Module 8

Lecture – 43

Limit of Functions at Infinity

Now, earlier we have discussed the limit of sequence of the real numbers. Now in this lecture, we will see, how the limit of the sequence of the functions can be introduced and whether the corresponding results, which are valid for the sequence of real numbers, can also be valid, in case of the sequence of functions of real number when you take the limits.

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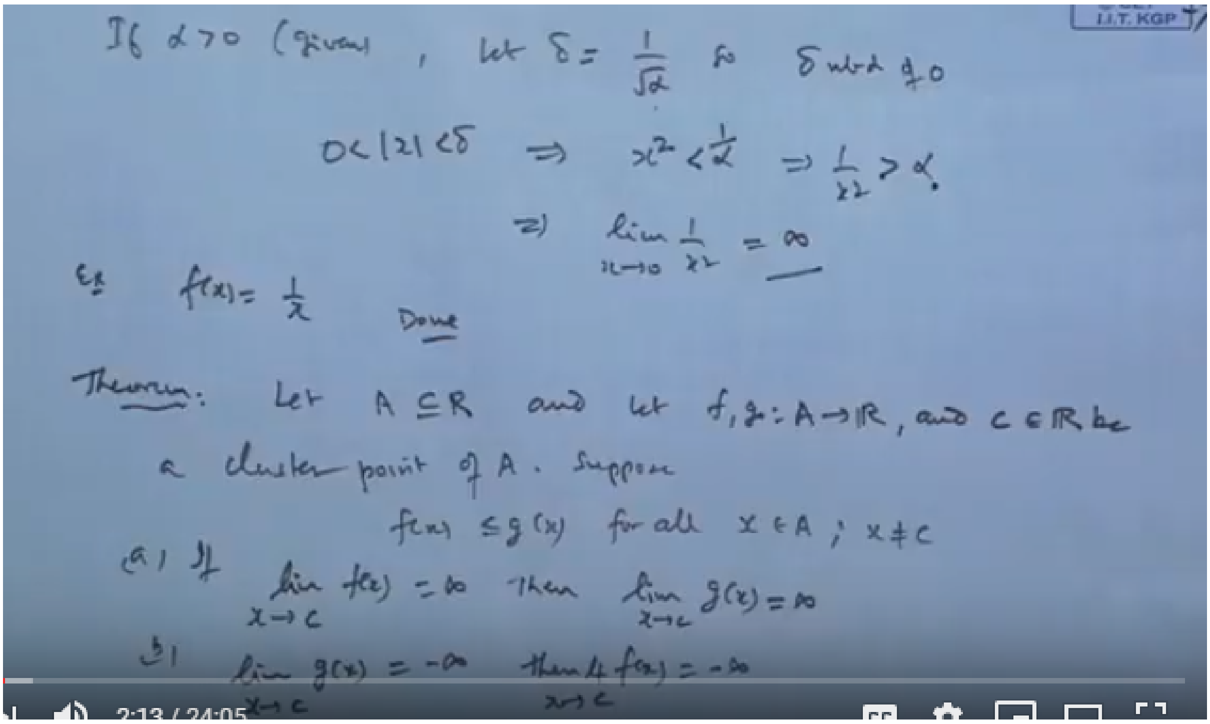

In the form, let a, which is subset of R, let F and G, which are the mapping from, A to R, A to R and let C, belongs to R, be cluster appoint, of A, cluster point of A. Suppose that, FX is less than equal to GX, for all X belongs to a, that is for all X belongs to a, but X is not equal to C. X belongs to A, but X is not equal to C. Then limit of this function FX, if limit of FX, when X tends to C, is infinity, then limit of GX, when X tends to C, will also be infinity. And b part is ,if the limit of G X, when X tends to infinity, X tends to C, is minus infinity, then the limit of F, if, when X tends to, then limit of this X tends to C, will be minus infinity. Of course this is obviously, one can say, so we are just skipping the proof for it and then. Okay?

Now, so far we have taken, when L is infinity. Is it not? L is infinity and then we have chosen this part. Now one-sided limit of this definition; Let A, belongs, which is subset of R and let F, f is a mapping from a to R and if C is a cluster point, belongs to R, be a cluster point of a, of the set, a intersection, C infinity, that is the left hand limit and the right hand limit, we are giving the concept here, X belongs to a, where the X is greater than C, then, limit of this F, when X tends to C plus, is infinity means, if, if, for every, for every alpha, belongs to R, there is a delta depending on alpha, positive, such that, such that, for all x belongs to a, for all X belongs to a, with 0, less than X, minus C, less than Delta?
Then the f of X, is greater than alpha, for this. So this is the left hand concept of the limit, the limit of the F, from, so the right hand side is infinity means, that if I picked up a set intersection C infinity and C is the cluster point of this set, then the, we say the limit of the function, when C approaches towards the right-hand side or right hand limit of A's infinity, means, that we can identify a delta, corresponding to every alpha, such that the f of X, will exceed by alpha for all X, satisfying this condition. Similarly, when it is the negative side, then we say simply, if it is negative, say minus then what is changes here is the corresponding C, it will be infinity C, that is in minus infinity C and then from here minus infinity C and from here it will be just a C minus, X, 0 less than X, less than Delta, then it will come to be FX, less than alpha, like that. So similarly the changes will be like this, accordingly, we are not giving that. Okay? Then, for example is, this, say 1 by a in Delta, FX, GX equal to E to the power, 1 by X, when X is not equal to 0? Now this we have already seen, that if we take for any Delta, for any point, any interval, 0 Delta, where Delta is positive? The right hand side limit of this function, tends to, this right hand, when X tends to Delta, the right hand tends to infinity, as X tends to 0 plus. And this limit goes to 0, as X tends to 0 minus. This we have already discussed limiting. Okay?
Now, next concept, when C, one of, the limiting point C, is not finite, if it is infinity, then also we can define, the limit as follows;

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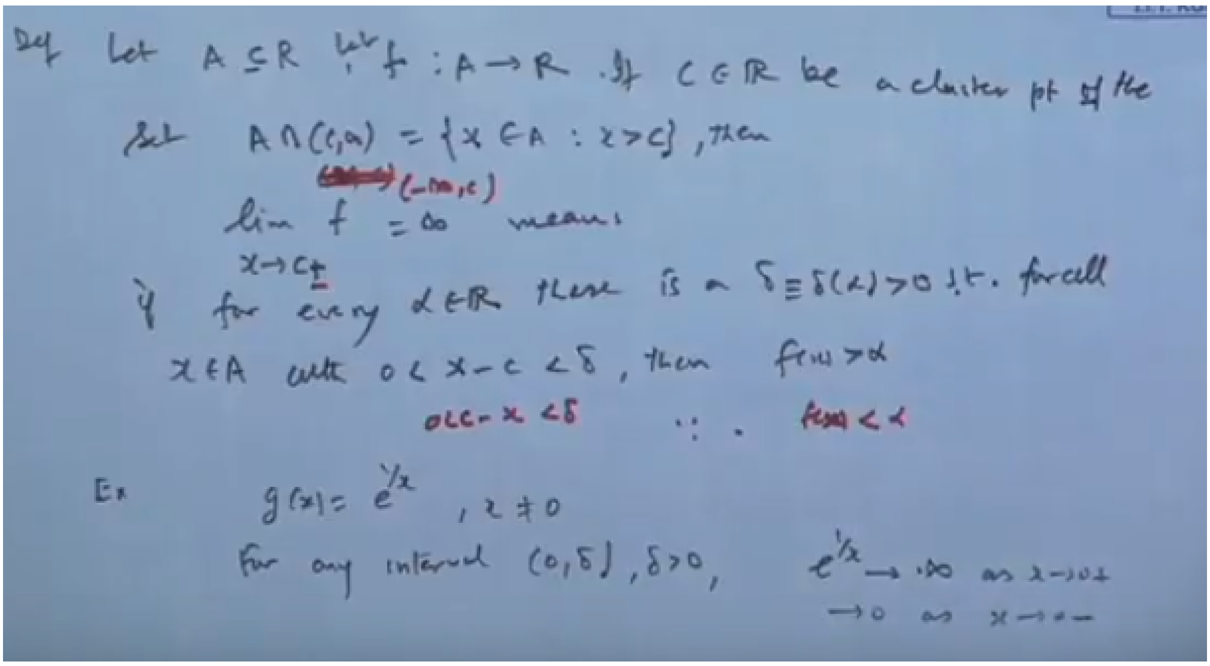

So limit at infinity, we can say the limit at infinity, concept, limit at infinity, limits at infinity.

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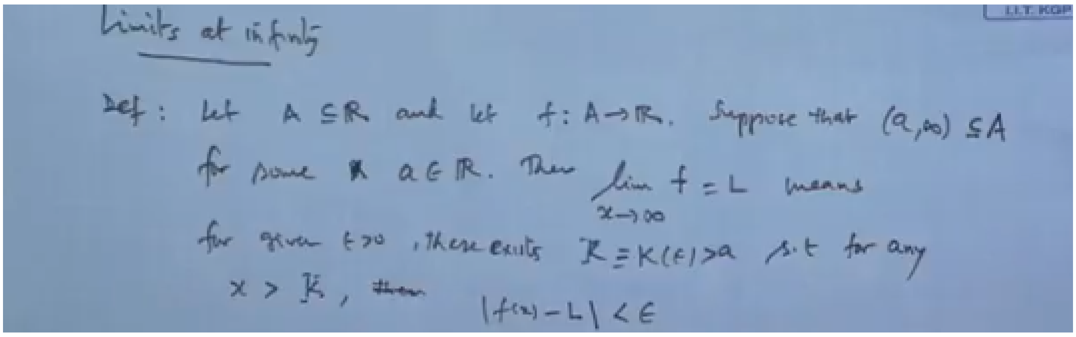

When we have the limiting point C, is infinity, we define as follows; Let a be a subset of a non-empty subset of R, and let F is a mapping from a to R, suppose that, suppose that, a infinity is contained in a, for some a, for some a, for some a, belongs to R. Suppose this interval, then we say, then, the limit of this F, when X tends to infinity L means, means, that for a given epsilon, means, for given epsilon greater than 0, there exist there exist a K, which depends on epsilon and for greater than a number, this K, greater than a. Such that, for any, for any X greater than K, any X greater than K, the f of FX, minus L, if for given epsilon is a there is a such that, for any, the thing, f of X, minus L, is less than epsilon. Then we say the limit of this function f, when X tends to infinity is L. So, when X is sufficiently large, means, at the Infinity we are touching, the finding the behaviour of the function, FX, function FX, what is the behaviour of the function FX, function FX, what is the behaviour of the function, at the point at infinity. This shows the limit. Okay? Then, plus infinity, minus infinity also similar way we can write. Then another result, going to, let a, which is subset of R and left F, is a mapping from a to R and suppose, and suppose, that a, infinity, is can, is a, a infinity is contained in a, for some a, belongs to R, some a. Then the following statements are equivalent, then following statement are equivalent.

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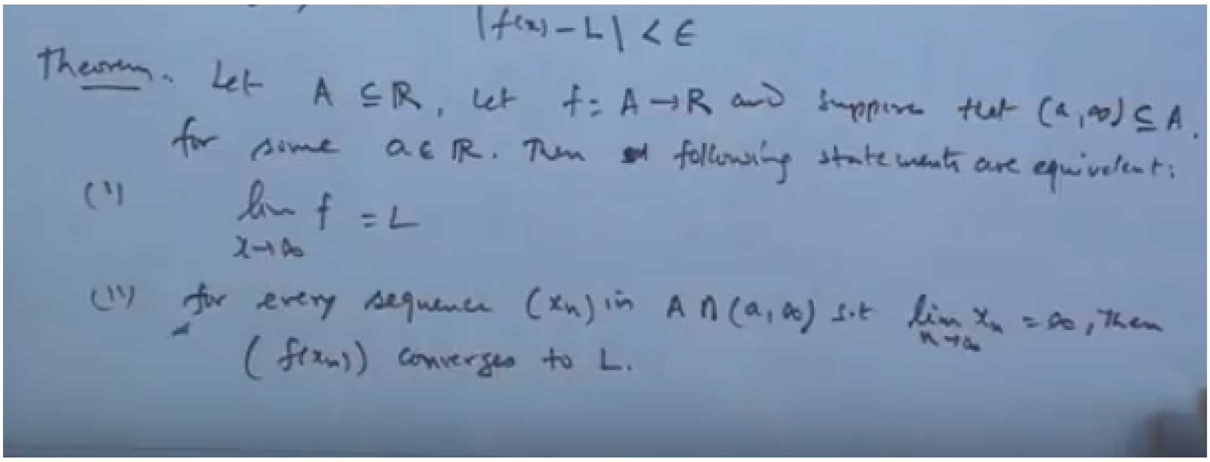

The first statement says; The limit of this function f, when X tends to infinity, is L and second statement says, that for every sequence, for every, for every sequence, xn, in a intersection, a infinity, such that, limit of xn, as n tends to infinity, is infinity. Then the sequence, then the sequence F of xn, converges, to L. So this is the equivalent definition in terms of the sequence, when we say limit F, as X tends to infinity, is L, then in terms of the epsilon delta definition we have taken, that for a given F, epsilon, we can identify a K, such that, when all X greater than K, the difference FX minus L, will be small. And for the sequence if in terms of the sequence we can say, there exists a sequence xn, which are, which are, which are tending to infinity and then the corresponding functional values, will converge to L, close to L, so that is the equivalent definition for this.

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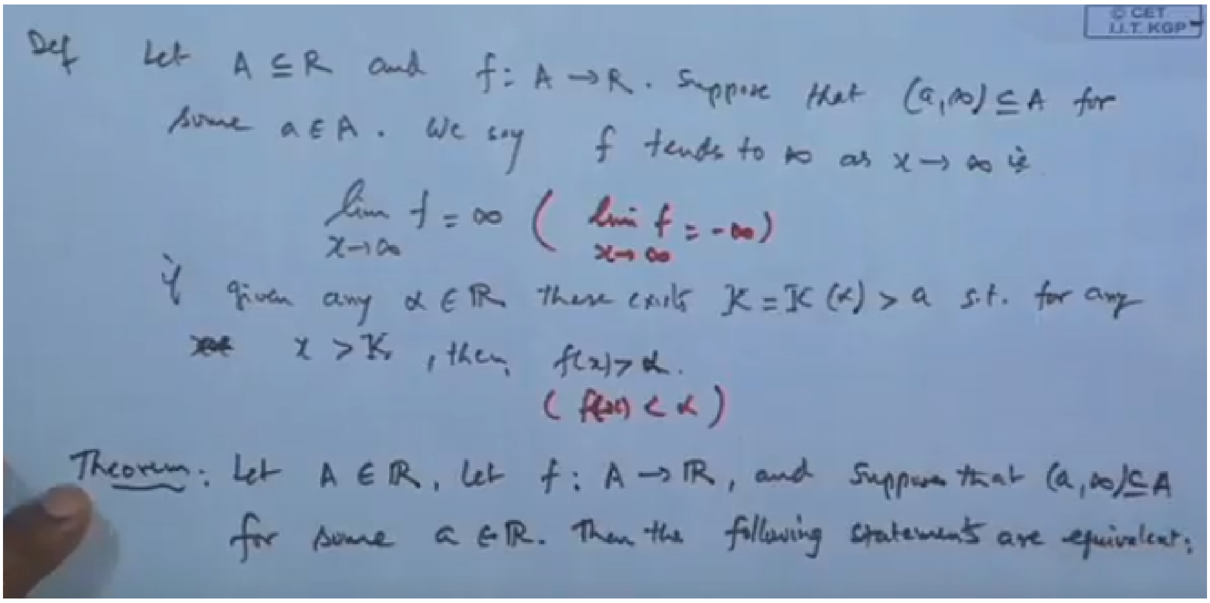

Okay? Okay, now if when both the limits, plus in, C is also infinity, a is also infinity, then we define the concept as follows; Let a, which is a subset of R and let F, from a to R, suppose that, suppose, suppose that, a infinity, this is contained in a, for some a, belongs to a, for some a belongs to a, then we say F tends to infinity, as X tends to infinity, we say, F tends to infinity, as X tends to infinity, that is, we write like this, limit of F, when X tends to infinity, is infinity, if given any alpha, belongs to R, there exist, a K, depends on alpha, greater than a, such that, for any, X, belongs to X greater than K, for any X greater than K, then f of X, will be greater than alpha. Okay? Greater than, alpha. So this is the concern, when both the, means, C is also infinity and L is also infinity. Similarly, we say, similarly if we take, limit of F, as X tends to, say infinity, is minus infinity, X tends to infinity, is minus infinity, it means, as X tends to infinity means, for any Alpha, there exist K, greater than alpha, alpha, K depend on, for such that K alpha, is greater than 1, then for any x greater than 1, then this, it will show, f of X, is strictly less than alpha. So, that will be that criteria, when X tends to infinity, F is minus infinity, this two. Again in the sequential form, we can say like this, so equivalent concepts, in the sequential criteria, that is in the form of theorem. Let a, belongs to R, let F, is a mapping from, a to R and suppose, that, a infinity, is contained in a, for some a, belongs to R. Then the following statements are equivalent, equivalent.

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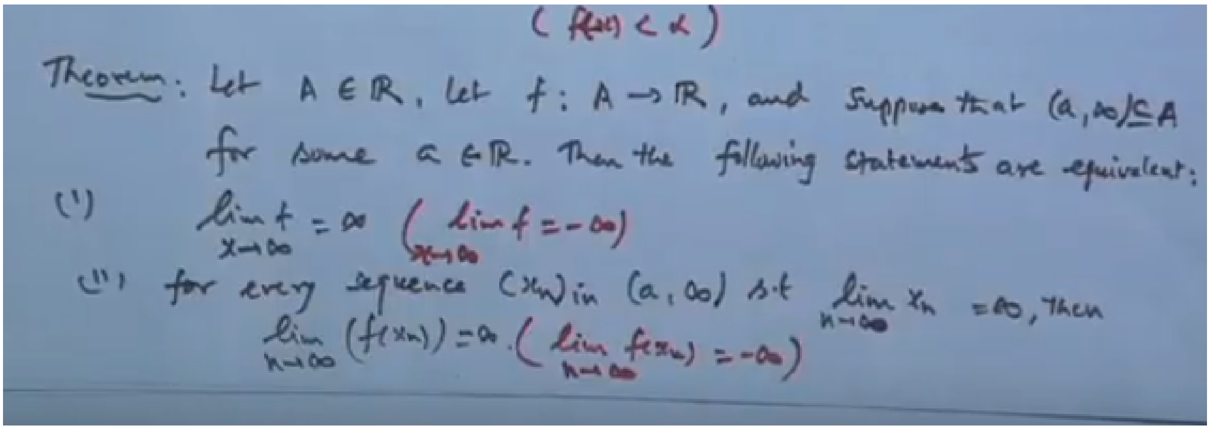

The first statement is, limit of the F, as X tends to infinity, is infinity. And second statement says, for, for every sequence, xn, in a infinity, such that, limit of that sequence xn, when any sufficiently large, is infinity, infinity. Then the limit of this, limit of, f of xn, as n tends to infinity, will be infinity, limit of this infinity. So this is the equivalent concept. If suppose we want that one, limit of this, is minus infinity, if the limit of F, when X tends to infinity, is minus infinity? Equivalently, we can say here is, for every sequence xn in a infinity, such that, limit of the xn is infinity, then limit of this F of xn, will be minus. Then here, limit of the F of xn, as n tends to infinity, will be minus infinity. This is the equivalent way, equivalently to or you can say the same, so concept is this one.

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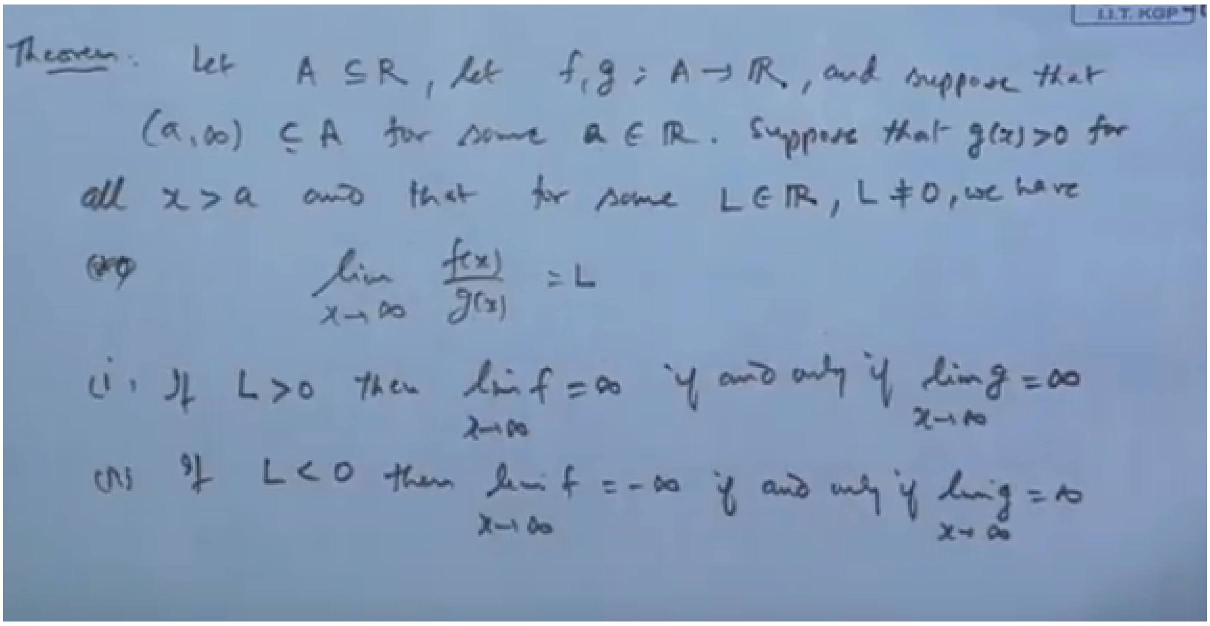

One more results, which will help in getting the sometimes limits. Let a, which is subset of R and let F G, are the mappings, from a to R, a to R. And suppose, and suppose that, a infinity, open interval, a infinity is contained in a, for some, a belongs to R, for some a belongs to R. Suppose that G X, is always positive, for all X, greater than a. So these are the restriction we put it. And that, for some L, and that for some L, which is in R, L is not equal to zero, we have we have one, we have, first is, limit of this FX, by GX, when X tends to infinity, is say L, this can. Then, depending on L, we can identify the limit for F and G. So first condition said; If L is positive, then, limit of F, as X tends to infinity, will be is infinity, if and only if, limit of G, as X tends to infinity, is infinity. So if al is comes out to be greater than zero, then both these function, f and G, when X is sufficiently large, will be infinity, the value will come out to infinity. And second is, second is, if L is negative, then, limit of F, if this is in, as X tends to infinity, if this is minus infinity, if and only if, if and only if, limit of G is plus infinity. That is if L is negative and limit of G is positive, then limit of F will be minus infinity, if limit of F is minus infinity, then limit of G will be infinity, if n is negative. But if L is positive, then if F is limit is infinity, G will also have a limit infinity and similarly visa versa, G has limit infinity, L will also have a limit infinity.

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(i) If $L > 0$ then $\lim_{x \to \infty} f = \infty$ if and only if $\lim_{x \to \infty} g = \infty$

(ii) If $L < 0$ then $\lim_{x \to \infty} f = -\infty$ if and only if $\lim_{x \to \infty} g = \infty$

Pf (i) Given $L > 0$. Let $a_1 > a$ s.t. (~~Choose $\epsilon = \frac{L}{2}$~~

$$0 < \frac{1}{2}L < \frac{f(x)}{g(x)} < \frac{3}{2}L$$

The proof of this, sees the proof for this, is very, is not difficult. Say first, given that, L is positive. Okay? So once L is positive, then, an a, belongs to this. So let us take a 1, which is greater than a, belonging to the interval, 0, a to infinity, this interval, which is. Such that, FX by GX, because this limit is L, so if I choose, the epsilon as L by 2, choose epsilon as L by 2. Then we can write this thing as, FX by L, is lying between, 3 by 2L, greater than, because F and G both are, giving to be a positive, G is also positive. Okay? So if this we can write it, FX and GX, is greater than half L, half L and when X is sufficiently large, which is positive. So we can, since L is, we can, suppose, there exist a1, such that, this condition hold. Okay? By, this is our. Okay? Such that, this condition hold, therefore, what will be, so multiply by GX, because G is given to be positive, so multiplied by G X,

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$$\left(\frac{1}{2}L\right) g(x) < f(x) < \left(\frac{3}{2}L\right) g(x) \quad \text{for all } x > a_1$$

Result follows

Ex 1 $\quad f(x) = x^n$

$$\lim_{x \to \infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & n \text{ is odd} \end{cases}$$

2. $\quad p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$$\lim_{x \to \infty} p(x) = \infty \quad \text{if } a_n > 0$$
$$= -\infty \quad \text{if } a_n < 0$$

we say, here is, multiplied by G X, we get half L, into G X, will be less than F X, will be less than 3 by 2, L into GX, GX. And this is true, for all X, greater than a 1. So obviously, this will be true for. Now take the limit. What is now, changes? If, f is infinity, then G is infinity, so here, if F is infinity, G will be infinity, if G is in, sorry, if G is infinity, F will be infinity, if F is infinity, from here G will be infinity. So follows, so result follows. Okay? So this will be the.

Now let us see some examples based on. We have X to the power n, n, another form, we have already discussed. That if function, FX is, X to the power n, then limit of this, function, FX, that is, X to the power n, when n tends to infinity, the limit of this, sorry, when X tends to infinity, when X tends to plus infinity or X tends to minus infinity, that will. So if I take the X tends to minus infinity, then when n is even number, this will go to, plus infinity, as, this will go infinity, if n is even. Because X is tending to minus infinity, but n is even, so it will. And when, n is odd, then, this will go to minus infinity, so we can get this, in this condition. Similarly, the polynomial, px is given, say AN, X to the power n, a n minus 1, X n minus 1, up to say a naught. Then of this P X, when X tends to plus infinity, this will be equal to infinity, if an's are positive, an's are positive. Okay? An's are positive and PN will be, negative minus infinity, if an's to be negative. So this can be proved like this. Okay? So in case of fraction also we can do it this way.
Thank you.