your further dividing it into two so now I am picking up this this is we call it a 2 this is B 2 because in this only the upper bound lies continue this process so if I continue this process what happens we are getting in this process when you continue you are getting the sequence of the points a 1 a 2 a hence such that a is less than equal to a less than equal to a 2 and so on well B is greater than equal to B 1 greater than equal to B 2 and so on so asnt increasing Simonton increasing sequence B is a monotonic decreasing sequence but the upper bound of a cannot exceed by Bhiemma and lower bound of this when you take the B1, B2, BN all these things the lower bound of this will be again when you take B, B1, B2, B1 is an upper bound here is it not so this is bounded above this bounded below and below will be at the most B it B okay so this will be the both are so.

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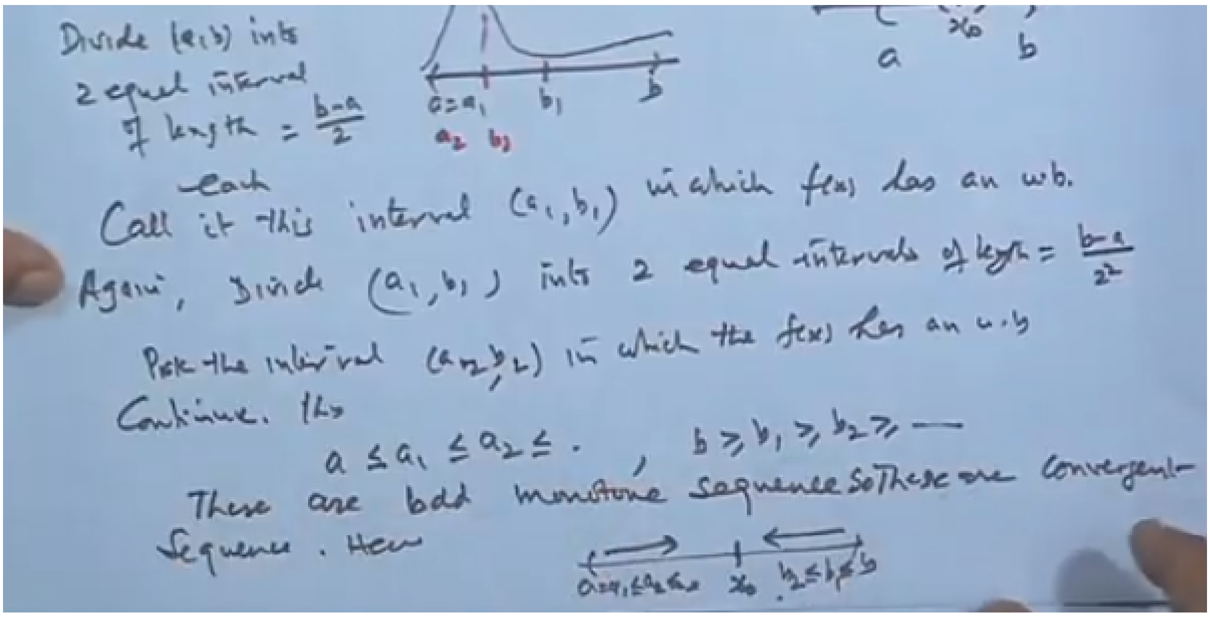

these are the monotonic sequence these are bounded monotonic, monotone sequences and we know the bounded monotone sequence of real numbers or rational numbers they are all convergent so they will converge so they converges so they are these are convergent sequence so these are convergent sequences hence n will converge to a point B n will converse from there so we get from here a B so a a 1 less than equal to a 2 less than equal to a 3 n be greater than B 1 greater than B 2 and so on so a n will go this side B n go to this type so a point X naught can be friend we have the limit of this will coincide ok.

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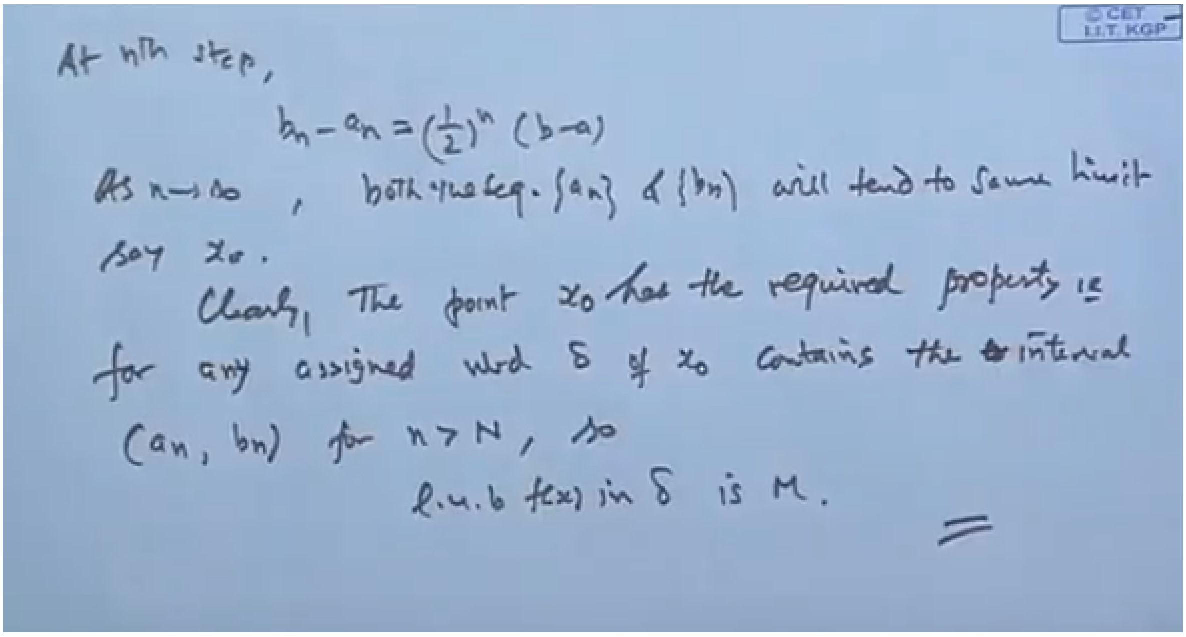

Why because at the Nth stas a step what we get the difference between B and minus n will be 1/2 to the power n B minus a so as n tends to infinity this difference is very very small therefore as n tends to infinity both the sequence a and B n will tend to the same limit same limit and that limit point is about a say X naught it means there exist an X naught and a neighborhood around the point X naught will be there in vista upper bound right ok so the point x naught the point clearly the point X naught has the required property required property, property that is for any sign for any sign for any sign neighbourhood Delta of X naught Delta of X naught contains the interval in comma B for large n so that so the least upper bound of the function FX in Delta each that's what it okay so this basically is proved and assuming the function is well defined at each point of this so the continuity is obviously is taken in consideration here so not looking of there now we come from for Murray the definition of continuity so there are two ways of defining the continuity one is given by the Cauchy another one is given by Henning. So Cauchy has taken in the form using the epsilon-delta definition that in terms of the intervals well that Henning is defined the continuity of the function in terms of the limit of the sequences but both these concepts both the definition are basically equivalent definition.

So we will say first what are these definitions and then we will justify that these two definitions are basically an equivalent definition.

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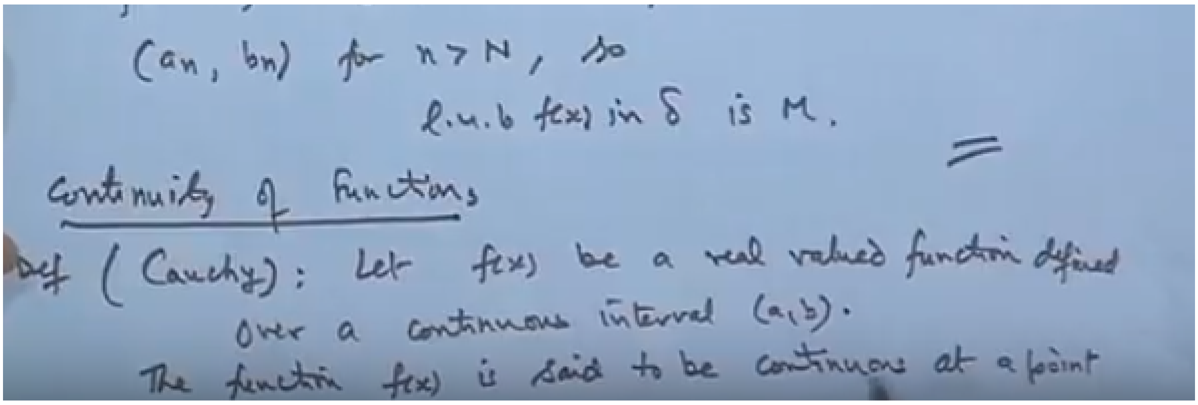

So let's see the continuity of the function of functions so first the definition by Kochi coach its definition what the course is efficient let the dominoes continue let FX be a function we have real value function define defined over a continuous interval a B of course here we are taking a continuous interval or but it may be the partly function will also be defined within a different way so over a continuous function interval a B and let by equal to FX be the given values of the function at this interval then the function FX the function FX is said to be continuous is said to be continuous at a point at a point X naught,

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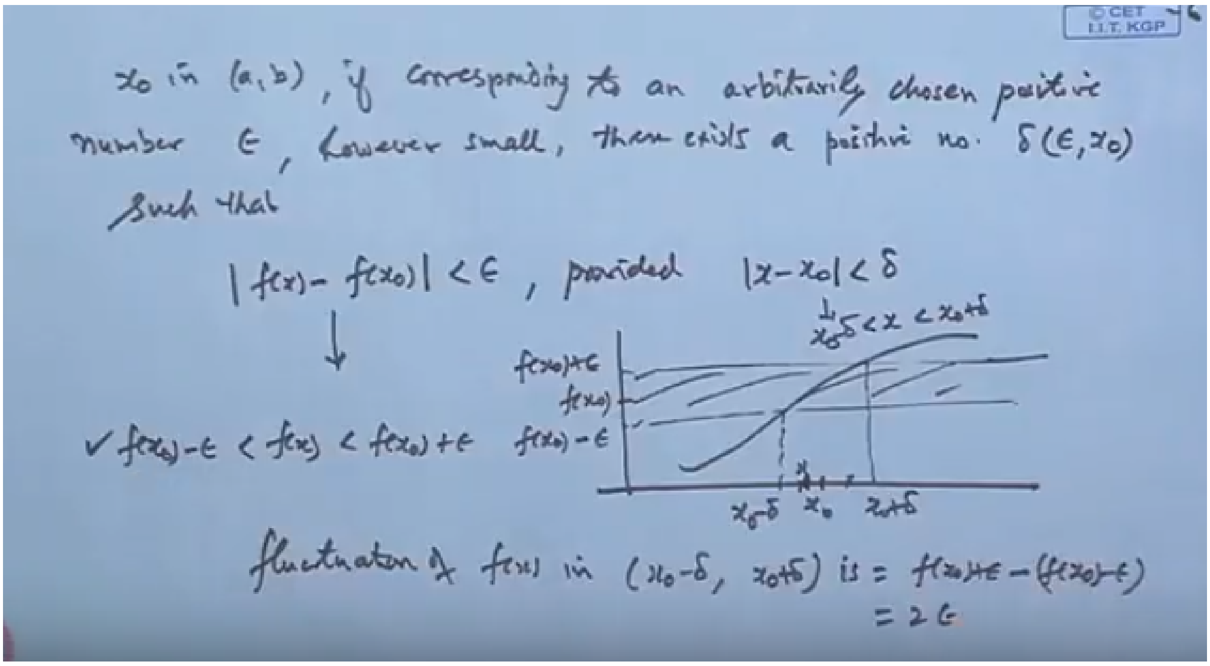

in the interval a,B, if for a given epsilon greater than zero or if corresponding to an arbitrary if corresponding to an arbitrary corresponding to an arbitrarily choosing arbitrarily chosen positive number epsilon positive number epsilon how small may be however is small her very small another there exist corresponding to this there exist a positive number Delta which depends on epsilon as well as on the point X naught such that such that the modulus of FX minus FX minus FX naught is less than epsilon provided mod of X minus X naught is less than Delta it means what the meaning is this a function f is continuous at a point X naught if for a corresponding to a sine arbitrary choosing positive number F signer corresponding to Arbutus choosing positive number F signer there exist or Delta there exist a delta depends on epsilon depends on epsilon so such that this condition hold it means that if we take epsilon is given then we can find out a neighbourhood of X not say X naught minus Delta and X naught plus Delta a neighbourhood can be obtained with respect to this given epsilon with respect to the bond which is given here such that the imas of any point X will always fall within this range will always fall within the same that is here is say FX naught then the range fluctuation of this will be FX naught minus epsilon FX naught plus epsilon of FX minus FX naught less than a this implies the FX lies between FX naught minus epsilon and FX naught plus epsilon and this we mean that X lies between X naught minus Delta and X naught plus Delta it means so if F is continuous then a neighbourhood of the X naught exists such that imas of any point X in this neighbourhood will always satisfy this condition that it will always fall within this to F signer bits of that step whatever the point is choose if so then we say it is reference it means the total fluctuation the fluctuation of the function of the function FX in the interval X naught minus Delta to X naught plus Delta H what the is basically the difference between these two upper and the lower bound so X naught plus epsilon minus FX naught - epsilon and that is equal to 2 Epsilon so if this fluctuation is less than.