Modeule 6

Lecture 39

Divergence Criteria for Limit

We will discuss the criteria, for the divergence, of the sequences and series both. So today we will discuss few chapters, in support of the divergence, that criteria, where the series diverges. Just like in f sequence, we have some criteria for the divergence of the sequence. In a similar way, here also we have criteria for divergence, of the limit, that if the limit does not exist, what are the criteria for it? Okay?

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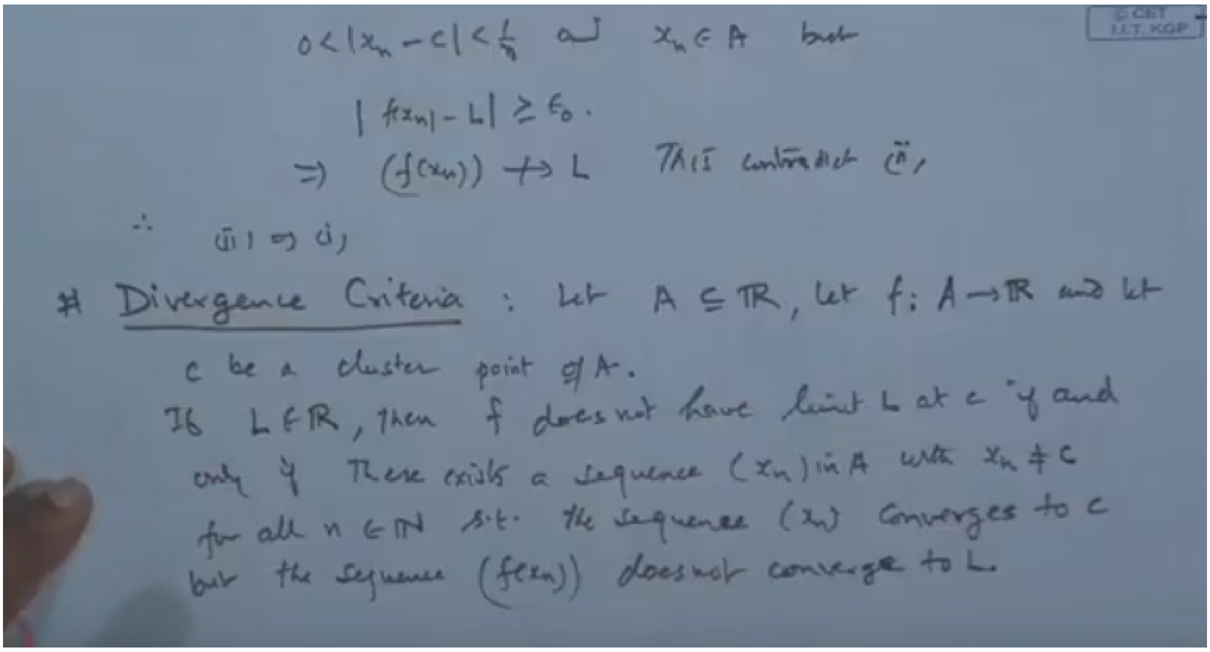

So let us see the divergence, criteria. Let A be a non-empty subset of r and let f is a mapping from, a to R and let C be a and let C be a cluster Point C a, cluster point of a point of a. Okay? Then the criteria says, if L if L belongs to R, then, F does not have limit L at C, if and only if, if and only if, there exists a sequence xn, if and only if, there exists a sequence xn, in A, with XN different from C for all n, belongs to the set of natural number, such that the sequence, the sequence xn, sequence xn converges, converges to C. But the corresponding images, but the sequence, of effects and means corresponding images f of, the sequence of corresponding images does not converge, does not converge to L. So this is the criteria for the diverge sequence. Limit of the function FX when X tends to is not equal to L, it means this will exist. Okay? Now let us see if you examples, where the sequence, the sequence of the function or the limit of the function does not exist. Which are very interesting examples and used very frequently in further study.

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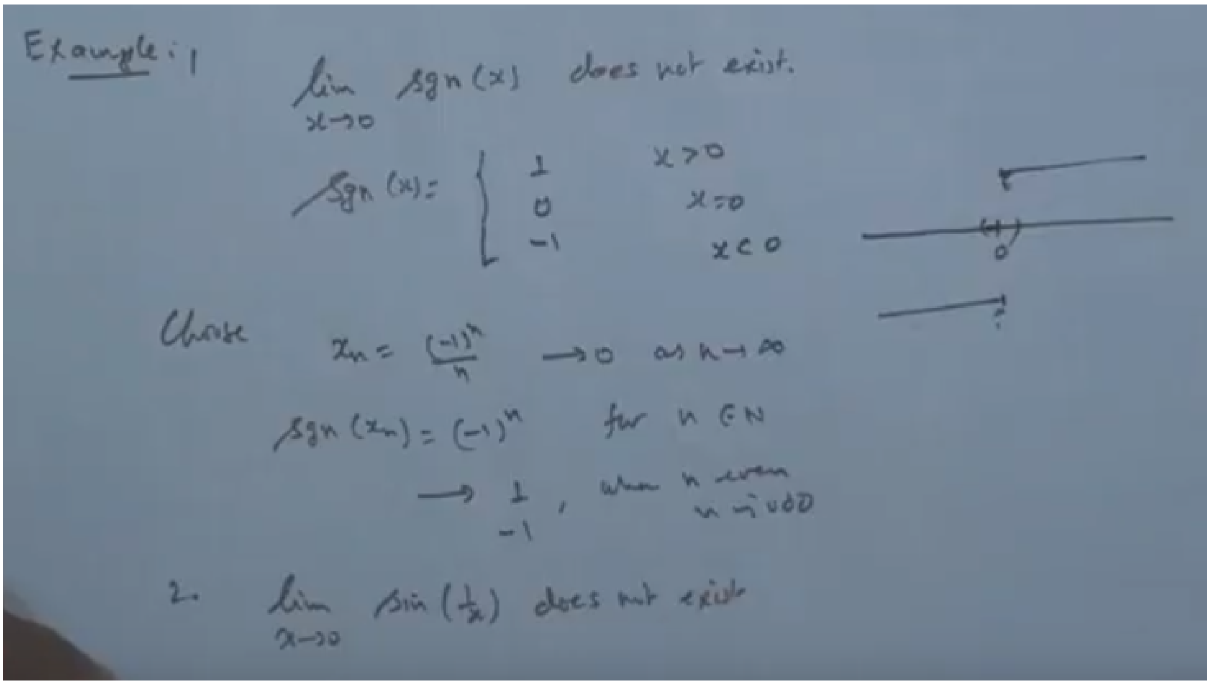

So first, is the limit of this, obviously the limit, limit X X, tends to 0, Signum of X, does not exist, does not exist. It mean what? First the Signum. What is the Signum of this? Signum function means, it is basically sign of the function. So when X is positive, this value will be 1, when X is zero, the sign of this we consider just a 0, then, when X is negative, we put it to you minus 1. So this is the Signum function that is a function is obviously. This is our line. So when this zero, when x is 0, the function is 1, X is negative, function is minus 1 and at the point 0, we take to be 0. The limit of this Signum function, when X approaches to 0, does not exist. So when we approach from the left hand side, or we approach from the right hand side, limit will not come out to be the same. Okay? In fact before this, I should prove that, whenever the limit of the function FX, exists, the limit will be unique. That I will show just after this. So here the limit will not limit. Let us see why.

It means if I take a sequence xn, which goes to 0, but thus corresponding function Sigma xn, does not go to that 0. So let us choose, the sequence xn, is minus 1 to the power n. obviously this sequence will go to 0, as n tends to infinity. What, what is the Signum of the sequence, XN? The Signum of this sequence X n, when n is obviously even, it is positive, when n is odd, it is negative and when, so with positive n, even number, it is a positive value, plus 1, when n is negative, it is a negative value. Okay? With one, we want minus one plus one. So basically it is the same as this. That is when n is even, you are getting plus 1, when n is odd you are getting minus 1 and this will limit, will we get it. So for all N, belongs to N. Now limit of this, it does not exist. Because it varies, when n tends to, it goes to 1, as well as minus 1, when n sequence approach towards the even, when n is odd. So there are the terms of the sequence, which goes to plus 1 or even minus 1 or. So nearby this 0 interval there is a Variation, variation of 2 and variation 2 it cannot be less than Epsilon, because, we want this thing to be less than Epsilon. I choose epsilon to be ½, how the variation can be less than ½? when the variation is from minus 1 to plus 1, there exactly true. Okay? So this limit does not exist therefore this.

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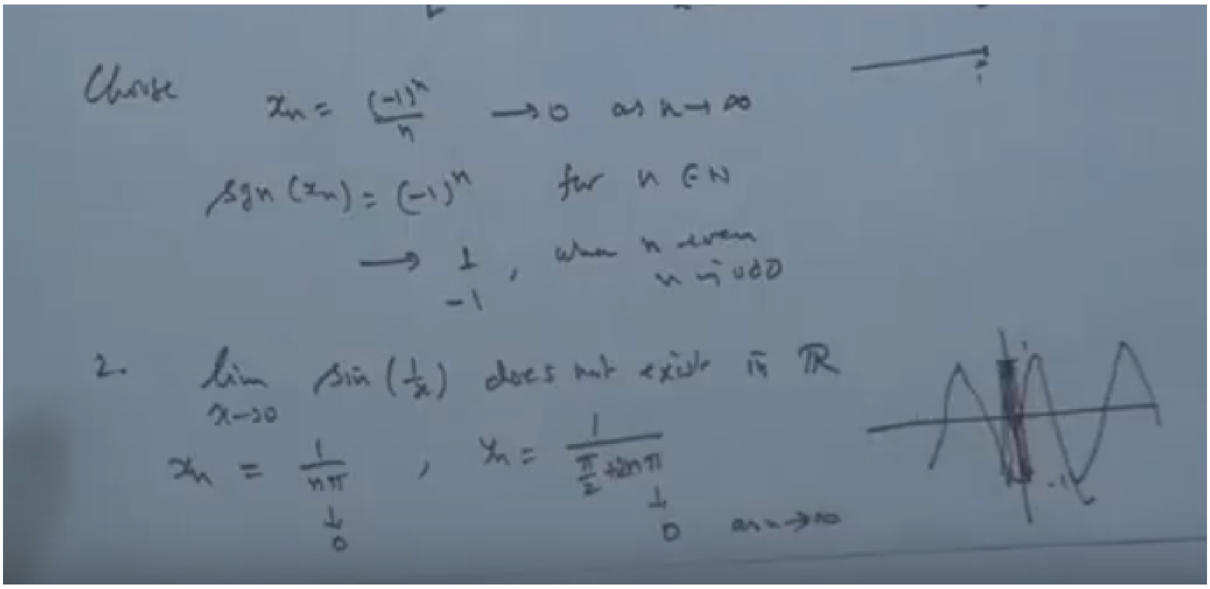

Similarly another example if I take. Suppose I take, limit of this, X tends to 0, sine of 1 by X, does not exist, does not exist, in R. Again sine function, if I look the sine function, the graph of the sine function is something like this. This is 0, it go plus 1, and minus 1, maximum. So we can get this thing, sorry. This is like this and something like that, then coming from here and the similarly, but as soon as zero, it fluctuates very. Okay? It fluctuates too much. And basically you are getting, this thing like this and so on, like this, something. Means nearby zero, it jumps, keep on Jumping, going up down, going up and down, like this, so we get the fluctuation and the friction a very tricky, around the point zero. We are claiming this limit does not exist. It means along the two different paths, if the limit of the sequence has different value and both the path tending to zero, then obviously limit will not exist. So let us take the limit X n, choose X n to be one by n y. And suppose another sequence y n, if I take one y. Y by 2, plus 2 n, y. Now both these things are tending to 0, this is also tending to 0, this is also tending to 0, as n tends to infinity. So these sequences are, converted towards 0. That is mode of xn minus 0, is less than Delta. Now for this sequence, what is the F of Xn?

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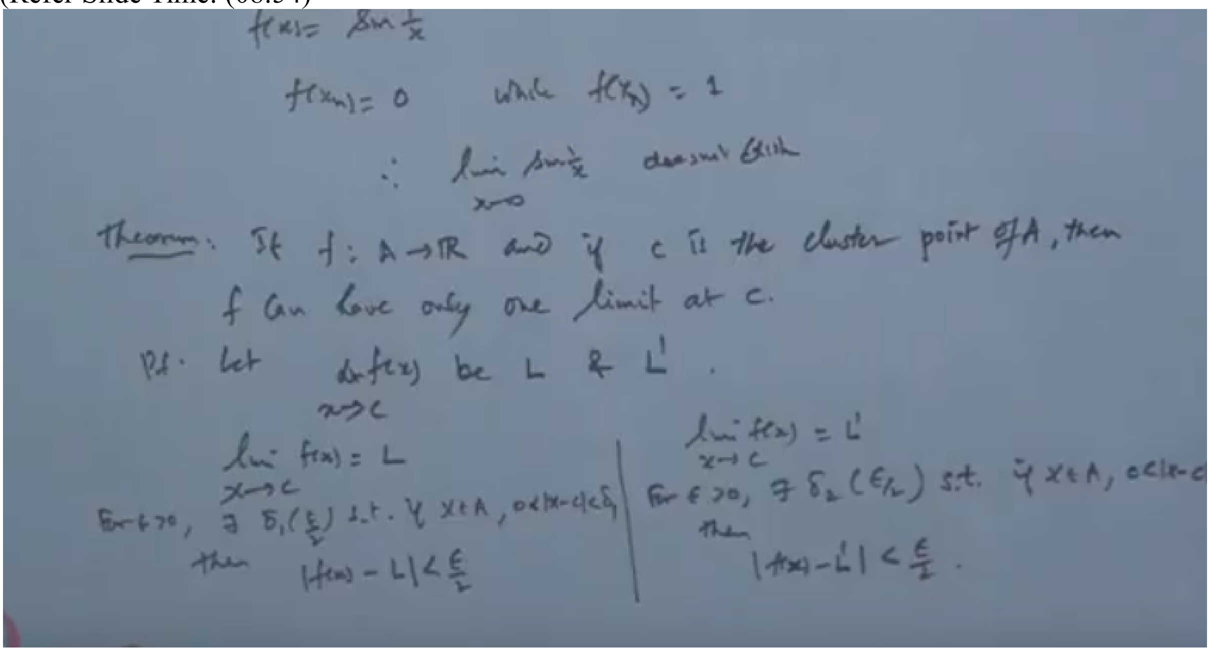

If I take F of xn, FX is sign, 1 by X. So what will be the F of xn? F of xn is 0. Because when xn is 1 upon n pi, the corresponding sign of n PI and sign integral multiple of pi, 0. While f of y n, this is sign of 2 plus PI by 2, that is always sign PI by 2, so this will be 1. So it means, the at along 1 point it is 0, sequence along another point the limit of this is 1, so it does not exist. Therefore limit of this does not exist, exist. Okay, now here as I told that, we are assuming, the limit is unique. Is it not? Limit cannot be to, if the limit of the function exists, it will be unique. So let us prove that result is and all these exercise which way depends on this only. Okay? So this we should prove it earlier what, however. If F is a mapping, from a to R, a to R and if C is the cluster point, of a and if, C is the cluster point of a, then F can have only, only one limit at C. We cannot have the two limits. So proof is, again we will prove by contradiction.

Suppose there are two, two limits? Okay? Then finally we will see that these two limits are not different, they are identical. So let the FX of this, when X tends to C, is suppose, let this limit be, L and L days'. Okay? L and L days, suppose this. So limit of this FX is L, when X tends to C, limit of FX when X tends to C is also L days'. Okay? So apply the definition now. So for a given epsilon greater than 0, there exists a delta 1, depending on epsilon, such that, if X belongs to a, and 0 is less than mode X minus C, less than Delta, then, Delta 1, then mode of FX, minus L, let it be epsilon by 2, here. Okay? Delta 1, it depends on epsilon by 2, suppose, FX minus L, is less than epsilon by 2. Okay? For all X, this. Similarly here, for the same epsilon, greater than 0, there exists a delta 2, which depends on epsilon by 2, such that if X belongs to a, and X lying between, X minus C, less than Delta 2. Okay? Then, then, mode of FX, minus L, is less than, L dash is less than epsilon Y 2, for all such. Okay?

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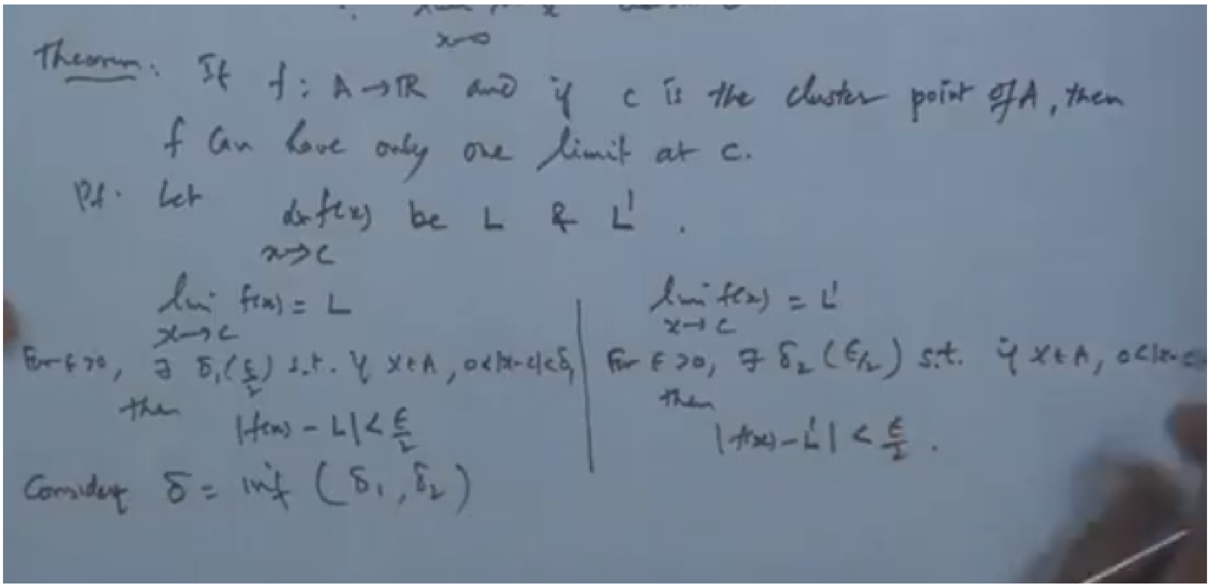

Now consider, mode L minus L, consider Delta, to be the infimum of, infimum of Delta 1 and Delta 2. So if I choose, replace Delta 1 by Delta, then this result will also hold? If I choose Delta 2 by Delta, again this result will hold.

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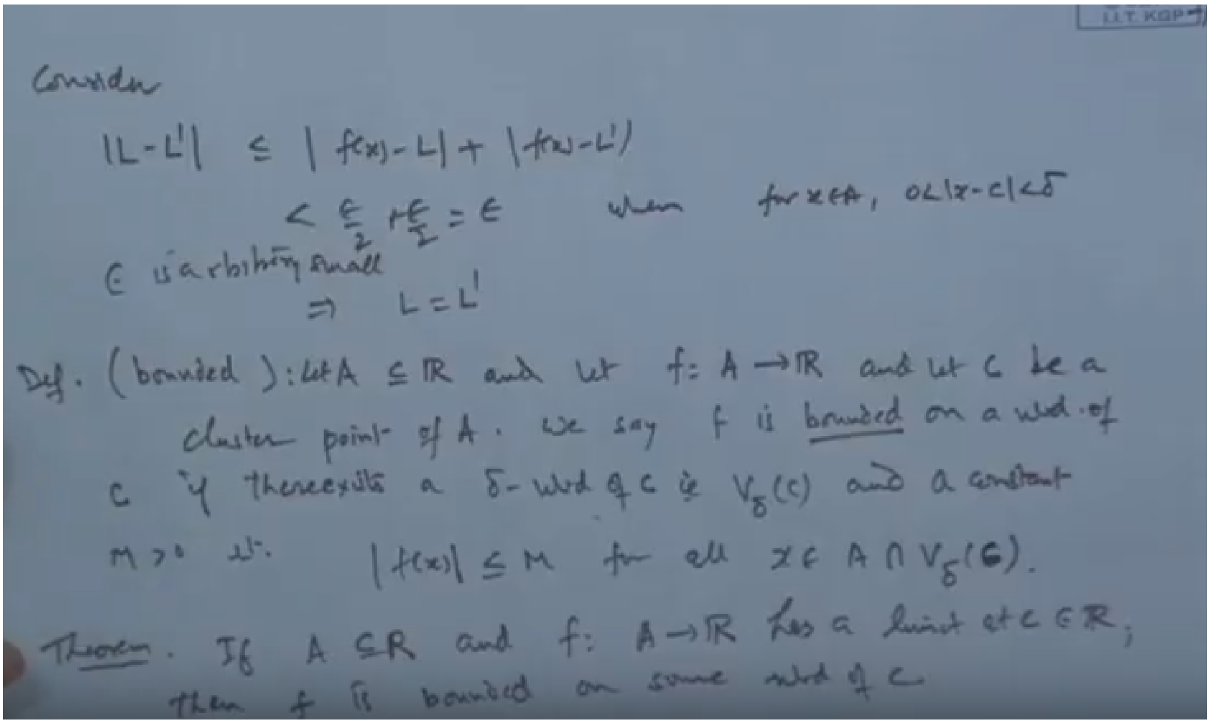

So now consider, consider, consider, mode of L minus, L dash. Now this is less than equal to, add and subtract this, so we can write FX minus L, plus FX minus L days. But FX minus L, is less than epsilon. So for all X, belongs to a, and 0 less than X minus C, less than delta, this is also less than epsilon y 2, this is also less than that, is this result, when this result shows. Okay? So what it shows? That this implies L is equal to L, because epsilon is arbitrary, arbitrary small. So we can get L equal L, that is, the limit will be unique. So if as functional value, FX, when X tends to C, has a, 2 different limit, along the two different paths, which are approaching to C, then the function will not have a limit at that point, so that is what. And then such a case, we say it is a diverging, function diverges, at XC. Okay? That is what, okay. Because diverge does not mean that functional limit of this must go to infinity only, it is also one of the criteria, in the limit of the function FX, when X tends to C, say infinity. It means the function itself is not defined at infinity, at the point. Okay?

And the limit is coming to be infinite, so it's not finite. But even if it is finite and different, we having different value, at a different along the different path, still we say, then we also, we say, the function FX, does not have a limit at X equal to C. So that is the criteria for this. Okay? Now, this we have discussed, now there are few more results, over the limits and that we have seen in parallel to, this will be parallel to our theorems, as we have discussed in case of sequences. Okay? So before that let us, see the definition of the boundedness, boundedness. Let a that A, is a non-empty subset of r and f and let f, is a mapping from, a to R, a to R. And let C be a cluster point, be a cluster point of A.

We say f is bounded, on a Neighbourhood, on a neighbourhood of C, bounded on a neighbourhood of C, if there exist, there exist and if bounded on the neighbourhood, there exist a delta, neighbourhood of C, that is, B Delta C, Delta neighbourhood of C and a constant, capital M greater than zero, such that, such that, mode of FX, is less than or equal to M, for all X, belonging to, A intersection V Delta C, V Delta C. If this is true, then we say the function FX, is a bounded function, is bounded, in the neighbourhood of C, bounded in the neighbourhood of C. Okay? If the function has a limit, then it will always be bounded. Okay? So the result is, if a, which is non-empty subset of R and F is a mapping, from a to R and let, has a limit at C, has a limit at C belongs to R, then the function f, is bounded, on some neighbourhood of C, some neighbourhood of C, is bounded on some neighbourhood of C. Let us see the proof of this result, so if the limit of the function FX exist. then the function f must be bounded, in some neighbourhood of C. Okay?

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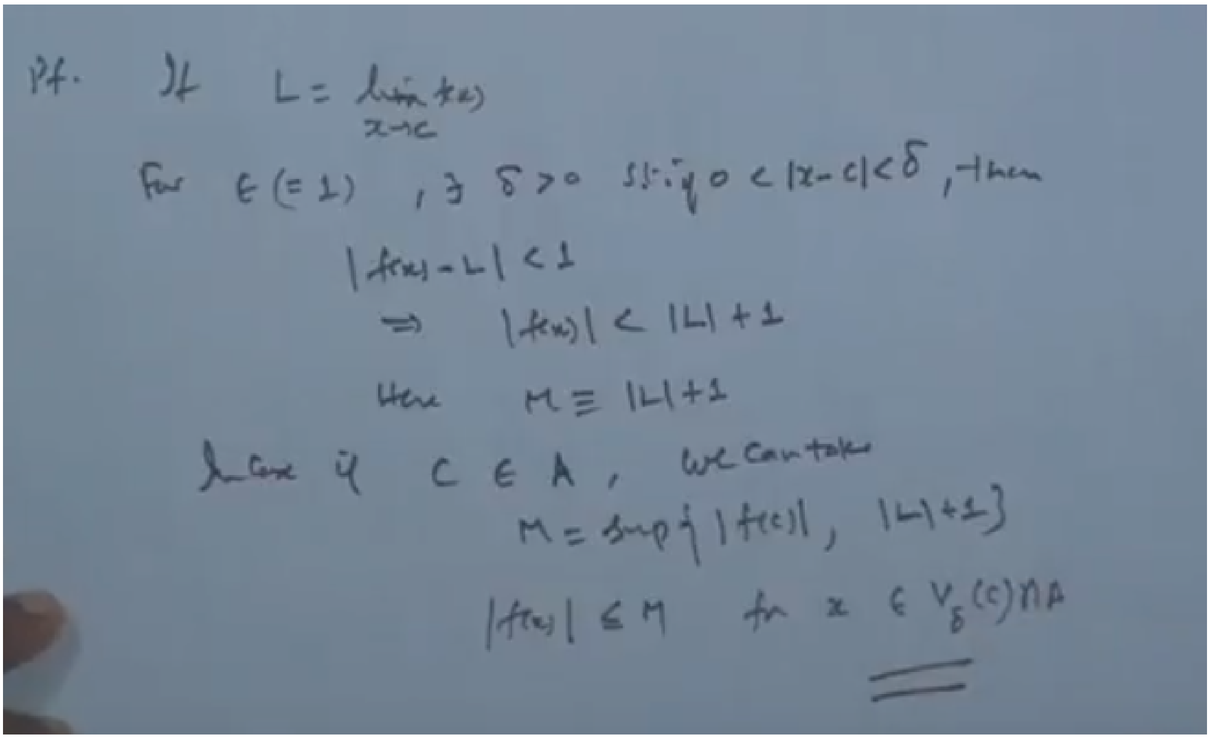

So suppose F is the limit, let if L is the limit of FX, when X tends to C, this is given. Okay? So let us by definition, so for a given epsilon, I choose epsilon to be 1, for a given epsilon, greater than zero, because it is already get 0, there exist a delta, which will depend on epsilon, positive, such that, 0, if 0 is less than mode X, minus C, less than Delta, then by definition, mode of FX, minus L, is less than 1. Now from here, can you not say the mode of FX, is less than mode L, plus 1? Okay? So I choose m to be, so here M can be chosen as, mode L, plus 1. Then for all X, in the neighbourhood of this, it is there. In case in case, if C, also belongs to a, in case if C belongs to a, then what we do is, we choose, we can take M to be the supremum value of mode FC and this bond and this one. So if I choose this, then all these function FX, will satisfy this condition, for all X, in some neighbourhood of C, that is intersection. Okay? So this work proved. Okay? Thank you very much.

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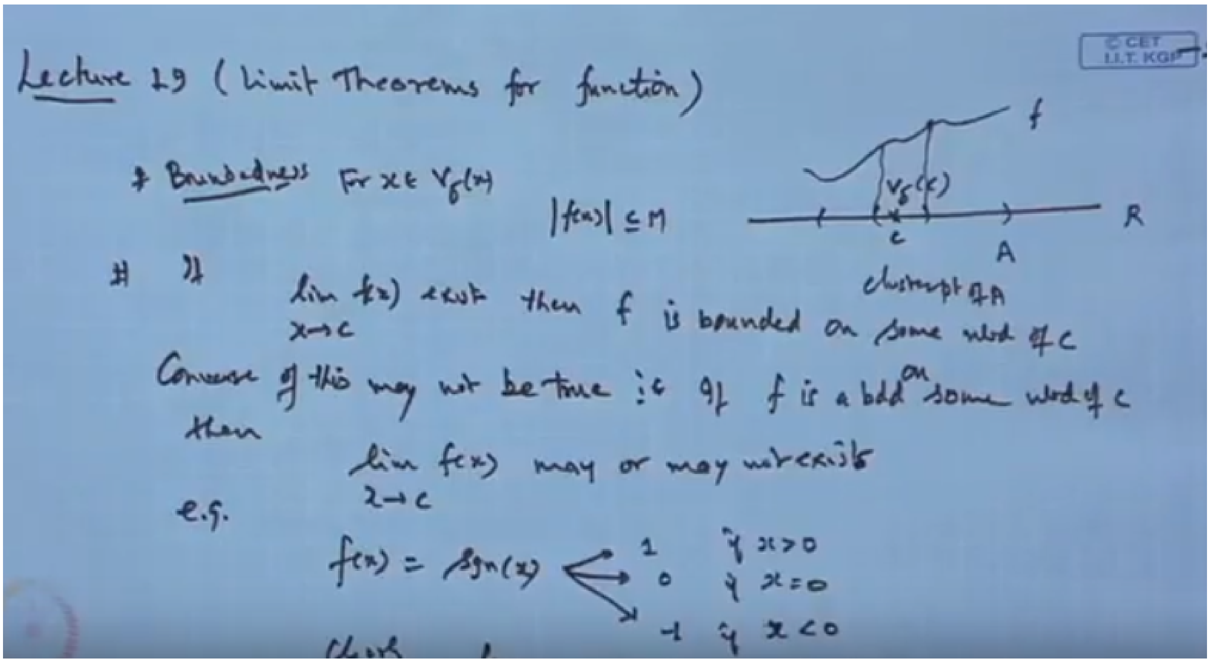

This is the point in R, which is say, cluster point of A and the function f is, defined, on this interval a and only set a, which is subset of R. Then we say the function f is bounded, in the neighbourhood, of a point C, means if there exists an neighbourhood, Delta neighbourhood of B Delta C and a constant M, such that, the values of the function, f X, for X belonging to, this V Delta X, is less than equal to M. Then we say the function f is bounded, over a neighbourhood, around the point C. And one more result, which we have seen. That if the limit of the function exists, if limit of the function FX, when X tends to C exist, exist, an Element, then, f is bounded, F is bounded, on some neighbourhood of C. This also we have seen the result, so, these two things, boundedness, boundedness of the function and this one. Now this result we have proved only one side, if the limit exists at the point C, that a limit of the FX, when X tends to C exists, then only we can say, function is a bounded function. What about the converse part? The converse means, if F is a bounded function, on some neighbourhood of a point, C, can you say the function has a limited C? The answer is, 'No'. So the converse of this may not be true, always. That is, if F is a bounded function, on some neighbourhood of C, then, limit of this function FX, when X tends to C, may or may not exist, may not exist. For example; If we take the function FX, as Signum of, say Signum of X, which we have already discussed, Signum function and we have seen that Signum x, which has a positive, value x, when, well 1, when X is greater than 0, equal to 0, when x is 0, and has a value minus 1, if X is negative. Clear? And this we have seen, that this function, the range of this function is bounded. So in the neighbourhood of the 0, it is basically a bounded set, because we can find a constant M, M is say 1, where all the values of the function,

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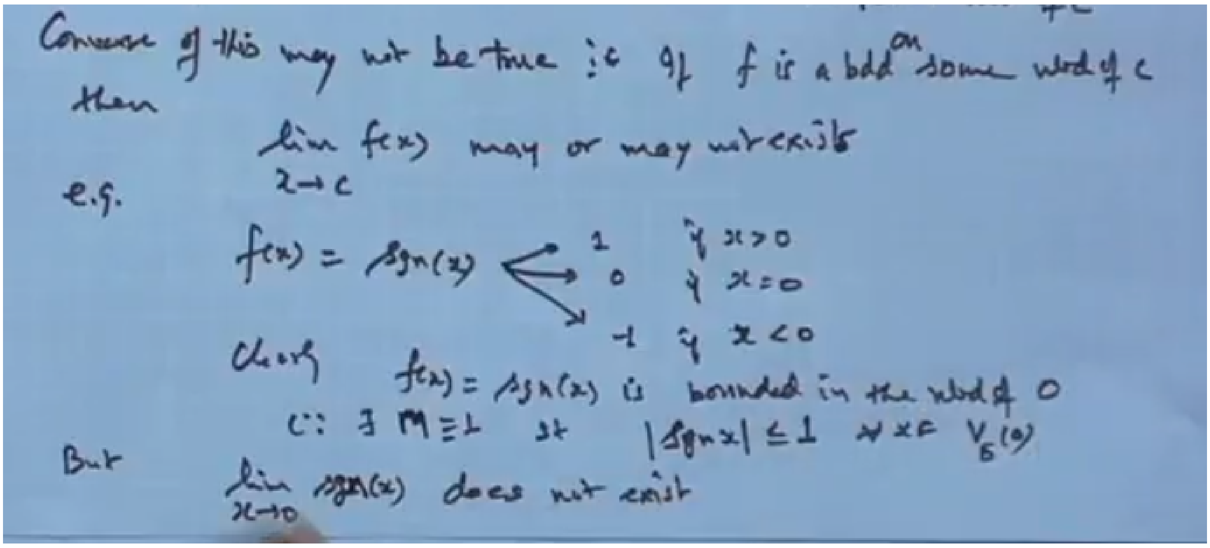

clearly FX, which is Signum of X, is a bounded function, is bounded in the neighbourhood of 0. Because we can identify M, because, because, we can, there exist an M, say equal to 1, such that, the value of this Signum X, you will always remain less than equal to 1, for all X belongs to the, Delta neighbourhood of zero. Okay? But, as we have seen, but limit of this Function, Signum of X, Signum of X, when X tends to 0, does not exist. Because when X tends to 0 from the positive side, the value will come out to be 1, from the negative side the value will come out to minus 1. So limit is not unique or we cannot identify, a delta, such that, the difference between FX, minus the number say 0, cannot be less than a smaller value epsilon, so it does not exceed, So, what we that, the converse of this system. However, if we say the function, if it is an unbounded function, in some neighbourhood of the point C, then obviously, function will not have a limit at that point. Because then the limit does not exist, there are two possibilities. Either the function is not defined at the point, the where the limit is required, it means the function goes to infinity or minus infinity, when X is approaching to C or at the Point C the function is not at all a finite value. Our second case may be, the function has a finite values, but at the limiting value of this, has two values or more than two values, along a different path. So in that case the limit will not be unique, so we say the limit does not exist. So when we say F is unbounded, it means at the Point C, the function is not defined, at that point. So it is just like 1 over, 1 minus X. At X equal to 1, the Function, it tends to infinity, in fact it is not defined, it goes to infinity, unbounded function.

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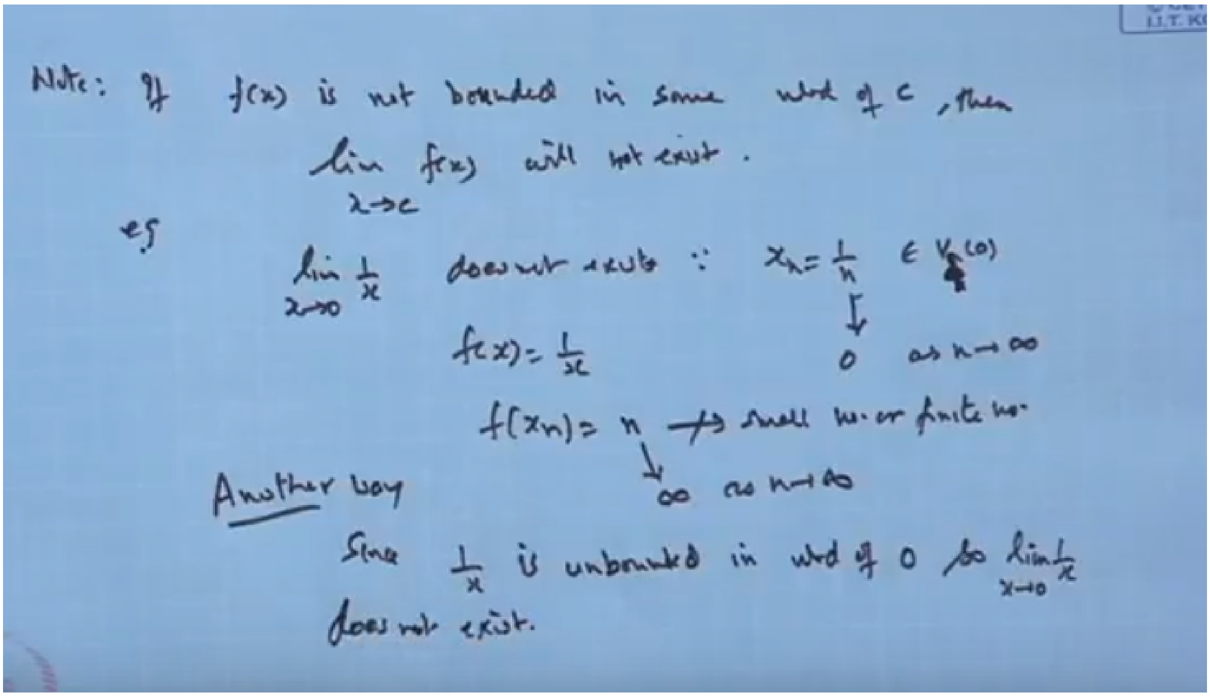

So we can say as a remark or note, if the function FX is not bounded, it is not bounded, in the neighbourhood, in some neighbourhood, of C, then limit of this function FX, when X tends to C, will not exist, will not exist, so that is. Clear? So that will be one of them. And that is region is, for example, if we take the limit of 1 by X, when X tends to 0, suppose I wanted to know whether the limit exists or not. Okay? So this limit does not exist. Why? because we have already seen. Because, if we take a sequence xn, say 1 by n, which is belongs to the sum Delta ,1 by n neighbourhood or Delta neighbourhood of 0, Delta neighbourhood of 0, say Delta is, such so that is the limit, though all the terms of the sequence xn, belongs to i,t after certain stage, say Delta neighbourhood of 0. Then this sequence tends to 0, as n tends to infinity. Okay? So this sequence can, but, what about the function? Function FX is given to be 1 by X. So what is the functional value? The functional value comes out to the N, which does not go to, the smaller quantity, a smaller number or finite number or finite number. In fact it will go to infinity; it goes to infinity, as n tends to 1. So limit does not exist. But this can also be, justified from here. Another way of justifying, that since 1 by X, this function is unbounded, is unbounded, in the neighbourhood of zero, in the neighbourhood of zero. So limit of this function 1 by X, X tends to 0, does not exist. Okay?
So this is one way, okay?

Now, thank you.