Model 6

Lecture 41

Left and Right Hand Limits for Functions

Okay. So the last lecture we have discussed the limit of the function FX when X tends to C.

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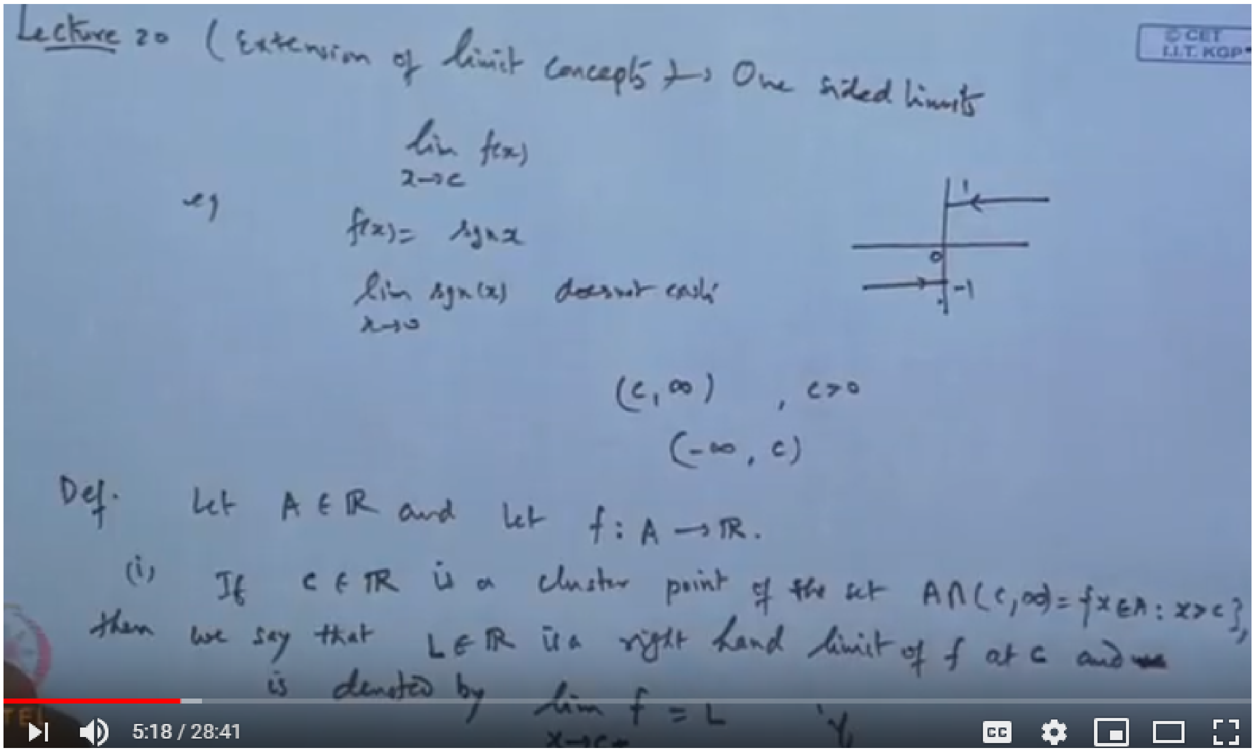

And in that case we have seen many examples, where the limit of the function FX, does not exist. And the reason for this, either the function is not defined at the point C. Means it goes to infinity or minus infinity. Or maybe it has a different limit, along a different path. Means when we take the right hand side, the limit comes out to be different, than the left hand side limit. Okay? Then we say the function is does not have a limit. For example, if you take the function FX, as the Signum of X, we have seen the limit of this function FX, Signum of X, when X tends to 0, does not exist because the reason is very simple. When you take the left hand side, the limit comes out to minus 1, right hand side, the reversed approach to plus 1 and at the point 0, it is 0. But however if we consider only the set a, over which the limit is taken, which is in either in the right hand side of this 0 or maybe the left hand side of the 0. Say if I take the C greater than 0, means C infinity interval if I take and then find the limit of the function FX, over the interval, C infinity, where the C is positive.

Then, the limit of this function right hand live exists. Because there no score for going below of this. Okay? Similarly when you take minus infinity, Say, C to minus infinity, then also the left hand limit exists. Is it not? so in such a case though the limit does not exist. But we can say partially, partially, the limit of the function exists, when you approach either, on the left-hand side or from the right-hand side, individually, limit exists and maybe different, that is a different matter. Now in such a case, when we define the limit of the function, we call it that thing, at the left-hand limit, when you approach the point, from the right-hand side, approach the point, from the right-hand Side, then it is called the right-hand limit, when it goes from the left-hand side, with the left hand, this one, goes to here, left hand and right hand limit and then we see, the if both the limits coincide, then only we can say the Function has a limit in general. Otherwise we can also introduce the concept of, we will introduce the concept of, left hand and right hand limit, which we can say, is an extension of the limit concepts. So that is also called, one side Limits that is one side limit, one side rate limits.

Okay? So let' us this see first. Let A, belongs to R and let F, is a mapping from a to R, then one, if F is in C, if C is in R, if C is in R,t is a crystal point, of the set a, of the set a intersection C Infinity. That is, the set of those point X, where belongs to a, such that, X is strictly greater than C; X is strictly greater than C, then strictly greater than C. Then we say that L, a real number, belongs to R. L blocks to R, is a right hand Limit, right hand limit, of the function f, at the point C and we write and we write, which is denoted as or and is denoted Y, limit F, X tends to C +, X tends to C plus, is L.

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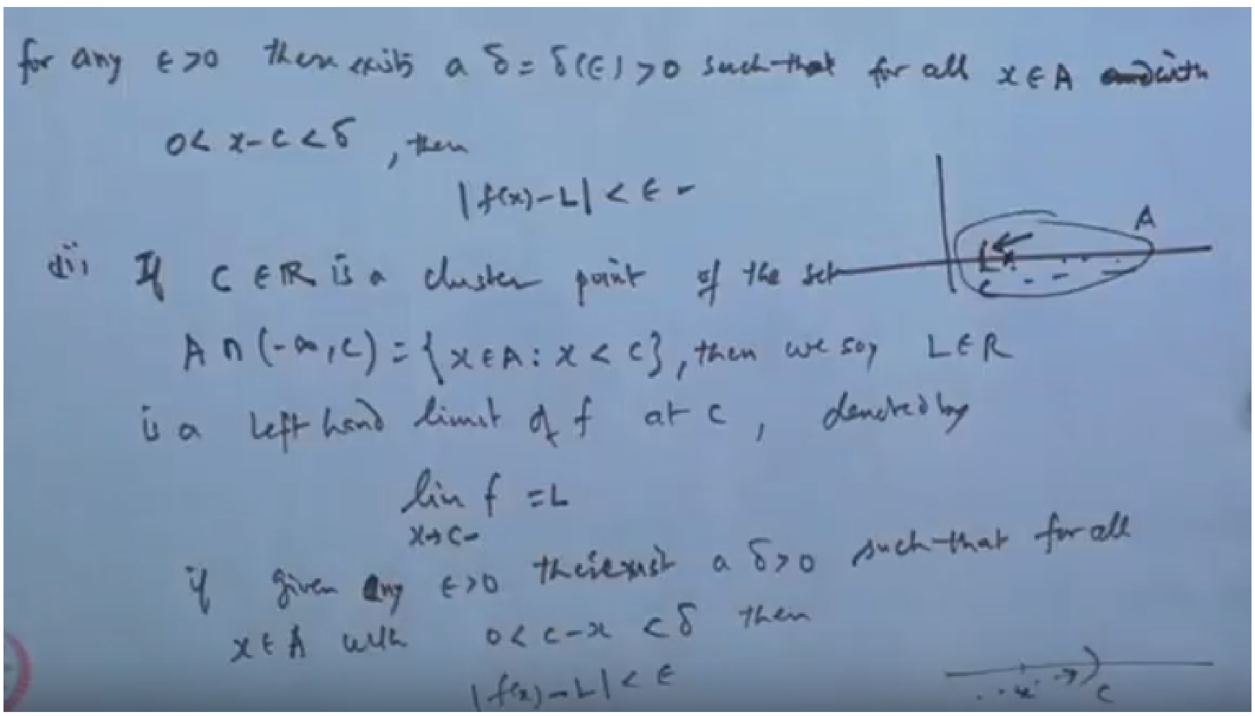

If, if, if for given Epsilon, greater than zero, for any, for if, for any, epsilon greater than 0, there exists a delta, there exists a delta, which depends on epsilon, greater than 0, such that, for all, for all, X belongs to A and X satisfied this can end with, so and, 0 is less than X minus C, less than Delta, that X belongs to n and as well as this condition. Then the mode of FX, minus L, is less than epsilon. So for all X will, with you can write with, 0 is less than, then this one. So this is called the left hand limit. So here this function, is there, this is our interval C infinity, C is this point and then infinity here, so we can choose here, and then a intersection, a is something, if we take the a intersection C. Say A is suppose somewhere here, this is our a. Then a intersection C will be this set. So if you picked up the point X here, which are greater than C and if the functional value FX minus L, lies between this interval, then we say, that, the limit of the function FX, from right hand side exist and equal to L. That is, we are approaching to see, from the right hand side, to see. Okay? Then second, be defined, and the if C, if C is a point in R, be a Cluster point or is a cluster point, of the set, a intersection, minus infinity to C. It means this is the set of those point X, belongs to a, where X is strictly, less than C. Then, then, we say L belongs to R, is a left hand limit, left hand limit of F, at C, denoted by, limit F, when X tends to C minus is L, if given any epsilon greater than 0, epsilon greater than 0, there exists, there exists, there exists a Delta greater than 0, such that, for all X, belongs to a, with, with condition is zero less than C, minus X, less than Delta. Then we have mode of FX, minus L, is less than Epsilon. So this is our right hand limit, left hand limit of this. So we are approaching the C, from the left hand side, that is, this is the Interval C. And the approach is from this side. So when you take the X point here, which approach to C from the left hand Side, then, and if this condition holds, then we say, the function has a limit, left hand limit, at the point C. Okay? So the left hand limit and right hand limit, this way we define. And we have seen the example, that if it looks the function sine X, the limit of this, left hand limit comes out to be.

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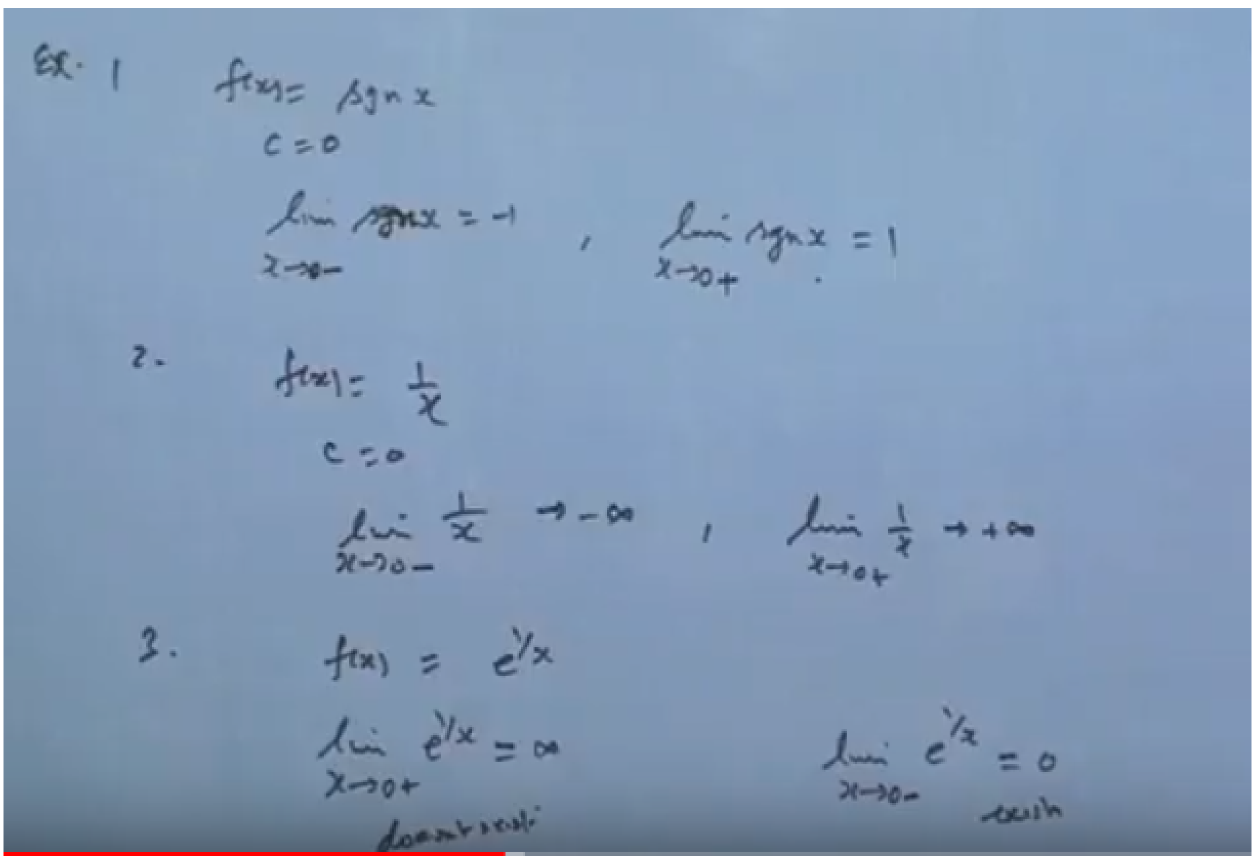

So example is. If we take the function FX equal to Signum of X and the C is a point 0, then we say, the limit of the function FX, that is Signum of X, when X tends to 0 minus, is minus 1, from the left hand side, while the limit of this Signum of X, Signum sgn, Signum of x, Signum of X, as X X tends to, 0 plus, 0 plus, this is first one. So this is the right hand limit, this is the left hand Limit, for this one, okay. Now when we introduce the concept of the left hand or the right hand limit, we look the set a and Intersection, with the C, infinity if the right hand limit or intersection with minus in infinity to see if the left hand limit and then if the limit comes out satisfy this condition, then we say limited. However it is not necessary, that always we will have a, either left hand limit or right hand limit or may be both. It may so happen, that we may not get there neither left-hand limit, nor the right-hand limit, hence limit will not exist. Or maybe sometimes, we get the left-hand limit t, but not the right-hand limit and vice versa. Or otherwise, sometimes we can get both the limits, but the values are different. So this is the case when both the limit exists, but they are not equal, but they are not equal. The second example if I look, the function FX, which is 1 by X, then what we says, when C is a point zero, then we say the limit of this function X, 1 by X, when X tends to 0 Minus, from the zero, it goes to minus Infinity. And, limit of this, 1 by X, when X tends to 0 plus, it goes to plus infinity. So, limit does not exist, here. Of course and they are different, they divert. Third case is. If we look the function FX equal to, say, to the power 1 by X, suppose if I take this function, e to the power 1 by X and then when we take the limit of this function, e to the power 1 by X, when X tends to 0 + or limit of the e to the Power 1 by X, when X tends to 0 minus, what we will see here, that when X tends to 0 plus, the limit will be infinity, where in this case, the limit comes out to be 0. So left hand limit exists, but the right hand limit, does not exist. Comes out to be finite, is infinity. Okay? The reason of this is, is simple. Why it is so?

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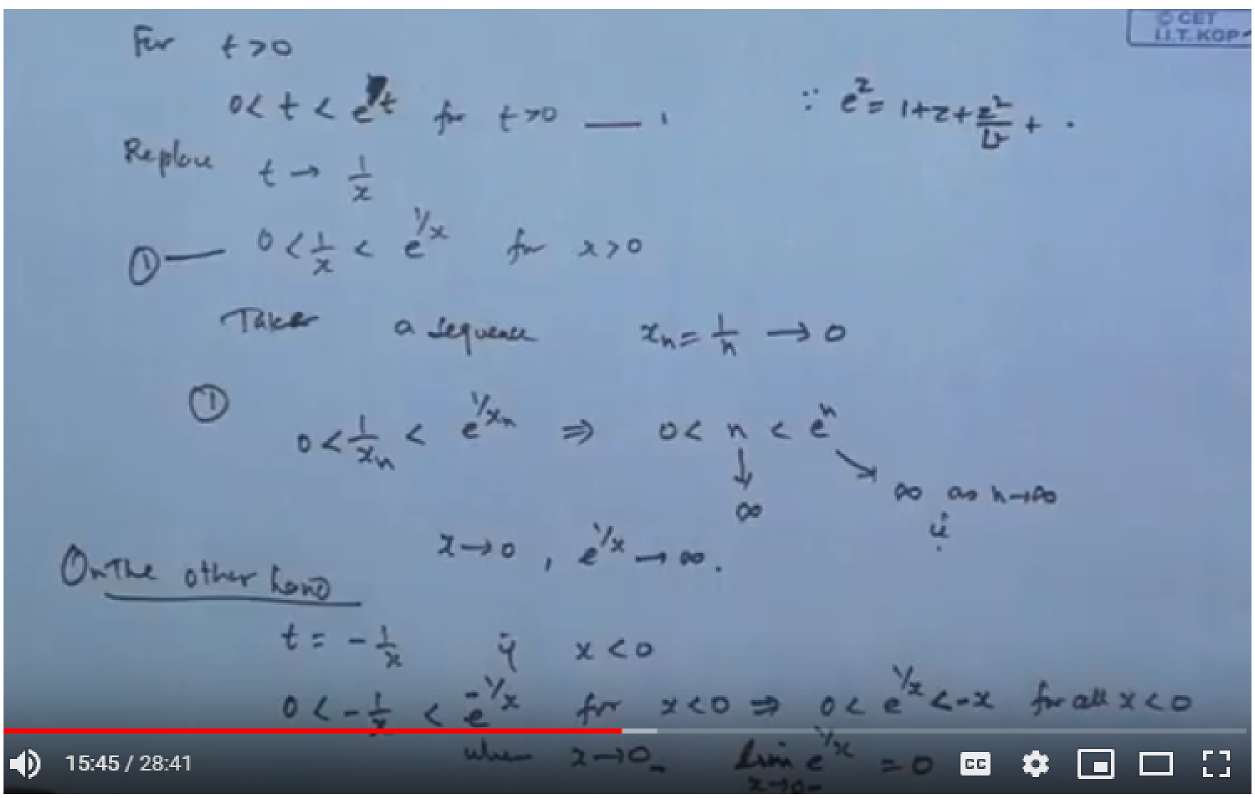

Because if we looked at any T, for any T, greater than 0, we have, 0 less than T, less than e to the power one by T, because the expansion of E, because e to the power Z or X is 1 plus Z, plus Z square, by factorial 2 and so on. So obviously, when you write this, Sorry, this is, not, this is T, I am sorry, E, zero is less than equal to t, less than T and this is a true for all T, greater than zero. So when you write T replaces, T by, 1 by X. So what we get is, 0, less than 1 by x, less than e, to the power 1 by X, for X greater than 0. Clear? for X greater than 0. Now this if we take a sequence, take a sequence xn, say 1 by N, which goes to 0, so this xn goes to 0. So what about this one? From here equation, from 1, what we get ? The corresponding sequence, 1 by xn, which is less than e to the power 1 by xn, greater than 0, will imply that 0 less than n, less than e to the power n, and since this tends to infinity, therefore this limit will go to infinity, as n tends to infinity. That is when X tends to 0, e to the power 1 by X, will go to infinity. Okay? Now on the other hand, on the other hand, if we replace in this expression, if you replace at t by, say minus 1 by X, T by minus 1 by X, if X is negative.

Then minus 1 by X, of course is positive, over here. Then we get from here is 0 less than minus 1 by x, less than e to the power, minus 1 by X, valid for X to be negative. But this implies, just a manipulation and this will give you the result, 0, less than e to the power 1 by X, which is less than minus X, for all X, negative. So when X tends to 0, when X tends to, sorry, yeah, when X tends to 0, then what happens this? Then, from the left hand side, this will be 0, so this limit comes out to be 0, because can. So this implies limit of this, e to the power 1 by X, when X tends to 0, from negative side is 0. Therefore this limit is done. So we have seen that three example. One is when the left-hand limit and like right hand limit, both exist, but they are different. Second case, when none of the limit lies, the left hand limit, nor right hand limit exists, it comes out to be infinity or minus infinity. Well in the third case, we have only the left hand limit exists, right here limit does not exist. So the concept of the left hand, right hand limit, basically depends on, the existence of the left or right hand limit depends on the function. Okay? And when both these limit coincide, then we say, the function has a limit at the point X is 0. So that is what. Okay? now the equivalent definition, in terms of the sequences.

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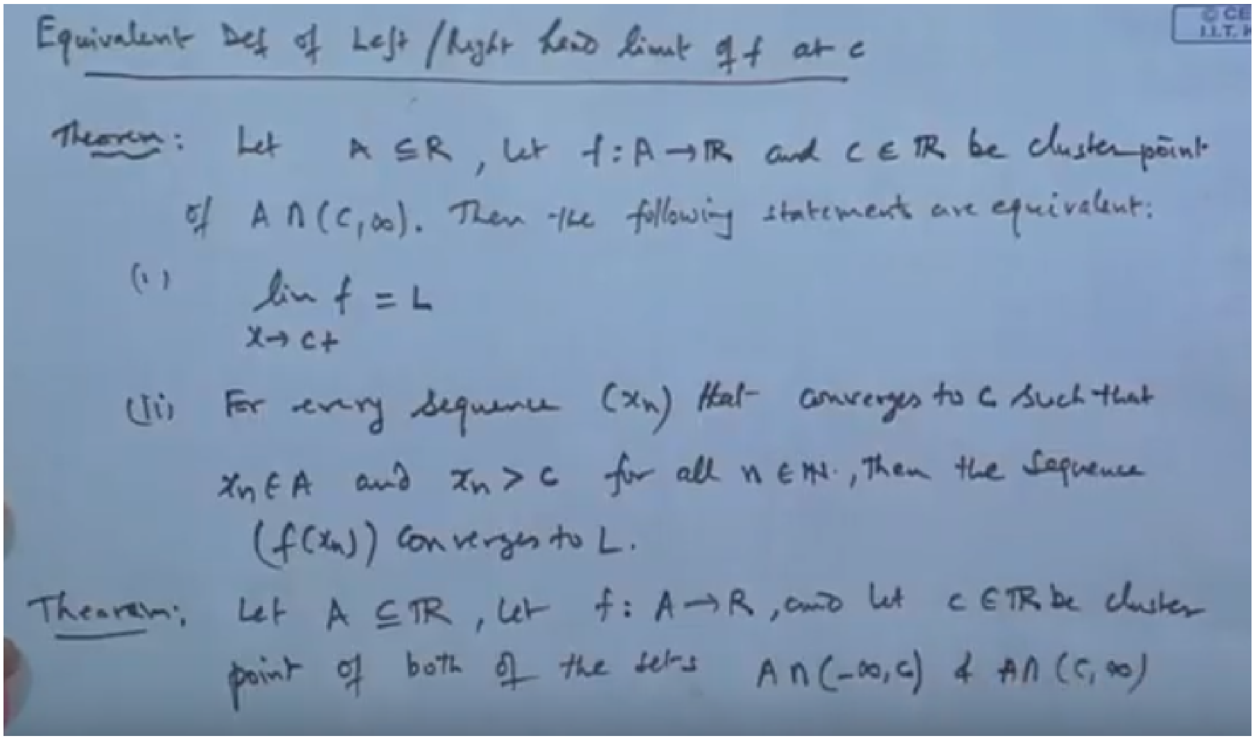

So this is the equivalence definition, equivalent definition, of left and right limit, left, oblique right hand Limit, of the function FX at C. This is in the form of theorem. Let A is a subset of R, let a be a subset of R and let F is a mapping from, a to R and let C, is an element of r, be a cluster point, be a cluster point of, a intersection, C Infinity, then the following statements are equivalent, equivalent. The first statement says, limit of the function f, when X tends to C plus, that the right-hand limit of f is suppose L, then it is equivalent to the second statement, that for every, sequence xn, that converges to C, such that, xn is in a and always xn, is greater than C, strictly. Means always towards the right-hand side of the C, for all n, for all n, belongs to Capital N. And then the sequence of its functional value, that is f of Xn. This sequence we converge, to L, converges to L. If this happen, then we say the right-hand limit of the function f, is L. So this is the equivalent definition, in terms of the sequences. Similarly, similar we can write it for the left-hand limit, if the limit of the function, f from left hand side exists means, there exist is you can xn converges, such that, xn is strictly less than C, for all N and then if xn converges to L. In a similar way, we can write it. Okay? Now, next with an interesting result, which we I taught earlier also the relation between the, this extension of the limit and the limits. What theorem says is, let a is a subset of R, nonempty subset of r and let f is a mapping a to R and let C be a cluster point, cluster point, of both of these sets, both of these sets. That is a, intersection minus infinity C as well as a, intersection C infinity, these sets.

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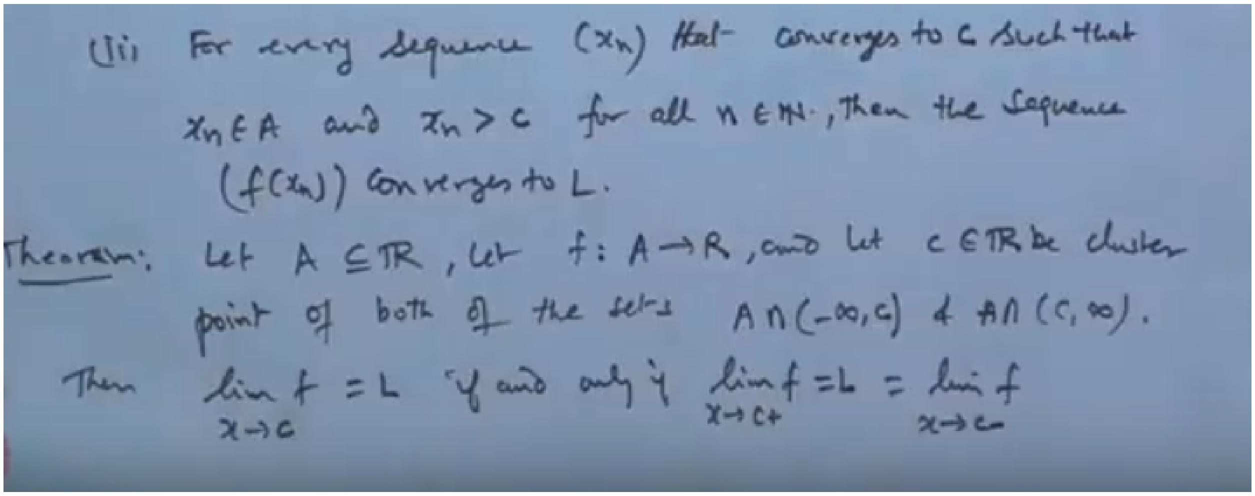

Then, then limit of the function f, as X tends to C, exist and equal to L, if and only if, if and only if, limit of f, when X tends to C, means right and limit of f is L, this is the same as the left-hand limit of the function, f at C. If both the limits coincide and equal to L, then we say the limit of the function exists and equal to L. Okay? So this will be the definition for. Okay. Then we, come to infinite. Now here, so far we have taken, only the concept of the limit when C is finite, limiting value is finite. L is also finite. So now we will take in the case, when limiting value is Finite, but limit comes out to be infinity. So what will be the form of the definition, when the limit L comes out to be infinity or minus infinity? Okay?

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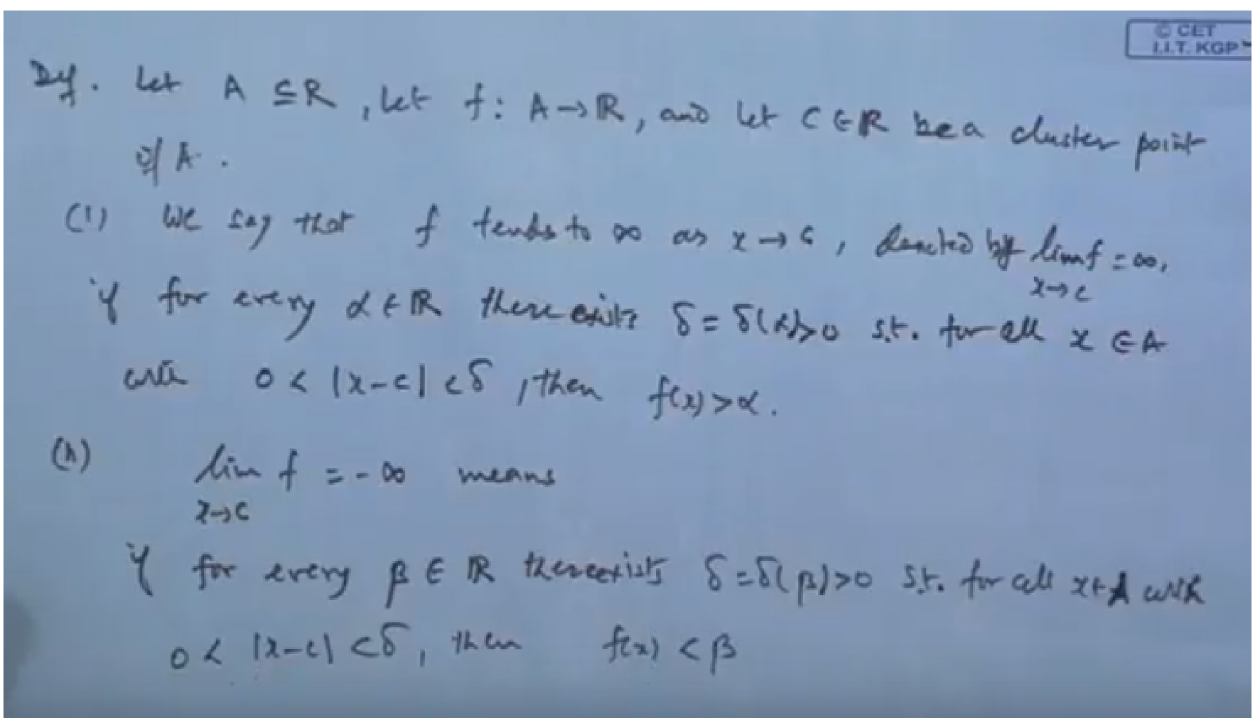

Now this is definition; Let A which is a subset of R and let F mapping from a to R and let C belongs to R, be a cluster point, be a cluster point of a, cluster point of a. Then we say F tends to, we say that F, tends to, infinity, as X tends to C, denoted by limit F, when X tends to C, is infinity, if, for every, if for every Alpha, belongs to R, there exist, there exist, a delta, which depends on Alpha, Positive, such that, for all X, belongs to A, with the condition, with, because it is a lying between, from both side, with X minus C, 0 is less than mode X minus C, less than Delta, in this neighbourhood, either from left hand side of C or right hand side from C and within the neighbourhood of the Delta, C neighbourhood of, Delta neighbourhood of C. Then, the functional value of this FX, is greater than alpha. And since alpha is arbitrary, it means, the value of the function f X, cannot be bounded, it is unbounded.

When we say the limit of the function f, when X tends to C, is infinite means, when X approaches to C, the function f is not bounded. This is you claim to say is, that if I choose a delta neighbourhood of C, then the point, in this delta neighbourhood C, will exceed, to any number, given number alpha. And this, once you decide any alpha, you can identify a delta, such that, when you choose the X, in Delta neighbourhood of C, then the corresponding value of the function at this point, will be greater than alpha. Clear? So that way we say, the limit of the function FX, when X tends to C, is infinity. Similarly we define the limit to be minus infinity, as follows; If we Say, F is the limit of the function f, when X tends to C, is minus infinity Means, means, if for every, every beta, belongs to R, there exists, there exists, a Delta, which depends on beta, positive, such that, for all X, belongs to a, with The, belongs to a, with zero is less than mod X minus C, less than Delta, we have then, f of X is less than beta. So when we say the limit is minus infinity, it means, then X is approaching to C, either from the left hand side or from the right hand side, the functional FX, is approaching to minus infinity. That is, it is unbounded towards the negative side. Then that is equalent to say is, that we, whatever the number you pick up, there will be a neighbourhood around C, Delta neighbourhood of C, such that, the value of the function will be, still lower than the that number beta. And this shows, there function FX, will go to minus Infinity, when X is sufficiently close to C.

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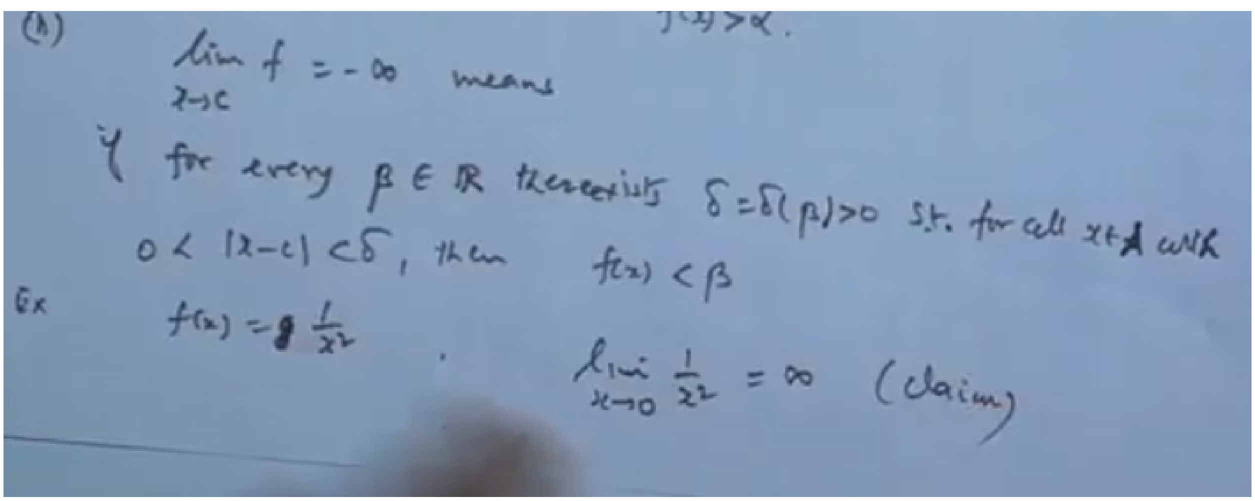

Now let us say example for. For example, the function FX, which is, which is 1 by X square, sorry, FX say, 1 by X Square, now if you look the limit, limit of this function, 1 by X square, when X tends to 0.

Now what happen? When X tends to 0, either from the right hand side or from the left hand side, both will go to 0. What, X tends to 0, so it will go to infinity. So we claim this limit is infinite. This is our claim.

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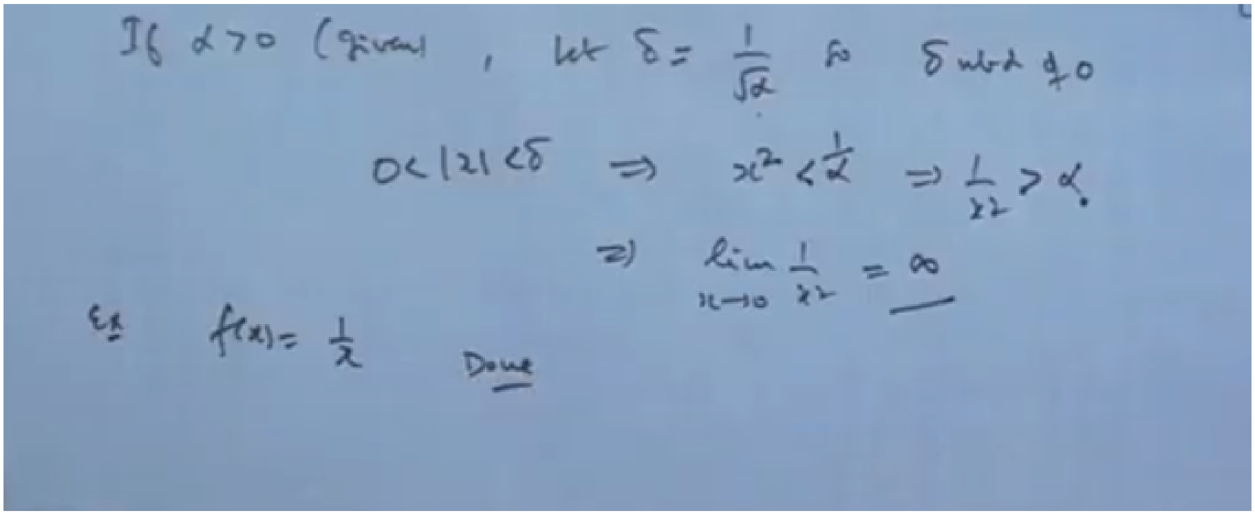

To justify it, we write like this. if alpha is greater than 0, is suppose given, and let, let us find Delta, as 1 by root alpha, so Delta depends on Alpha. Now once it has, it follows that, but the Delta neighbourhood of this, so Delta neighbourhood of 0, Delta neighbourhood of 0 means, mode of X, is less than Delta, greater than 0. Now then, this implies That, X square must be less than, the Delta is 1 by root alpha, so X square must be less than alpha and hence 1 by X Square will be greater than alpha? No, this is 1 by alpha, sorry, 1 by alpha so it is greater than alpha? It means when X is sufficiently close to 0, the value of this, can exceed to any number alpha, for a given alpha. Do a table the Alpha is arbitrary. So when you choose any alpha, we can find a neighbourhood of zero, such That, the value of this exceed by that number, it shows, the limit of this, 1 by X square, when X tends to 0, is infinity. That is what it shows, all satisfied. Another example if we take, suppose I take the, 1 by X also we have seen, that is okay. So take the, okay, 1 by T, FX equal to 1 by X, this also we have seen. The limit of this, when X tends to 0, from right hand side, is infinity, left hand side, plus infinity, so this is already done. So, need not worry Okay?