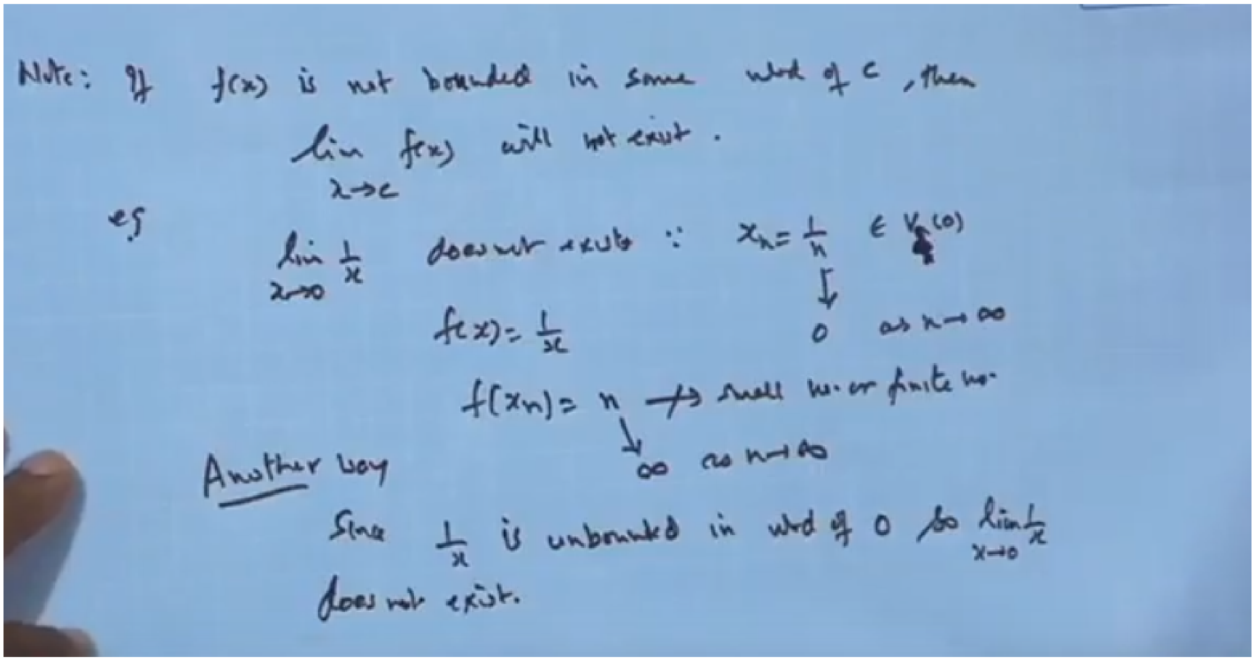
Module 6

Lecture 40

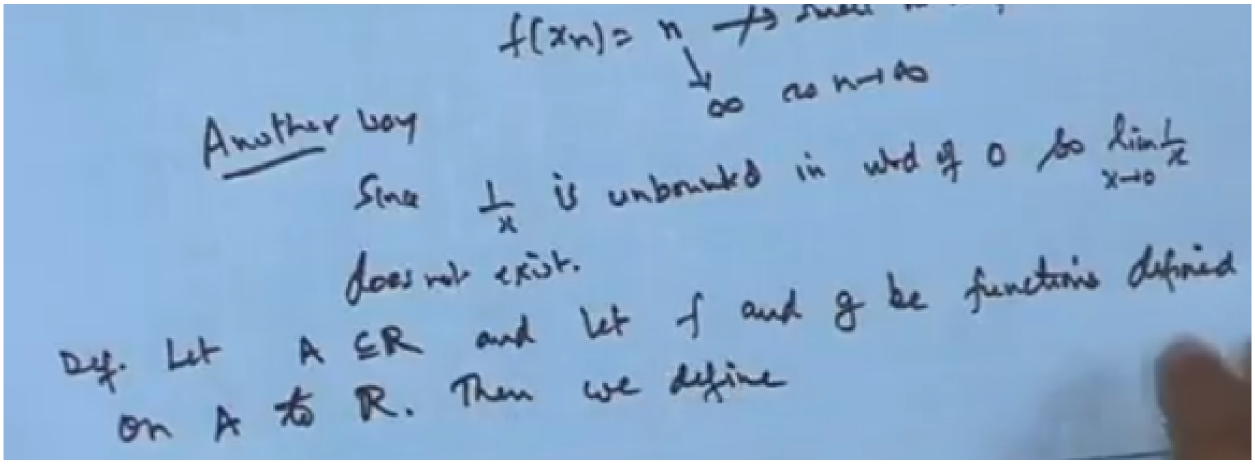
Various Properties of Limit of Functions

(Refer Slide Time: (00.17)



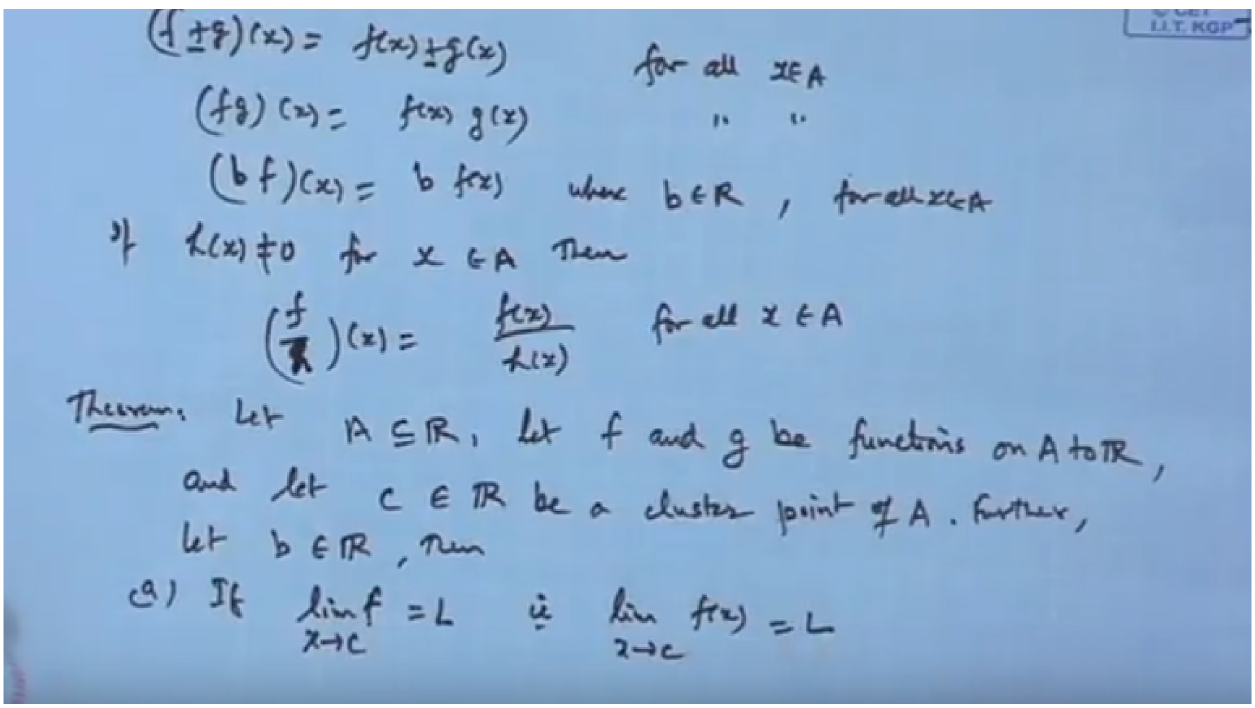
We have few results theorem, which are parallel to the theorem Sequences, just like in case of sequence, we have established some results, the sum of the limit, of the sum of the sequence, is the sum of the limits difference product and like this, similarly the similar results holds Good in case of the function, in fact, we have seen one result ,that if the limit of the function FX when X tends to C is exist, say then it can be equivalently written in terms of the sequential form, that is we can get the sequence xn which goes to C and xn is different from C, such that limit of this F of xn will go to n, so all the results which are valid the proof of all the results are exactly parallel as we have done in case of the sequences, however we can also establish that prove with the help of external data mechanism, so I am just stating the results without proof, because it goes runs h1 layer as follows and by the on the same lines as we have used in case of sequences. Okay?

(Refer Slide Time: (01.34)



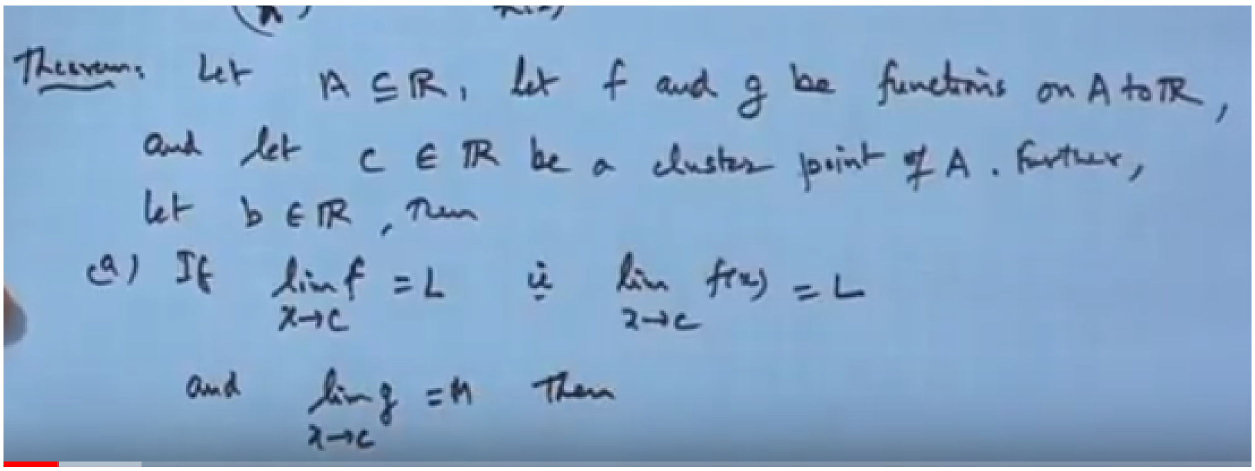
So these are our before going for the results we define of course it is a very simple obvious thing, but it still I will complete it, let A is a non-empty subset of R and let, let F and G be functions, we functions define on A defined on a to R, then the addition subtraction etcetera we define as
Follows,

(Refer Slide Time: (02.30)



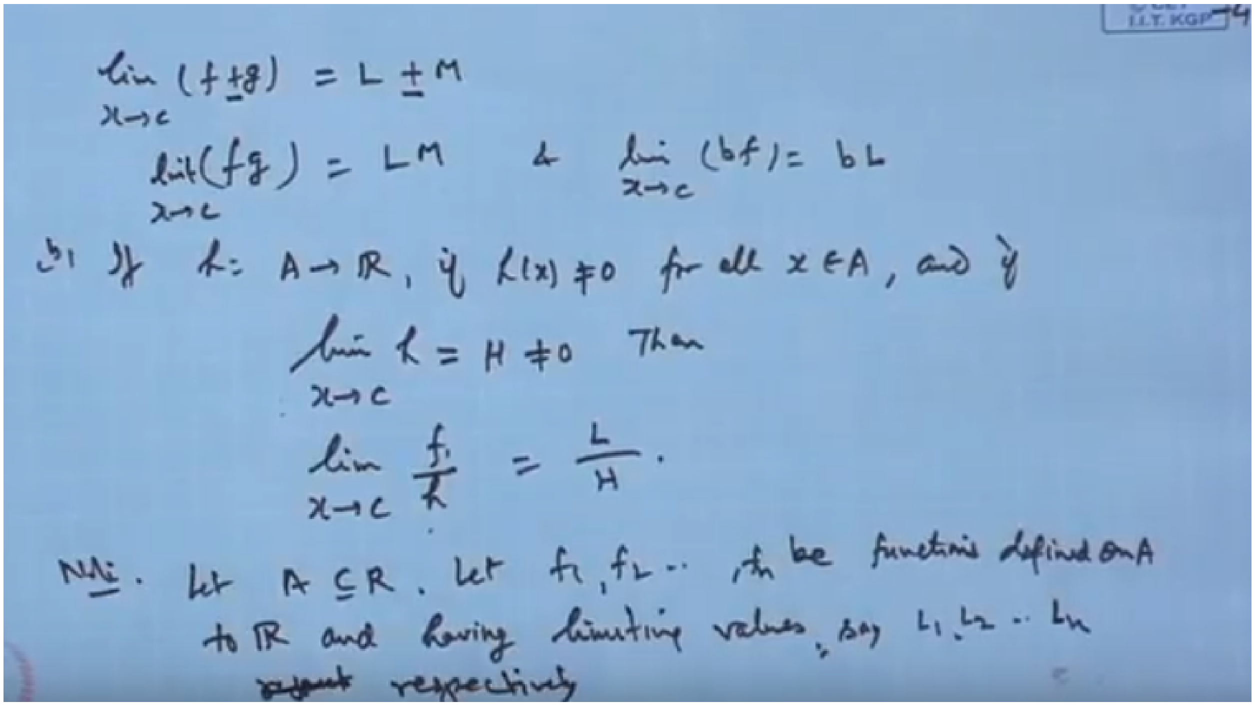
then we define the sum of the two functions F plus G X H FX plus GX, the difference of this is defined like this and the product of this is FX into GX, if B is a constant then V of F X is defined as B of FX, will be is some real number R, for some constant real number and if HX is if HX is not equal to 0 for X belonging to a then we can define the sequence quotient F by h, FY h x h FX over HX fi for all x belonging to a and this is also true for all x, belonging to a this is also true for all X, belongs to a all X belongs to A, so this way we are defining, so similar results theorems also limiting theorems can also be written like this, let A which is subset of R and let F and G be functions, functions on A to R, A to R and let C which is a point in R, be a cluster point we a cluster point of A, cluster point of A, further let B is a real number any real number R belonging to R then the following result we can say if the limit of F, when X tends to C is L meaning is that is, that is limit of FX when X tends to C is L, so I am doping the X of A,

(Refer Slide Time: (05:04)



so that limit is this and limit of G and limit of G run X tends to C is M say then,

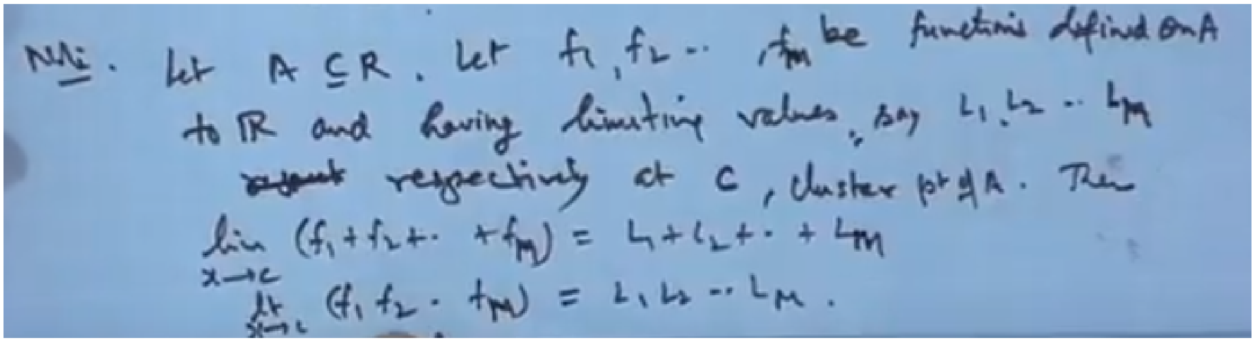
(Refer Slide Time: (05:23)



then the limit of the Sum, F plus G, X tends to C, will be L plus M, limit of the difference will be L Minus m limit of the product F G, as X tends to C is LM and limit of this, be f X tends to C is B times L and further if If, H is a mapping from A to R a function and if H is not equal to 0 H X is not
equal to 0 for all X, belonging to A and if the limit of H exists, limit of H when X tends to C is H which different from zero.

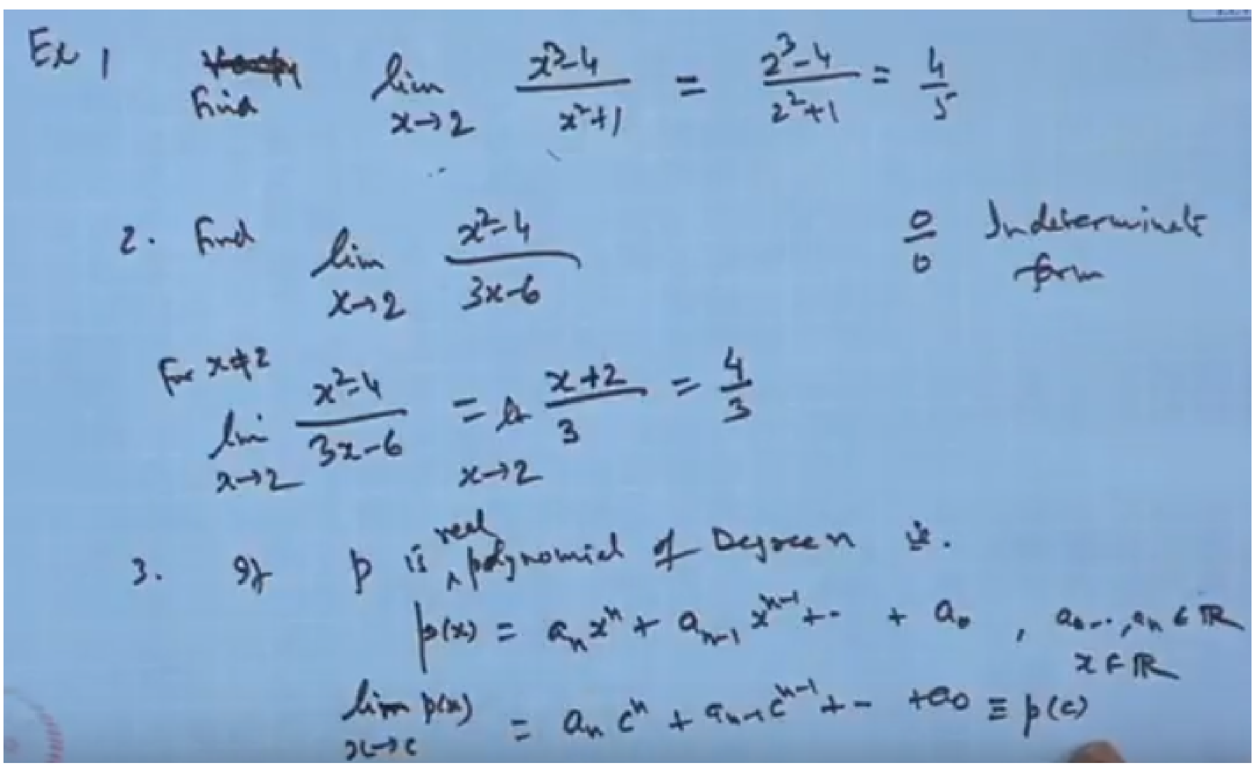
Then the limit of this ratio F over H, when X tends to C is nothing but L by H. Okay? So proof is okay, no now just go join me, now here we will make one remark, the remark is the in the Part B, we have put a restriction that limit of H X, when X tends to C should be different from zero, because if it is zero, then this limit is not defined. However, even that suppose F is also zero H is also zero, then we cannot define the limit of F over H, because that comes out to be a in determinant case the limit may or may not exist, if it is support limit or L is also zero and H is also zero, in that case the when you substitute the value or take the limit it should come out to zero by zero, you cannot say the limit is one, or limit is zero, it may be anything so all may not exist also, so what we say then whenever this limiting behavior of the function f of H is different from zero and particularly in the case of ratio, when they are different from zero H is different from zero, then only we can say the limit exists, because if H is different from zero, even L is zero, the limit will come out to be zero, but if L is also zero and H is also zero then the problem occurs, so we put this restriction on it. Okay? So that is what we may not, another note so we can say note, suppose there are these functions, which are defined on A and having let f 1 f 2 say FN be a functions, defined on a 2 are and having and see with the Crystal point of this and having the limit, limiting values say L 1, L 2, L n respectively, respectively, respectively

(Refer Slide Time: (08:54)



when at the point, limiting value at the point C, which is a cluster point, cluster point of A, then these limit of this F 1, plus F 2, plus F n, when X tends to C, will be the sum of these limits and similarly F n say, F n so here n and when it finds the product of this, again when you take the here it is I will say space, so here it should not be suppose it is M. Okay? M so here is also M, this is M, otherwise it will confuse, because limiting value is taken it's game okay, so when X is 10 okay, it's okay, then m and this is M and this comes out to be L 1, limit of this X tends to C comes out to be L 1 l 2 and as a result when all L 1 l 2 L n are same then it comes out to be X to the expects to the power n and these are all identical, in such particular case.

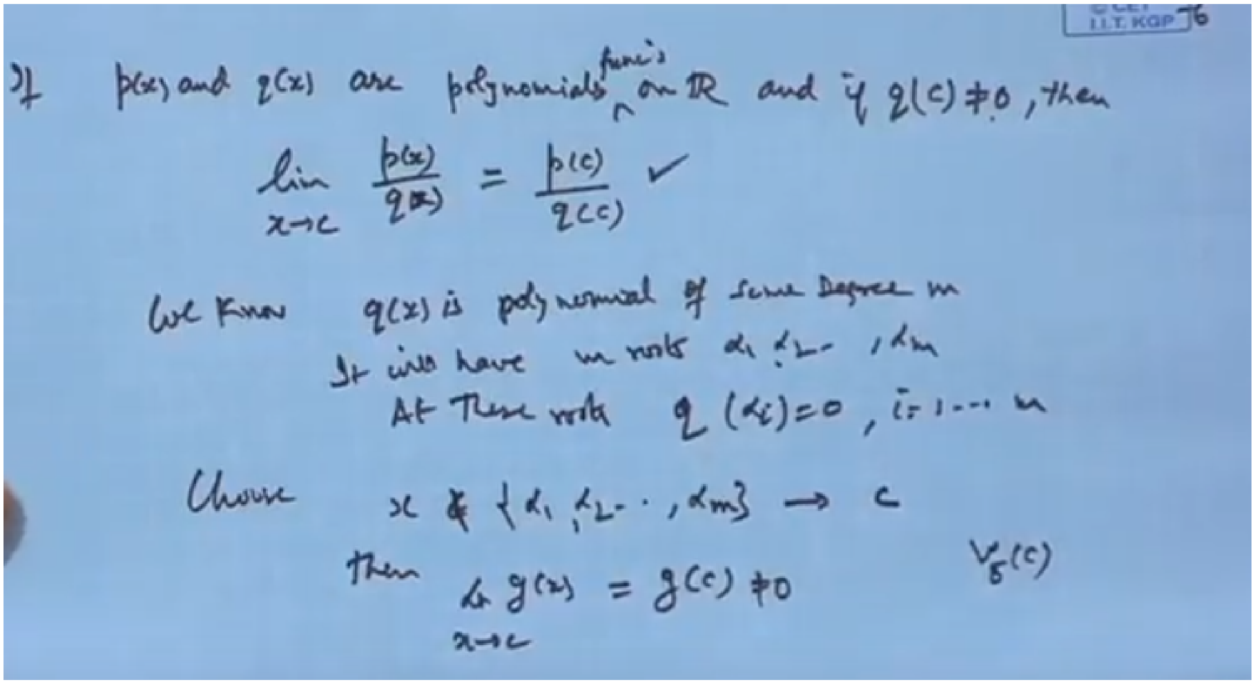
(Refer Slide Time: (10:20)



Now let's see this few example, based on these previous results and we can, suppose we wanted to so say, limit of this verify the limit of this X tends to say 2, X cubed minus 4 over X square plus 1, or find the limit or find the limit of this, now it's not as to use the epsilon-delta definition, so what we do is because it is of the form FX by HX, HX is X square plus ,1 which is different from 0 in the neighborhood of 2.Okay?

So obviously we can use the result, we can find out the limit of this function, we can find the limit of this function and then value so we're getting the limit of this function, the substitute X tends to 2 means it will go to 2 cubed minus 4 over 2 squared plus 1 and that comes out to be 4 by 5, so just substitute it, provided everything goes with doesn't if I take the limit of this, say X square minus 4 over 3x minus 6 when X tends to 2, find the limit of this? Now here when we substitute X is 2 the denominator is vanishing, ok so it will not define for that, similarly when you take X equal to 2 it the numerator is a subtenant, so basically it is coming to with a 0 over 0 form, when it's replaced X equal to 2, which is known as the indeterminate case, indeterminate form that we will discuss later when we go for the ellipses will you know that how to compute this el limit of such a ratio which comes in the form of indeterminate, but here this is a function very simple function. We can also use some trick to find out the limit of this, obviously when we say the X tends to say to the numerator is 0 denominator 0, it means there must be one factor involving X minus 2 in the numerator as well as X minus 2 in the denominator, then only numerator and denominator both are tending to 0, so remove that X minus 2 factor so if I write the function X square minus 4, 3x minus 6 for X, different from 2 then what happen you can write this thing as X plus 2 over 3, just because X minus 2 get canceled and then when you take the limit of this now, as X tends to 2 it is the same as limit X tends to 2 which comes out to be 4 by 3 and that okay, so that way we can easily find.
Now third is, if suppose P is a polynomial, if P is a polynomial, of degree n, that is PX is of the form real polynomial of degree N, so it is of the form say A N, X to the power N, A N minus 1, X N minus 1, plus a naught where a 1 a 2 a a naught a 1 a 2 n these are all real numbers, ok X belongs to R, so P X is a polynomial degree n if we are interested in computing the limit of this P X, when X tends to C then obviously it is a sum of these functions, because individual each one is a function, so when you take the limit of this as X tends to C then X to the power n will go to C to the power n, so it will come out to be the a C to the power n, plus n minus 1, C n minus 1 plus n naught, which is equal to the polynomial P at the point c, a constant P C, so limit of the pillion P X when X tends to C this.

(Refer Slide Time: (15:00)

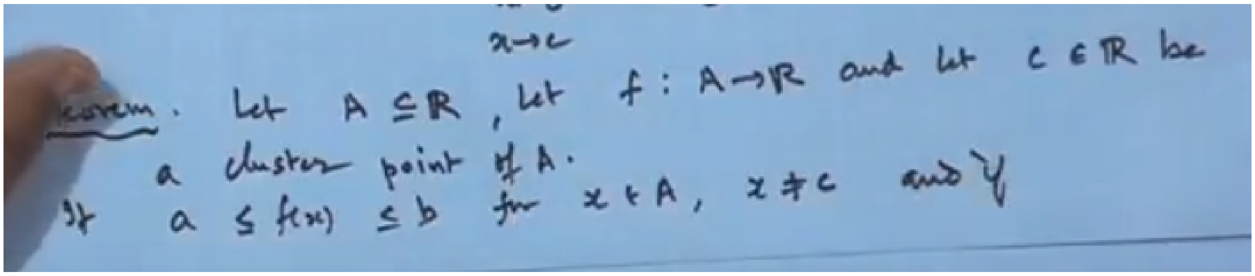


Then if suppose, we have a two polynomial, px and QX if px and QX are the two polynomial, then if P X and Q X are polynomials, polynomials or function polynomials, on our polynomial functions, functions on earth and if the polynomial at the Point C, Q polynomial at the Point C, is different from zero, has a value different from zero, then the limit of this ratio px over Q X, when X tends to C, is equal to PC over Q C. Okay?

So, what we do here is, be sorted out for the point, P X is a polynomial of any degree say N and Q X is a polynomial degree say m and it is given the Q at the Point C is different from 0, but that it does not solve our problem. Why? Because we know Q X is a polynomial, of some degree okay say degree says suppose I take say M, it means it will have at most M roots, so it will have M roots, some may be repeated some real some completely like this, it may have M roots, it means at these roots alpha 1, alpha 2, alpha M, at this point at these roots, the value of Q must be 0, I am assuming all the roots be distinct and real, say distinct and real, so Q must be 0 so it means when it approach to C then path should not contain this point, because if it can't at this point, then you cannot find the limit of Q X when X tends to C, so what we do is we separate out we take the choose X, which is not in this set alpha 1, alpha 2 ,alpha, which is different from this ok and then this X is tending to C, so obviously if I choose all the point in the neighborhood of the Delta neighborhood of C which does not involve these points.

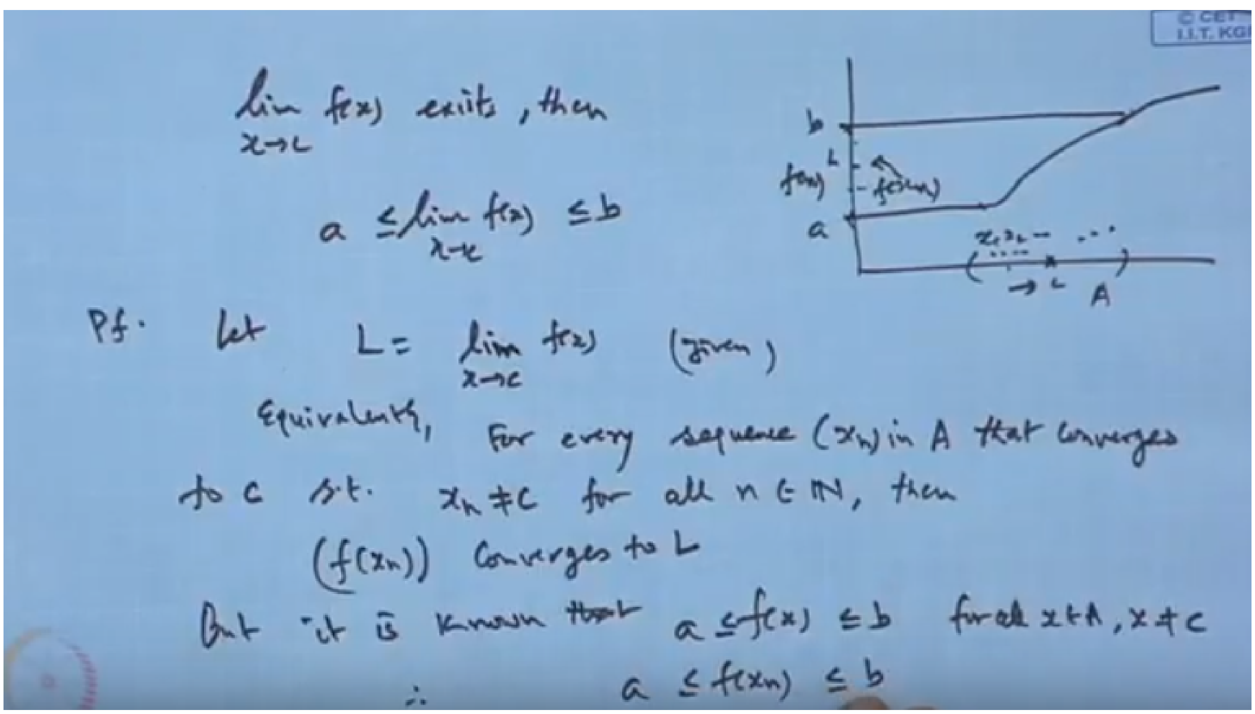
Then the limiting value ofQ X when X tends to C must recreate, then the limit of GX when X tends to C, in that case it is equal to GX and GX already given to be nonzero, so this one, then C in once it is non zero, then you can apply this result and get this, ok so when you are very careful when you find the limit of this px / QX the point should not be available we have the Q as a H, this one roots, those point in that neighborhood all the roots of the QX must be out, then only you can get the limit of this ok, so that is the one thing which we have.

(Refer Slide Time: (18:42))



Now this result is also interesting, the legend says let a which is subset of R and let F, F which is a mapping from A to R, A to R and let C belongs to R, C belongs to R, be a cluster point, the cluster point, be a cluster point of A and suppose if a is less than equal to F X, less than equal to B, means the all the images of F, in the neighborhood of the C lies between this, FX for all X belongs to a and X is not equal to C, all the point F, has a image over the set a except let's see which lies between a and B, then and if limit exists and

(Refer Slide Time: (20:12))

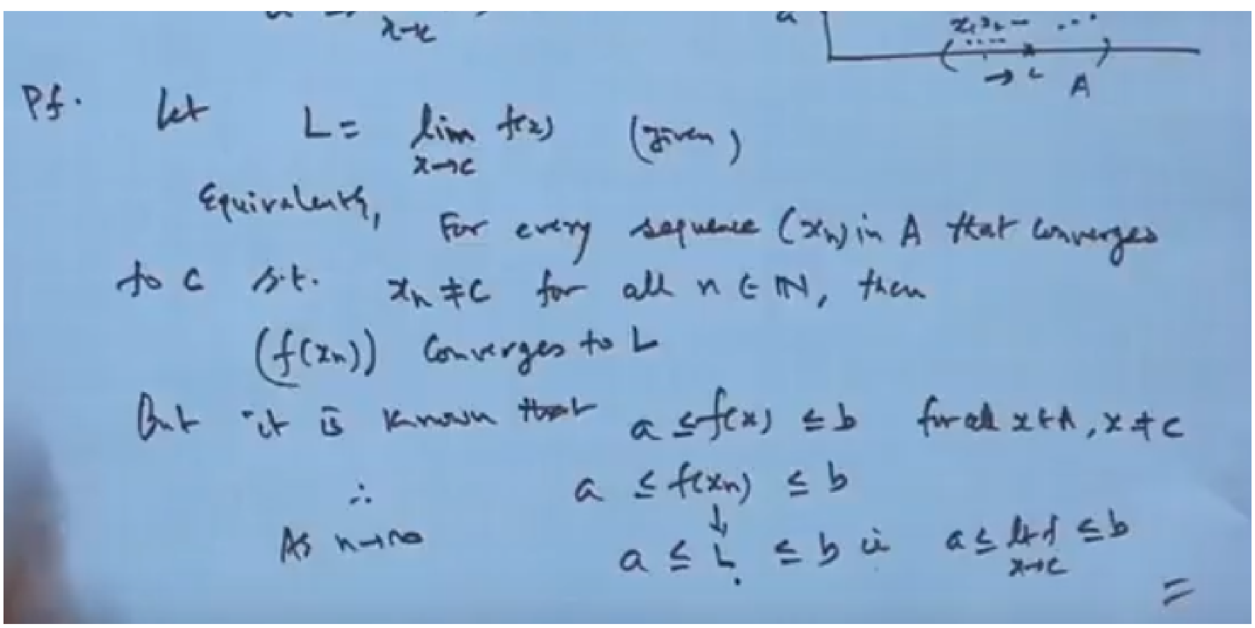


If limit of this FX, as X tends to C exists, then the limit of this FX, when X tends to C ,will lie between a and B, means the lower bound cannot be, FX limit of FX will lie between the two 1 A and B. So what it says is, if suppose we have this set a, here is some C and this is our function f, this is a function say if. Okay?

There it has a lower bound a and this is the upper bound B for this function, may be something, so what it says is that if all the images FX, lies between a and B and function is also have a limit, at the Point C, then the limiting value of this say L, will lie between a and B so proof is simply, assumed that given F is the limit let L is the limit of the function FX, when X tends to C, this is given. Okay? Because limit exists means this will be given. Okay? Now as we have seen that limiting epsilon-delta definition in sequential way Convention Bay, is also the same this means we can also define the limit of the function, by means of sequence and the sequence limit says, that this result if I go through what this result says, this is equivalently, we can say for every sequence, for every sequence xn in a, every sequence xn in a, that converges, that converges to C that converges to C, such that, xn is not equal to C, xn is not equal to C, for all n, belongs to kept, its natural number, then the sequence, of the functional value F of xn. This sequence, converges, to L. So when we say the limit of the function FX, when X tends to C is L, the equivalently we can also say, that there will be a sequence xn, in a. Means x1, x2, xn, these are available in this or from here also, which are different from C. Then the corresponding images FX 1, FX 2, F XM , this sequence will converge to L. This is way, equivalent definition of this. So using this definition we can now prove like this.

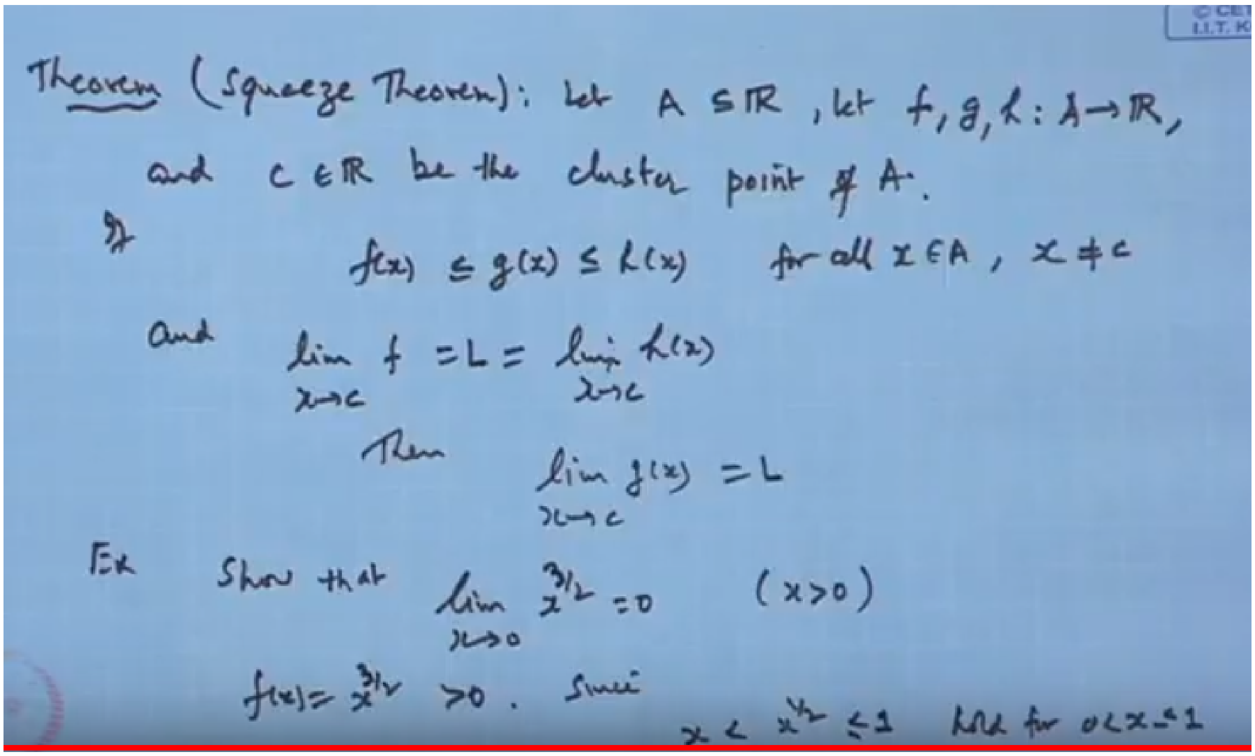
So, let us suppose, a sequence xn in exists, that converges to C, but all the point xn is different from C, then the corresponding F xn, will converge. Now what is given is, that FX will always lies between, this is given, FX always lie between a and B, but given is. It is known that, the f of X, will always lie between these two want. For all X belongs to a and X is of course not different C. Now here xn is in A, but they are different from C. So for such X n,

(Refer Slide Time: (24:13))



This value, f of xn, will also lie between a and B. Is it not? So when this two bound is there, so when you take the limit of n, as n tends to infinity, this sequence fxn, will go to L and this would want lie between NP. So limit of FXN, when n tends to infinity, below, that is the a is less than or equal to limit of f, when X tends to C, is less than equal to B and that's proved the result, is enough, that proves, that it is. Okay?

(Refer Slide Time: (25:01)

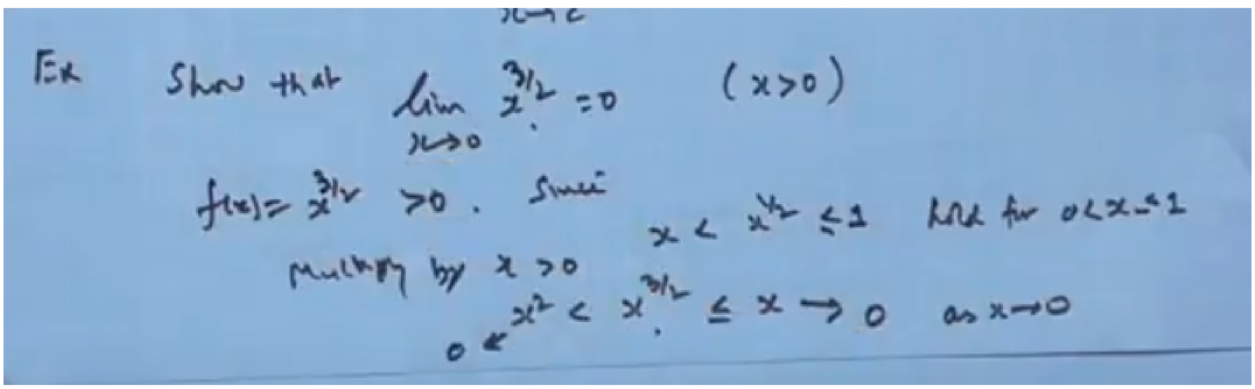


We have another result, which is the Squeeze Theorem, just like as, in case of sequence we have shown, that similarly for functions also, we have the theorem, which is known as the excuse theorem, excuse theorem. What this is, theorem says, let a, is a subset of R, non entry subset of R, let f G H, is a mapping from, a to R and let C be   C belongs to R, with a cluster point of a, point of a. Now if this inequality horse, FX is less than equal to GX, is less than equal to HX, horse, for all X, belonging to a, except say, C. X is not equal to C. And if the limit of this, if the limit of f, limit of F, as X tends to C, is L, which is limit of HX, when X tends to C. Both the limits exist and they have the same value say, L.

Then this Excuse theorem says, the limit of GX and when X tends to C will also be L. Okay? So this was proved, earlier also and even it follows from the previous result also. Because the limit of this is L, limit of this is B, so a and B are there. And this GX always lie between a and B, for all. Is it not? So therefore, by excuse theorem, by the previous result also, limit will exist and since both are equal, left and a and B, are so this limit will also be a, excuse theorem. Or, otherwise also GX minus FX, you can write it and a check we can get the results for it. Okay? Like this, so this is nothing too much, Let us see the, use of

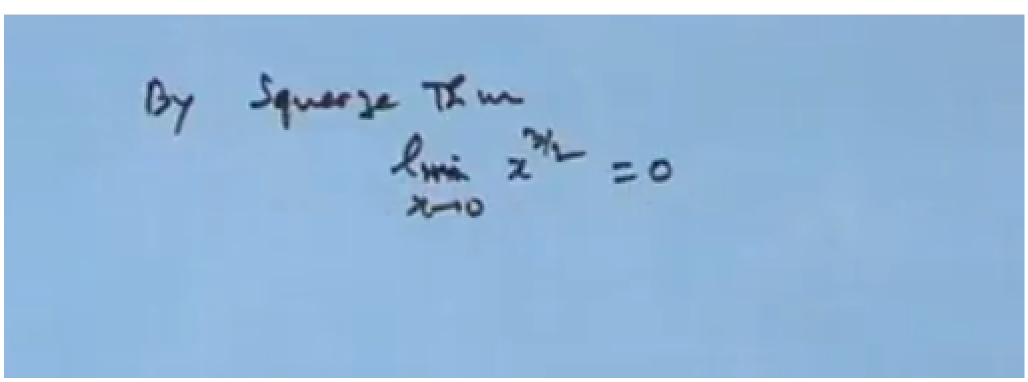
application of the Excuse theorem, where, we can apply the excuse theorem, to get the limit of a function, easily, quickly. Suppose I would say, so that, limit of this, limit of X to the power half, when X tends to 0, is 0. We’ve already seen, when X to the power n, when N is a positive Integer, then it can be written, X into X, X into X, up to N and limit X tends to 0, it will be 0, but, when this power is not in teacher, then, X raise to 8 a fraction. Then you cannot write in the form of X into X or something like that, we don't. So we have to show that. Of course this will be discussed, when we, did for the Cantor story that, that X to the power alpha, alpha is rational, alpha is a sequence converges, then alpha, will go, that is already discussed.
But I still by using this excuse theorem, we can prove this, so, how? So suppose fx and FX is basically X to the power half, for all X, this is, we want less, 3 by 2, let us be 3 by 2, okay, for this X is greater than 0, X is greater than 0, sorry, so that this is greater than. Now FX is 3 by 2, when X is positive, it is positive, throughout. Now, we have this, since X is less than X raised to the power half, which is less than 1 or maybe at the most equal to 1, horse, for 0 less than X, less than equal to 1, this is result, result is true.

(Refer Slide Time: (29:23))



So if I further multiply by X, then multiply by X, multiply by X. Because X is positive, so it will not reverse the inequality. So what we get it? X square, is less than X, is 3 by 2, is less than equal to X. Now apply the Squeeze theorem. Here the FX is this, which tends to zero, this GX, HS is also which tends to 0, where the X tends to 0. Therefore limit of this has to go to 0.

(Refer Slide Time: (29:58))



So this proves the result. So this shows, that by Squeeze theorem, limit of this X raise to the power 3 by 2 and X tends to 0 each.