Lecture – 4

Concepts of Metric Space

Now real analysis is the branch of mathematical analysis. Which is primarily concerned with the limit process and continuity of the real valued function defined on the subsets of the reality. this concepts of limit continuity of the function convergence of the sequence depends for the meaning on the notion of absolute value of the difference of two numbers when they are regarded as the points on the real line, which is nothing but the distance between the between the points. when this notion of the distance all metric for a real number is extended to abstract sets in general then the resulting mathematical structure are known as meeting spaces. The meeting space was first introduced by fregets and in fact what we want to discuss here is we know the real line are when we take any

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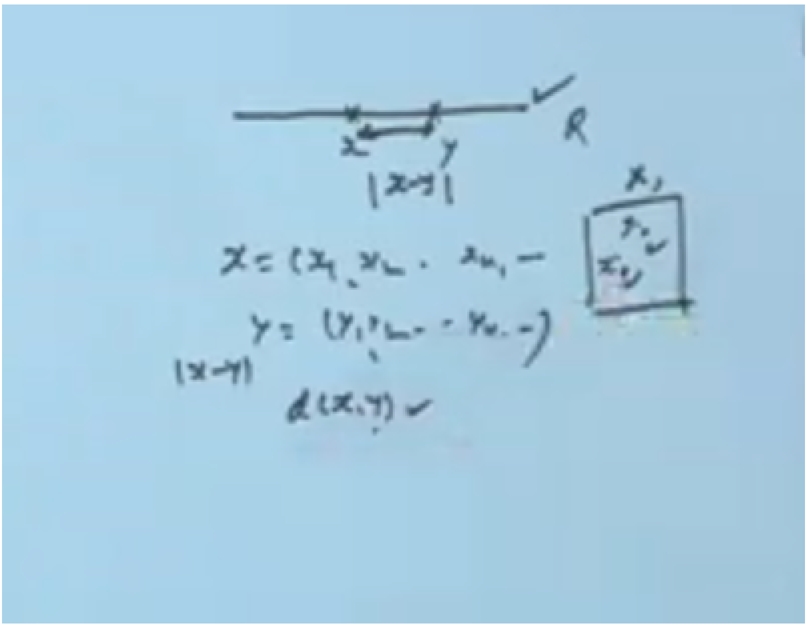

two arbitrary point on it, then the distance between X and Y is the absolute length between x and y, that is when we measure from X to Y either from X to Y or from x to x this absolute length we call it the distance between the two point x and y. now this concept we wanted to extend to an arbitrary set X, because once you take the set X whose elements are certain points how to introduce the notion of the distance between any two element of the set say for example if we take a x to be a infinite sequence set of all infinite sequence and which are bounded then if we pick any two element x n by both are infinite sequence x is also x1, x2 xn and so on y is also y1, y2, yn and so on, so what is the how to find the distance between x & y? if I just take x minus y then there are several points will it will it be mode x 1 minus y 1 mode x 2 x 2 etcetera or something else so in hurt in what way you will introduce the notion of the distance between the points of an arbitrary set. but we should keep in mind that notion of the distance on the all satisfy those condition must be satisfied by our newly introduced function d. then only this d we can say it in an extension of the distance concept over a general arbitrary set.

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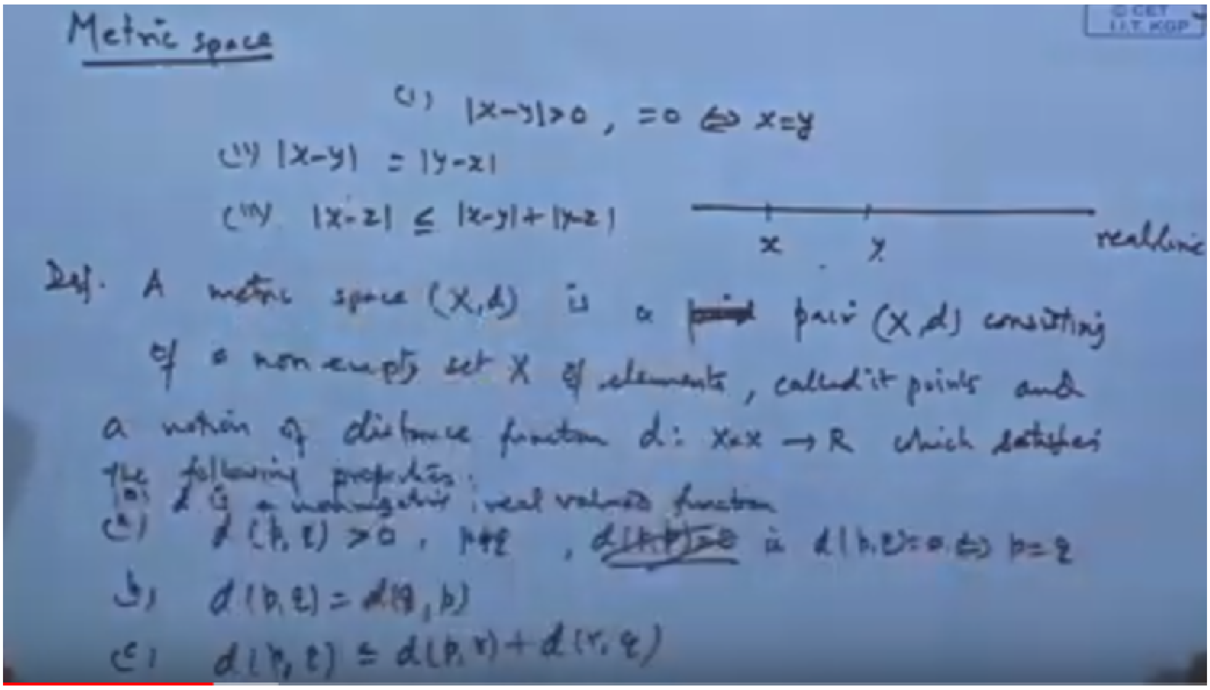

Now let us come back to our meeting space now so this is the optional one but we will discuss it here. as we have seen that in case of the real line when we pick up the 2-point xn then the distance between X and Y is defined as mode of X minus pi this is the absolute difference between the two point on the real line and this distance will always be greater than 0 and will be 0 if and only if the X equal to Y then if we measure the distance either from X to Y or from y to x, both will be the same. Then third is when we take the any point z either in between x and y or maybe outside then this will remain less than equal to X minus y, plus y minus z.

So these properties are satisfied by the episode function or we say the distance nose function defined on the real numbers or in the real line then we wanted to extend it to an arbitral metric arbitrary set because this is the set of real numbers we are having this property now this property we wanted to make it as an exam for an arbitrary set and that leads to the concept of the metric yes so we define the metric space as a metric space x,d is a pair is a pair XD consisting of a non-empty set X of elements we call it a here is point call it points and a notion of distance function d define over X cross x to our set is which satisfies the following properties the following properties.

The first property the distance between the two point p, q is greater than 0, if p is different from q and distance between P, P is 0. in fact other way round also we can say by Vies versa also deep, so second is distance is this and d of p, q is the same as d of q, p then c is d of p q is less than equal to d of p r plus D of R Q so distance function is satisfy the first is you can write this 0 property, d is a non-negative real valued function, non-negative real valued function and this property we can also write it that is d of p q is 0 if and only p equal to we can replace it in fact this is so when a P and Q are identical both are equal to 0 and vice versa so if this properties are satisfied. Then we say this set X together with the notion of the distance forms a meeting space.

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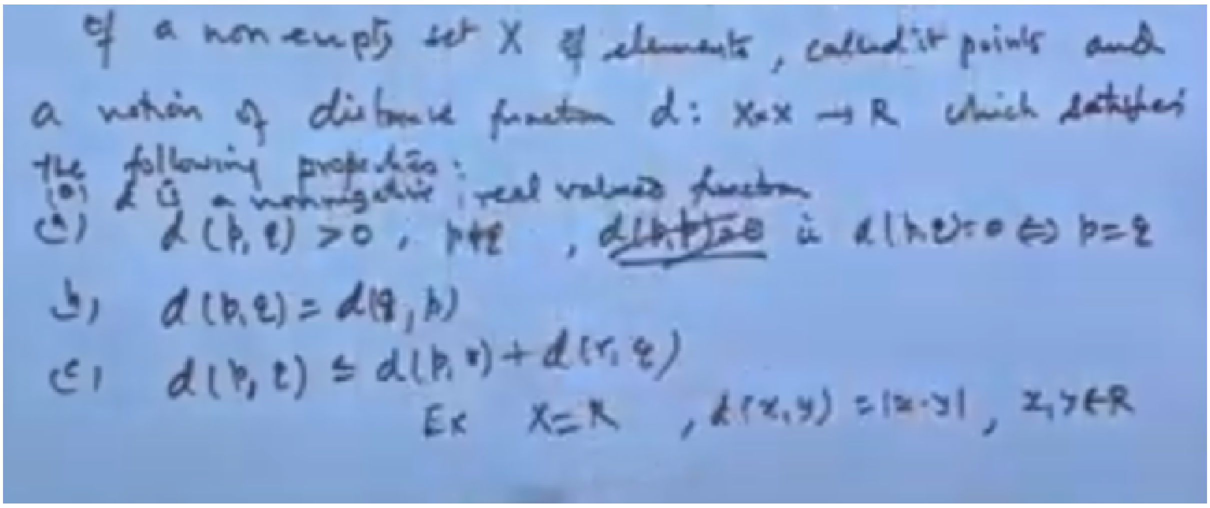

And one of the example is our X is R set of real line and D of x,y is defined x minus y, where x and y are real, then as we have seen earlier satisfy all the property.

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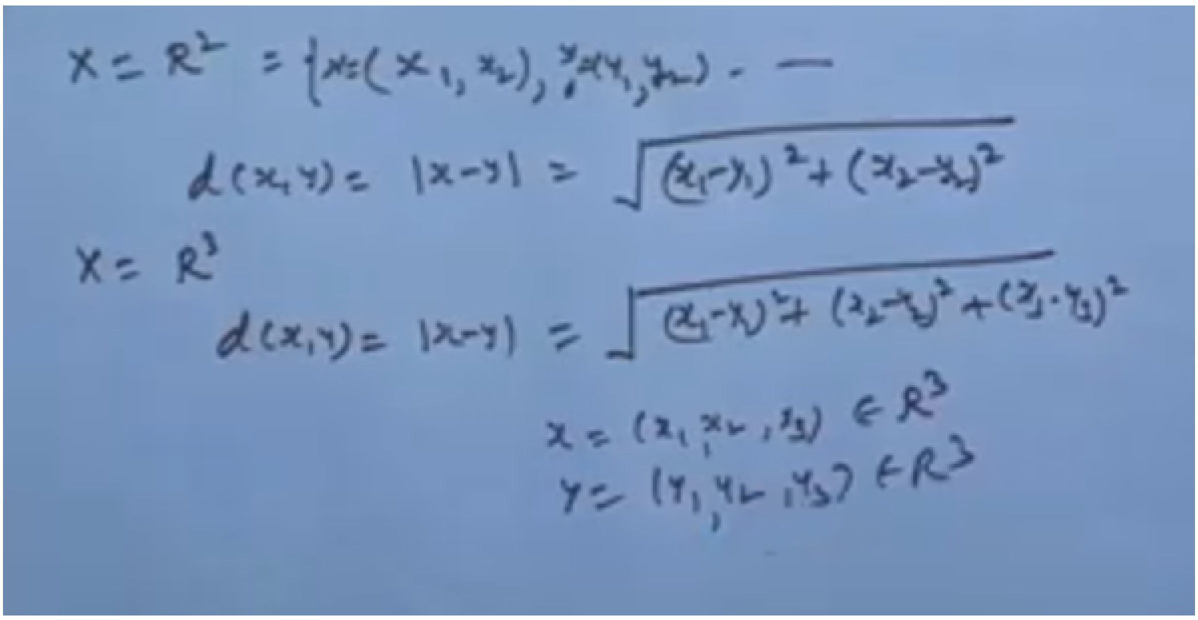

then another examples is suppose X is set r2, r2 is the two-dimensional plane we are the elements is of the form say x1, x2, y1, y2 and like this so on so suppose this is X element this is another element and continue then d of XY which is written as mode x minus y i am writing just the same using the mod but meaning of this is in two-dimensional case it will be x1 minus y1 whole square plus x 2 minus y2 whole square under root in case of three-dimensional the distance between X Y is defined as we write the same way mod X minus y but thing is the is x 1 minus y1 whole square x2 minus y2 whole square, x 3 minus y3 whole square, this x1, x2, x3, y1,y2, y 3 these are the ordinates belongs to R3, y- y1, y2, y3 belongs to R3 and it can be shown that this DS satisfy the condition of the meeting so we are not going in detail it's a part of the function insist so there it will be discussed.

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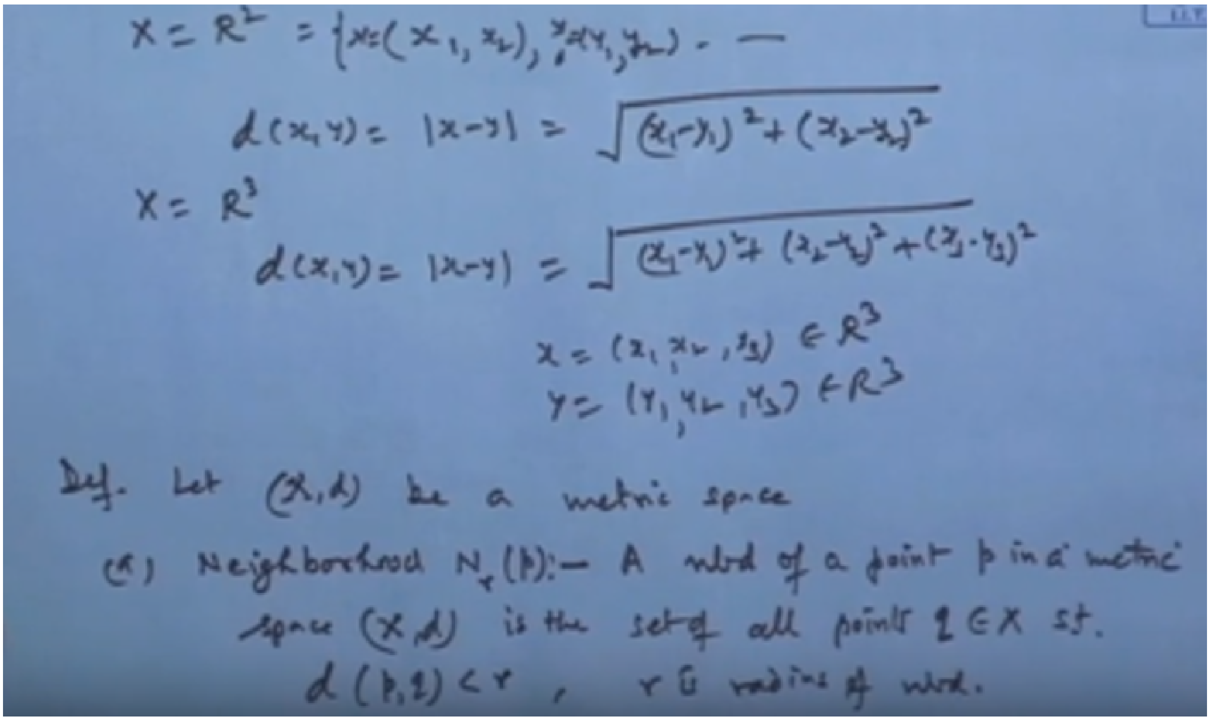

What we are concerned is that since we have discussed the concept of the neighbourhood open sets, closed sets open ball closed ball in real line with the help of the modulus functions. So this we wanted to extend with this concept to a arbitrary metric space orbit set with the notion metric space. So let us see the few say concepts which are useful for further. so we define now let x,d be a metric space, let XD i am writing XD be a mythic space ok and then be defined like this first definition is the neighbourhood, neighbourhood which we denoted by Nr P that is neighbourhood of a point P with radius R so this P right is our neighbourhood I am writing NPD neighbourhood of a point P in a metric space in a metric space x,d in a metric is the set of all points is the collection or is the set of all points q belongs to X such that the distance between P and Q is strictly less than r then r is called the radius of this then r is the radius of the neighbourhood.

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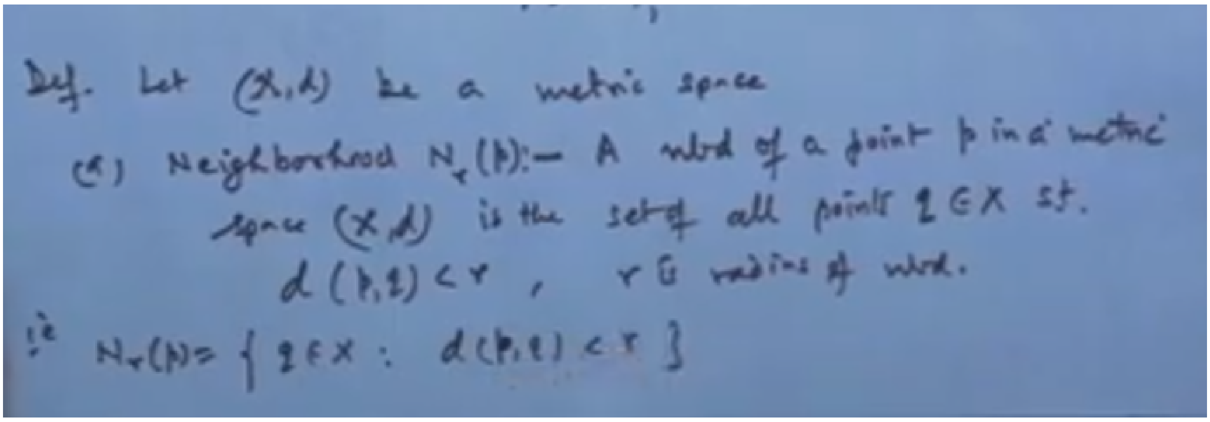

Okay it means our neighbourhood of a point in a metric space which we denote by NRP that is the set of those point Q belongs to X such that distance of P and Q is strictly less than R.

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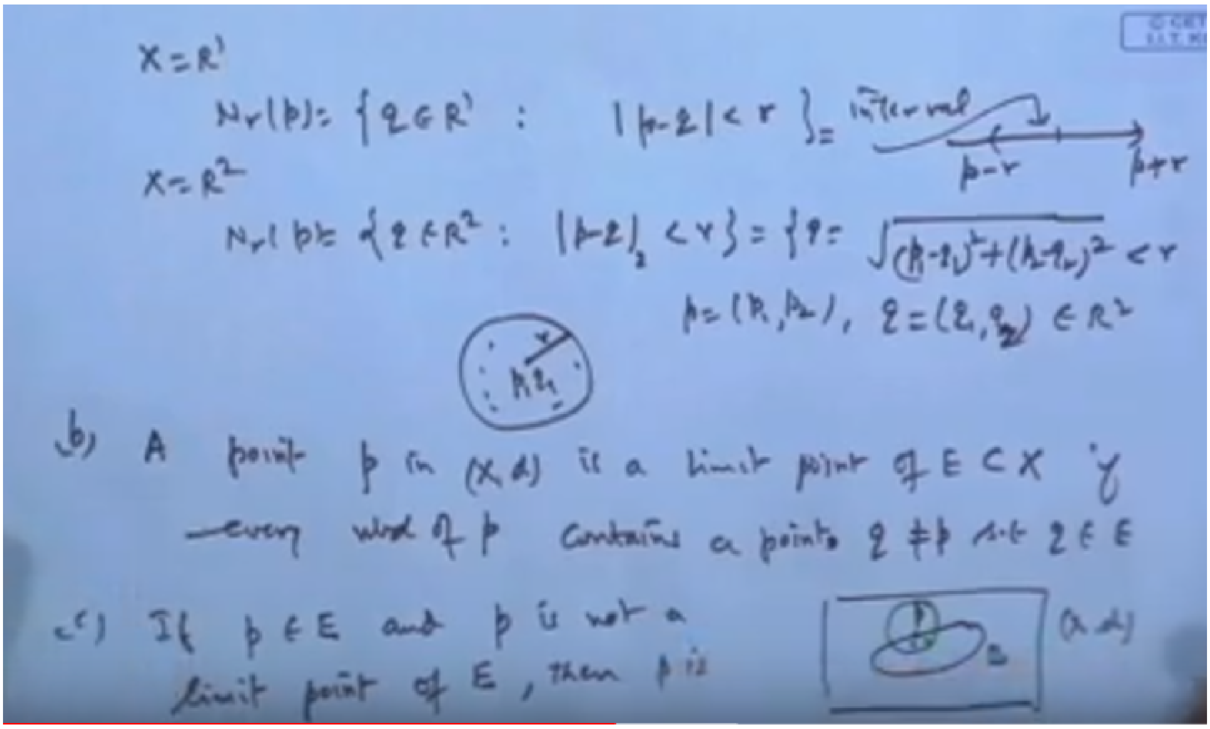

so this in a for example in case of real line if x is r1 then this neighbourhood Nr p is nothing but what? Q belongs to R1 such that mod of p minus q is less than R that is it is an interval is interval line between P minus r2p plus r and q will be a point somewhere here, so this is the interval equalant to the interval this. this interval when X is r2 then Nrp which is Q belongs to r2 such that mod of p minus q the distance i am just writing the distance okay d2 is less than this is the to distance 2 is less than R that is it is the set of Rq such that under root of this and out of P 2 points are there then p1 minus q1 whole square plus p 2 minus q2 whole square is less than R, that is where p is p1, p2, q is q1, q2 both belongs to R2. so it is a ball centred at p1, q1 with the radius say are so all such point will be here, so this is the neighbourhood of this similarly we can go for other. Then second definition is the limit point of the set A point p in a metric space Xd is a limit point is a limit point of the set E which is subset of X limit point of the set E. if every neighbourhood of P if every neighbourhood of p if every neighbourhood of p contains a point contains a point Q different from P such that Q is in E, such that Q is in E.

So this is the point and never limit point means if this is a space metric space XD and here is the set E we say this P this P may or may not be belongs to e but it will be a boundary point of e, if it is a limit point then p is called the limit point of this if every neighbourhood around the point if we draw neighbourhood means suppose in order to we get the neighbourhood in this pain in like this so if every neighbourhood of the point P must if in course the point cube other than P which is in e then weaker say P is the limit point of the set E. okay then the definitions third, this is the definition about the isolated point if P belongs to E, and P is not a limit point. NP is not a limit point is not a limit point of E, is not then p scored an isolated point p is called an isolated point isolated point.

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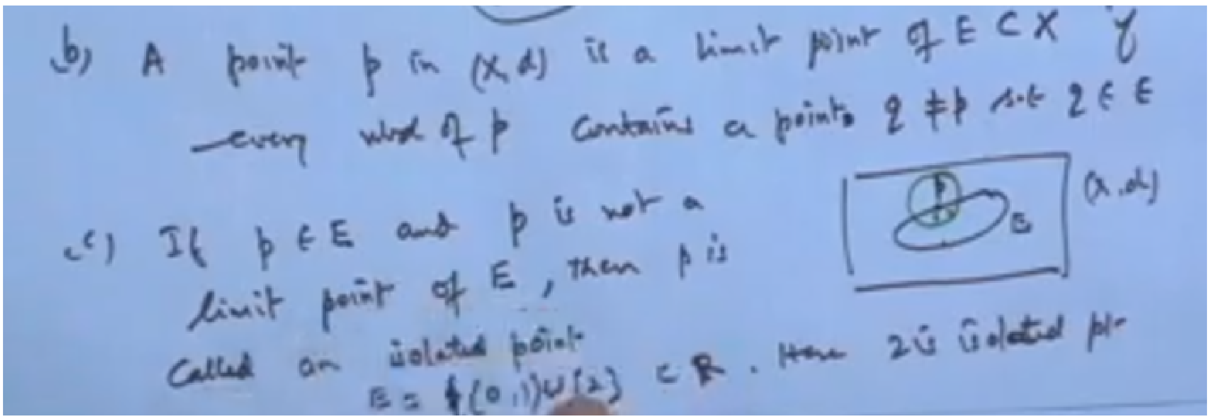

for example suppose I consider the set E as this suppose I Take set E as the set or say 0,1 Union Set 2, suppose I take the set 0,1 in neon, 2 in r2 which is subset of r 2, r subsets of our so interval this not to here 2 is an isolated point I so let it point why because if we draw a neighbourhood around the two then it does not include any point of E so two cannot be a limit point of E but two is a point of E so it is an isolated point.

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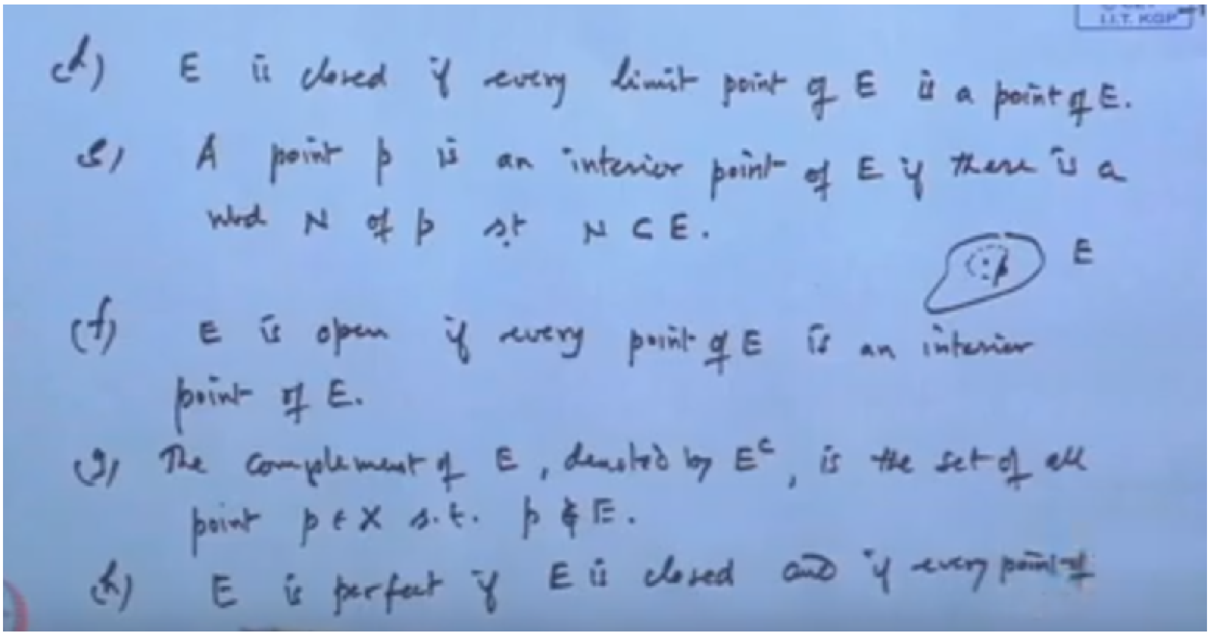

and E said to be closed is closed if every limit point of E. if every limit point of e is a point of E is a point limit point is a limit point of view then E a point P a point P is an interior point of interior point of E, if there exists or if there is a unneighborhood, if there is a neighbourhood n of P there is a neighbourhood N of P, with a suitable radius such that this neighbourhood is totally contained in E. So what he says is this is our set E, we say this point p is an integer point if there exists a neighbourhood around the point p which is totally lies inside E then such a point p is called an interior point. then E is set to be open a set E in a metric space is said to be open, if every point of E if every point of E is an interior point of E. then this curve then G complement the complement of E denoted by Ec is the set of is the set of all points P belongs to X set of all points P belongs to X such that p is not in E.

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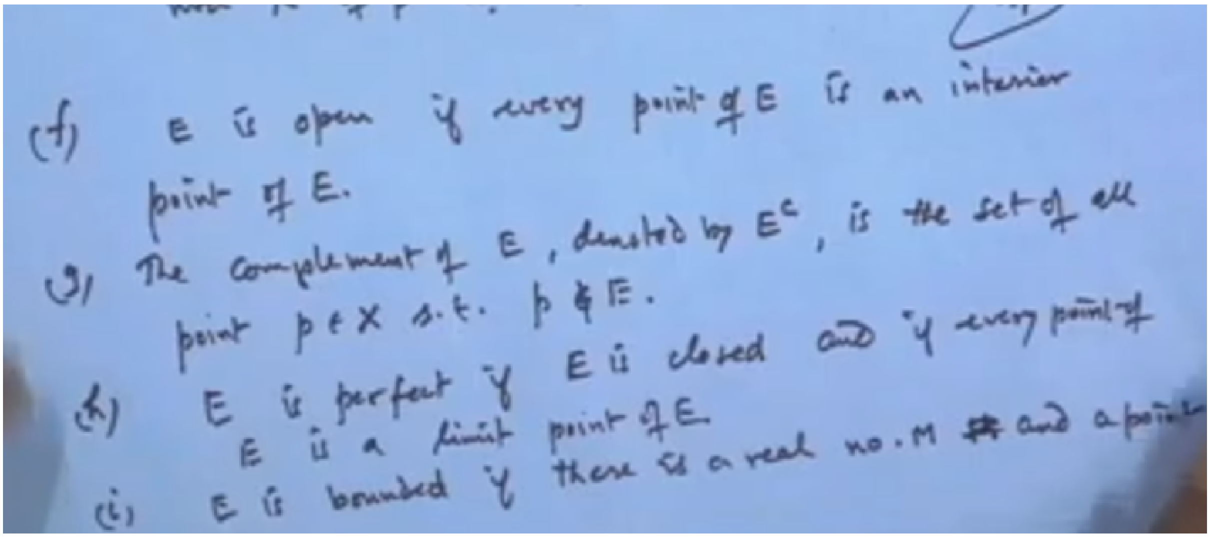

H a set E set to be perfect in a metric space XD, if E is closed and if every point of E point of e if every point of e is perfect if every point of E is a limit point of E. I a bounded set we will discuss so is E set is said to be bounded if there exists or if there is a real number M such that

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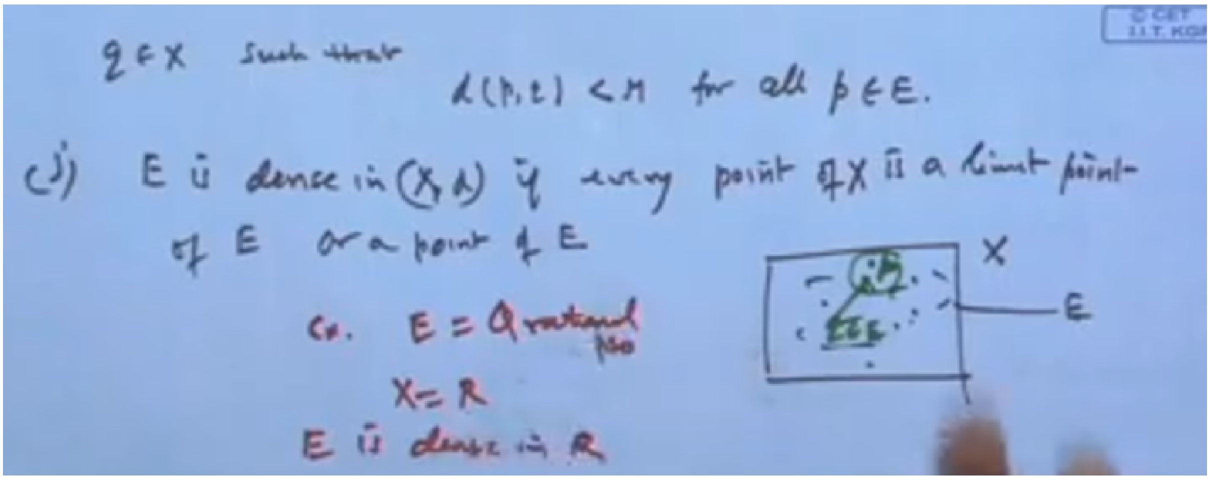

and a point and a point Q belongs to X such that such that the distance between P and Q does not exceed by M, for all P belongs to E. and lastly we say, J) a set E is dense in a metric space XD in a metric space XD if every if every point of X every point of X is a limit point every point of X is a limit point of E or a point of E or a it means what? When we say a set this is X a set E this is E is set to be dense in this. It means every point every point of X every point of X is a limit point of E or a pointer. so if we take any point X say here suppose I take any point X here then this point will either be a point of E or if it is not a point of E then it must be a limit point of E so limit point of E, if we draw a neighbourhood around the point P there must be some point Q of E which is different from P. so in this case the element of X an element E they are so close to each other that you cannot separate out as soon as you take any X and draw a neighbourhood around the point X you will always find at some point of E different from this or maybe the point itself is a point in E so such a case be stated dense E. for example the set of real line for example our E which is a set of rational number and X is R, then this E is dense in R because any real number can be approximated by means of rational number so if you draw any real neighbourhood around the point real number we get another real number is rational point which is different from this.

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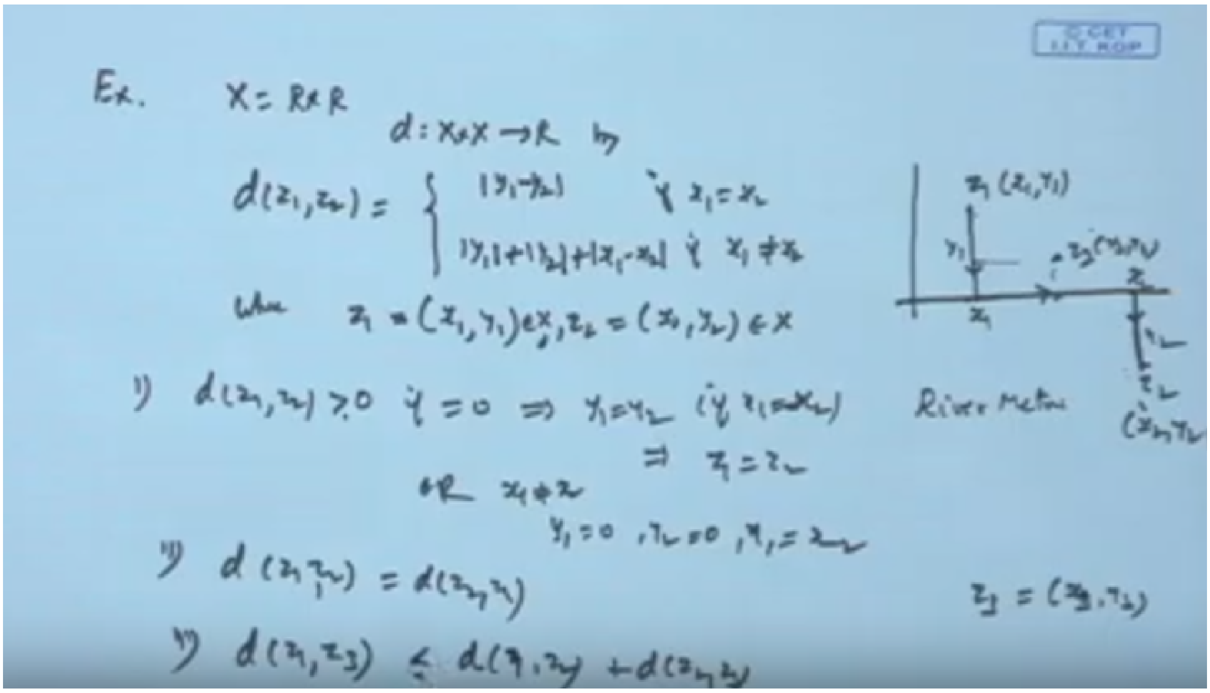

so this becomes in the last lecture we have introduced the metric space and also discuss few of the examples of a metric space let us take few more example which is slightly a different way suppose X is Rx R, this is our space X and notion of the distance d define on x cross x2, R by d of z1, z2 is the length y1 minus y2 if x1 equal to x2, otherwise y, y1 mode y1 plus mod y2 plus mod x1 minus x2 if x1 is not equal to x2. Where z1 is basically the point x1, y1 belongs to x, z2 is a point x2, y2 belonging to x, so this is also an x this is also an x. now what is this can basically suppose we have the points, 1 point is say z1 another point is z2, ok so what we are doing is when x1 and y1, x1 is equal to x2, the distance is y1 minus y2, otherwise it is this so if we go from here then this is our x2 this is our x1, y1, so x2, is different from x1 so what we say we take this is y1 this is y so take the absolute distance x 1 plus this distance this is our x1 this is our x2 so take this distance, so a person is coming from here following this path going this way and approaching gate so this is the length between z1 and z2. We can also find the length directly ok so that will also give another metric that is under root x2 minus x1 square plus y2 minus y1 square and square root of this so this is also admit but this is metric is known as the river metric. Because if a person wanted to cross the rebel he has to first approach up to the first that River one side of the river then he has to cross the river and then compared to the side, ok. so that is called the river metric. This D which is defined in this way actually satisfies all the condition of the meeting. the first is obviously d1z, dz1, dz2, is greater than 0 because every term is absolute and when it is 0 and if this is 0 then what happens then it implies if x1 is equal to x2, y1 is equal to y2, if x1 is equal to x2 so in that case xz1 equal to z2 or if this is not if x1 is different from x2 then we are getting this thing.

Now some of this is zero sum of this is zero it means individual term must be zero because they are absolute term so y1 is 0, y2 is 0 and x1 is equal to x2. So again this shows the z1 equal to z2 so this way d, x, y is greater than equal to 0 is satisfied. second is d of z1 z2 is the same as d of z2 z1 no change because of the modular science there will not be change in the value and third is d of z1, z3 where z3 is a point x3, y3, so z3 is another point say x3 by y3 here we can say z3 another point x3 y3 okay. so if you find the z1, z2, and z2, z3 we see z1, z2 in this distance while z, 1z3 will be this distance this and this and then z3, z3 again this and this so it will be less than or equal to this therefore triangle equity is also satisfied so this forms a instance on X cross a metric on this X. another example of this myth on the are we can also introduce the metric D of z1, z2 h mod x 1 minus x2 plus mod y1 minus y2, say Z is x1, y1. z is x2, y2, take this design so mod of this plus mod of this ok mod of entire thing z y1 minus y2. Now this also forms a metric is metric and we see all the conditions are satisfied therefore this will be meeting ok now this shows that over a set ax one can introduce in more than one base.