Module 6

Lecture – 38

Limit of Functions (Contd.)

Okay, so we were discussing the limit of functions

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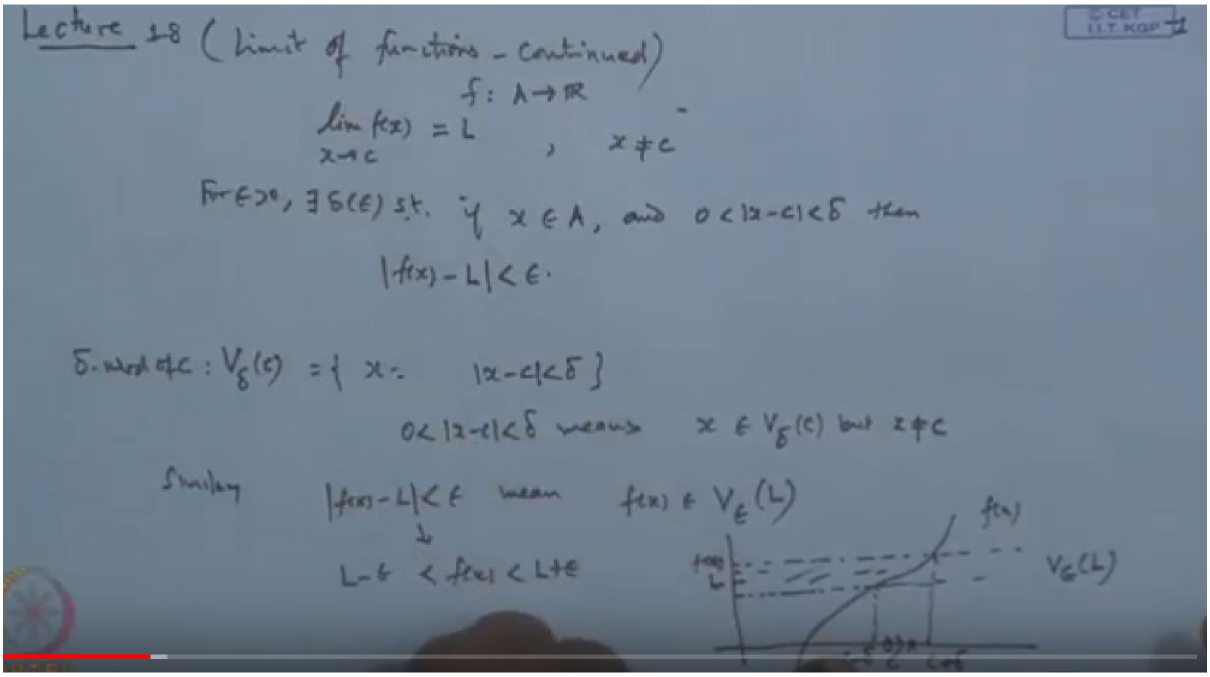

And the way we have defined, either, we say the limit of this function FX, when X tends to C, is say L. Where X is not equal, to say C, need not be equal to C, the meaning of this is, that for every given epsilon greater than 0, for a given epsilon greater than 0, there exists a delta which will depend on epsilon, such that, if X belongs to the set a, a is any set, f is a mapping. F is a mapping, from a to a to R. Okay? So X belongs to a, but X is not equal to C, X and X and X, satisfy this condition, X minus C, is less than Delta. So for a given epsilon, greater than, there exist a delta, depend on epsilon, such that for all X which lies between this intervals, the value of FX, minus L, will remain less than epsilon. So this is the way, we have defined this. Now if we look this definition, then it can easily convert in the form of neighbourhoods. Because what is this? the neighbourhood of C, or Delta neighbourhood of C, which is, nothing but what? If the set of those point X, such that mode x minus C is less than delta. Okay?

Now in this now in this if I remove the Delta C if I remove C then we Say the point X Is not equal to C, so 0 is less than mode X minus C less than Delta means, means X belongs to V Delta C, but X is not equal to C, so it lies in the Delta neighbourhood of C and the FX minus L similarly FX minus L is less than epsilon means that l FX belongs to the epsilon neighbourhood of L, because the X Is not equal to C, so 0 is less than mode X minus C less than Delta means, means X belongs to V Delta C, but X is not equal to C, so it lies in the Delta neighbourhood of C and the FX minus L similarly FX minus L is less than epsilon means that l FX belongs to the epsilon neighbourhood of L, because the meaning of this is that FX lies between Say the point X Is not equal to C, so 0 is less than mode X minus C less than Delta means, means X belongs to V Delta C, but X is not equal to C, so it lies in the Delta neighbourhood of C and the FX minus L similarly FX minus L is less than epsilon means that l FX belongs to the epsilon neighbourhood of L, because the meaning of this is that FX lies between L plus epsilon and L minus Epsilon, So FX lies between this. It means that if we have this function, say this is our Function FX and here is the point C, the value this is L, so what he says is, if the limit of the function FX when X tends to C be mean, for a given epsilon Greater than 0, it means, if I consider a neighbourhood, neighbourhood B epsilon L, that epsilon neighbourhood of L, then corresponding to this neighbourhood. We can find a delta neighbourhood of C, C minus Delta, C plus Delta, where C may not be included in it. Then what this limit says is, for any arbitrary point X, which lies between this interval and different from C, the corresponding image FX, FX will always fall within this   name, ok then we say limit of the function FX exist, ok if it is not then obviously this will not the limit of the function FX when X tends to here will not exist. It means, that for all the Dell will exist some epsilon, neighbourhood of a such that for any X we choose their chorus or there will be an X, where the limit of the value of the function will lies outside of the stretch if it does not exist, so, so what we can that we can also write the equivalent definition of the limit in terms of the we can also express limit concept in terms of the neighbourhood. Okay?

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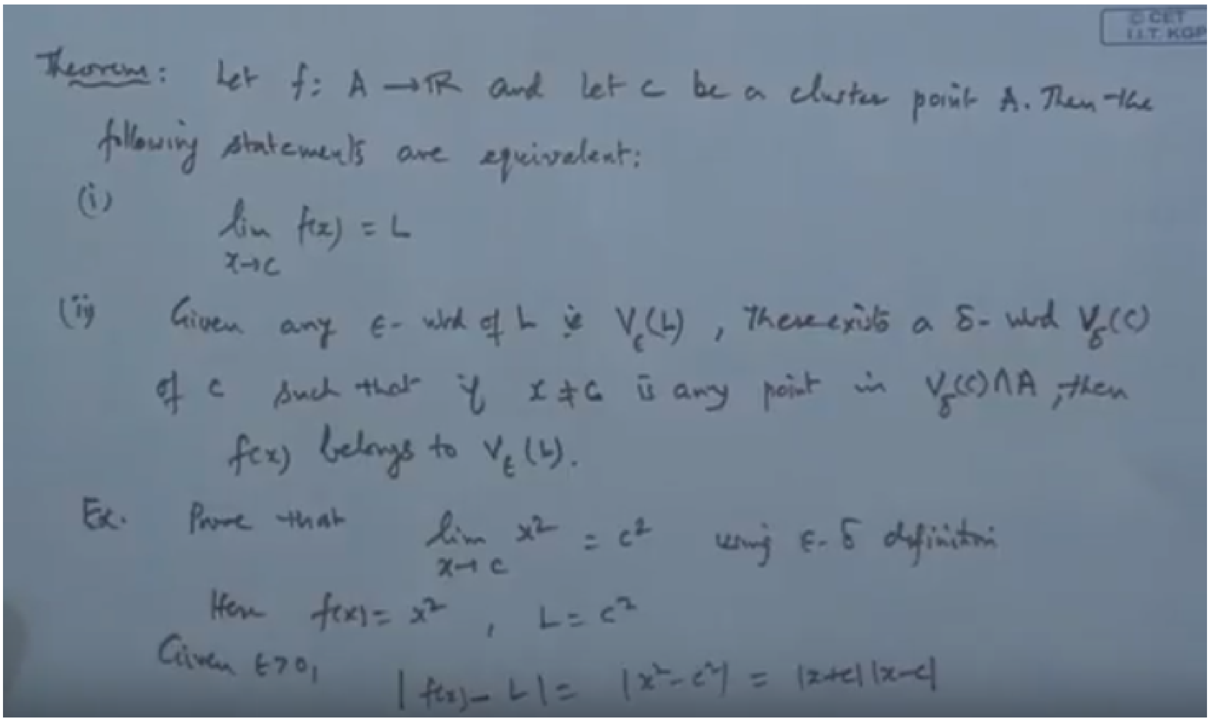

So this we can write in the form of the theorem the theorem is, let F is a mapping from a to R and let's see and let's see be a crystal point of a, then the following statements, statements are equivalent, the first statement is limit of the function FX, when X tends to C, is say L and second statement says, given any epsilon neighbourhood, neighbourhood of L, neighbourhood of L, which we denoted by that is V, F signer l VL, there exist our delta neighbourhood, denoted by V Delta C, Delta neighbourhood of C, such that such that, if X is not equal to C, is any point in the common portion of V Delta C intersection a, in a part of this then F X, belongs to epsilon neighbourhood off L and that what I explained in this figure Is, that for any epsilon greater than Zero, of or in the FC any epsilon neighbourhood of F correspondingly we can find out the Delta neighbourhood of C, such that if X, which is different from C lies in this interval then the image will fall here, that's the meaning, so both are equivalent definition. Okay?

Let's take few example using the epsilon Delta definition, suppose I wanted to so that prove it, the limit of the function X square, as X tends to C, is C square, it's C square, using epsilon Delta definition. Okay? So what is our function? So here the function FX is, X square, L is C square. Okay? So what we want is for epsilon we will choose first, so let epsilon I'll be given, so given any epsilon greater than zero, then identify the Delta we have to identify Delta, so that this condition holds. It means, we want this   difference FX minus L should like should lie in this neighbourhood, so what is the FX minus L so FX minus F L means, C L that is equal to X square minus C square, basically this is X plus C into X minus C, this will be the okay, now when we take X sufficiently close to C, then the limit of this must be C is called that one, so house of how close she is means any number which is less than one we can choose, so suppose

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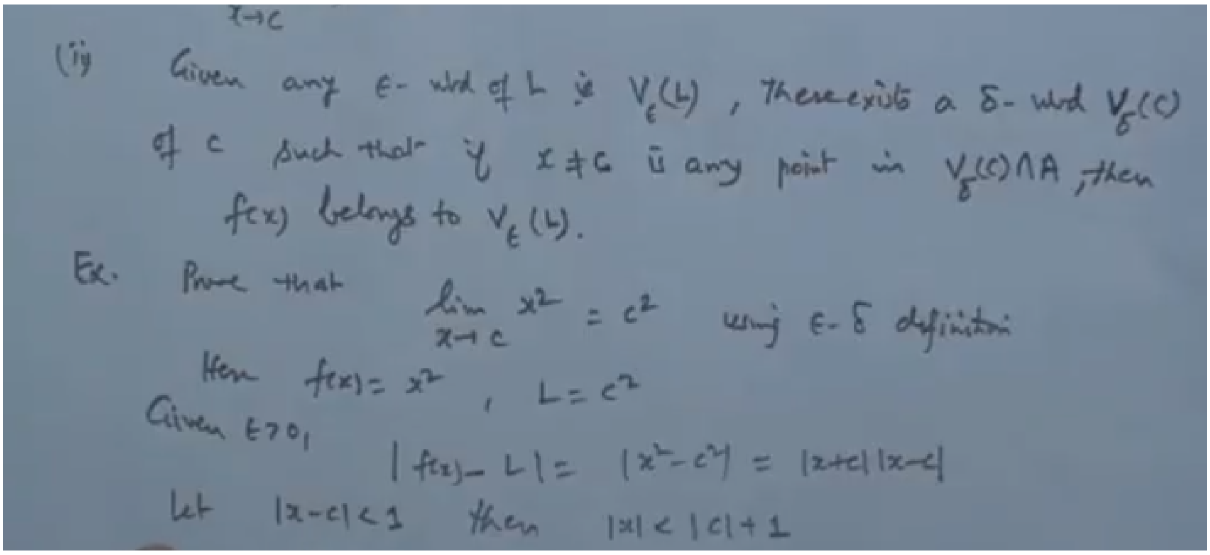

I take let us take mode X minus C less than one first, why because this X in this case X is lying close to C, which is less than Okay? So if I take this one then, the want for this X plus one, then we get from here is or mode X, can be less written as this because mode X minus 1 is less than one, so it is greater than equal to mode X minus mode C and then mode X will be less than is not C plus one. Therefore, therefore mode X plus C,

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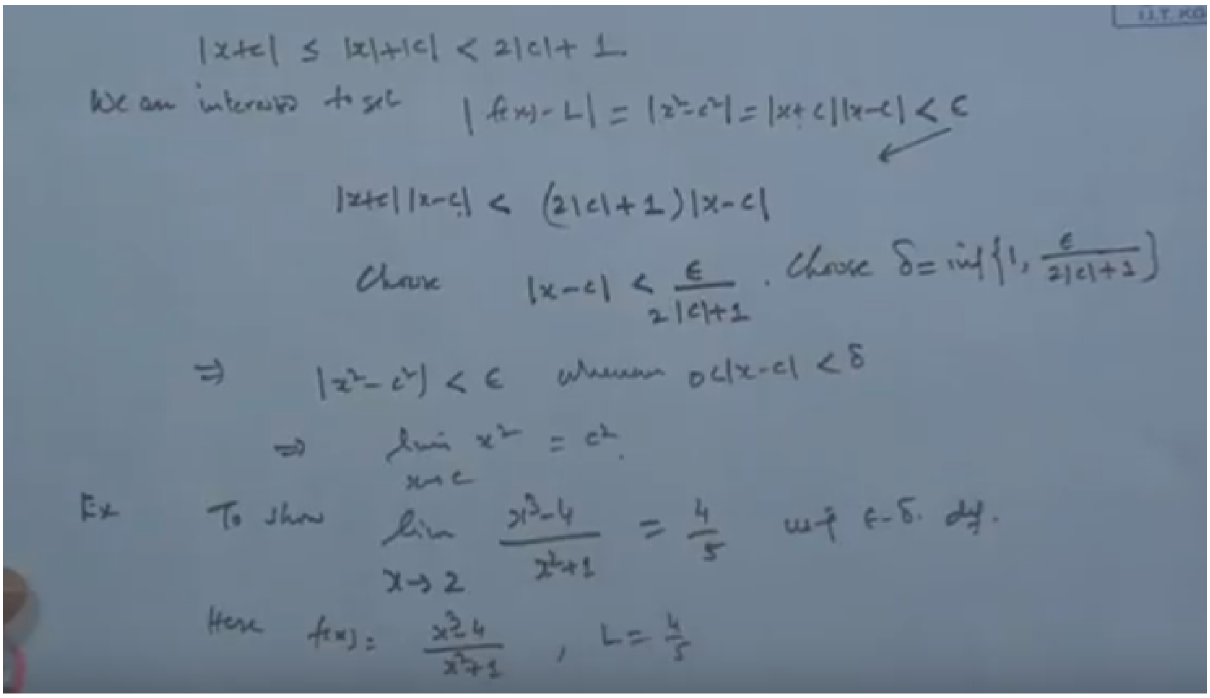

the value of this will be, less than equal to mode X plus mode C, but mode X is less than mode C, so it is more C plus 1, this we get it from here, just this by manipulation we are doing so that we can get the bond for it, now what we are interested is, we are interested, we are interested to get this FX minus L, which is the same as X square minus C square, which is the same as X plus C, X minus C to be less than Epsilon, we are interested in this, now X plus C bond is already less than 2, so basically this mode X plus C, mode X minus C, when mode X minus C is less than 1, then the bond for mode X is obtained from here, so we can get this is less than basically 2 times mode C plus 1, this is less than mode 2 times mode C plus 1 into mode X minus C. Okay? Now this entire thing, we want it to be less than Epsilon this we wanted, so if I choose, mode X minus C to be less than epsilon over 2 times more see plus one and then use this bond here, then what we get is this element less than Epsilon, it means this will imply that, mode of X square minus C Square will remain less than Epsilon. Is it right? So that is FX minus L is less than Epsilon, but what should be the Delta.

So it means that if we picked up from here, then from here then choose Delta, to be the value, which is infimum of infimum of 1 and epsilon over 2 more C plus 1, because you are getting the 2 want for Delta 1 is X minus C is less than Delta, Delta is 1, here also you are saying it's minus C is 2 this, so if I choose the Delta which is minimum of these two, then for this particular Delta this result will also hold, this result will also hold, that is for a given epsilon we can identify this Delta, that if any X belongs to this, a mode X minus epsilon are less than Delta, then this will continue. Whenever mode X minus C is less than Delta and of course greater than 0 because not, so what is so this shows the limit of this function X square when X tends to C is C square, that's the answer for it. Okay? So men idea is mainly you when it is asked to prove, or estimate is the limit by using the epsilon-delta definition, what we consider we start with the given f sign and then with the help of extract now we will identify delta which depends on epsilon, so the use idea is, that find out FX minus L, try to get the bond for this in terms of F mode X minus C less than some number, so this bond can be obtained as this, so Delta we can identify, once you find it by Delta obviously it depends on epsilon so corresponding to this epsilon if I choose this Delta, it means if I take this neighbourhood X minus C less than Delta, so obviously this Delta is also less than one as well Delta, is less than this number so we can take the 1 we can definitely find out the bond for this and one can say this limit, if this difference is less than Epsilon and it's the proved, so this is the way to compute the two so the limit in by using the epsilon Delta definition, let's take one more example, we are, we are also using some trick to get, suppose I asked to find two so, limit of this say X cube. Okay? X cube minus 4, divided by X square plus 1, when X tends to, 2 is 4 by 5, suppose we are interested, using two so this using epsilon Delta definition. Okay? Suppose I don't say using Delta, because these two so you can substitute X equal to 2 and one can get the value of this each, but if it is asked to you apply the epsilon Delta definition then you have to start with a given epsilon greater than 0 and then suitably identify the Delta that's. Okay? So what is a this is our function FX, here FX is, X cube minus 4 over x squared plus 1 and L is 4 by 5,

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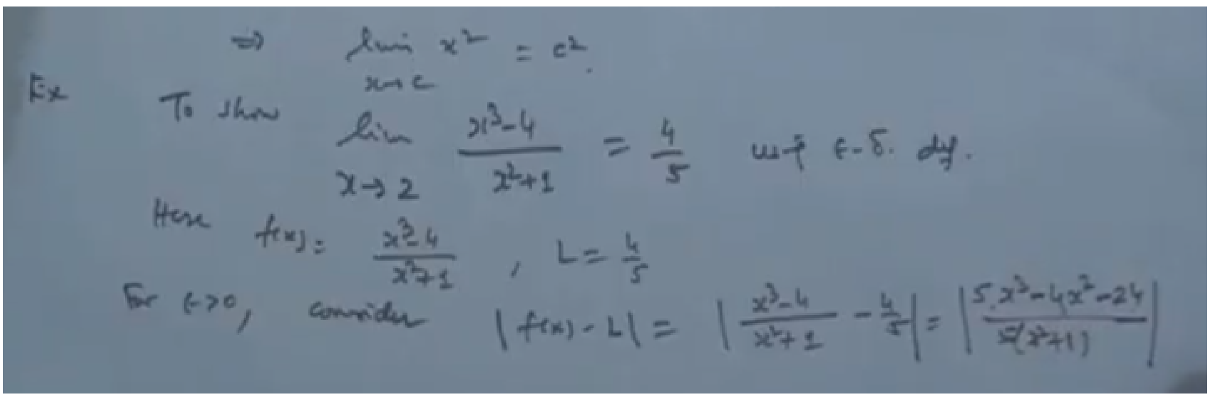

so let for, given epsilon greater than 0, let us say consider first the difference of FX minus F, this FX minus L is X cube minus 4 X square plus 1 minus 4 by 5 and if we find that Soviet, then the value will come out to be 5 X cube, five X cube minus four X square minus four x square minus 24 divided by 5 X square plus one this model, now here the numerator is one unit higher than the denominator so what we do first we I separate out the factors, now if we looked at genome numerator X equal to two satisfied if X is two means 80 minus 16 minus 24, so it's only 40 - 60 miles it satisfies means x squared - 2 will be a factor of this equation equal to zero, so we can separate out the X minus 2 factor from the numerator,

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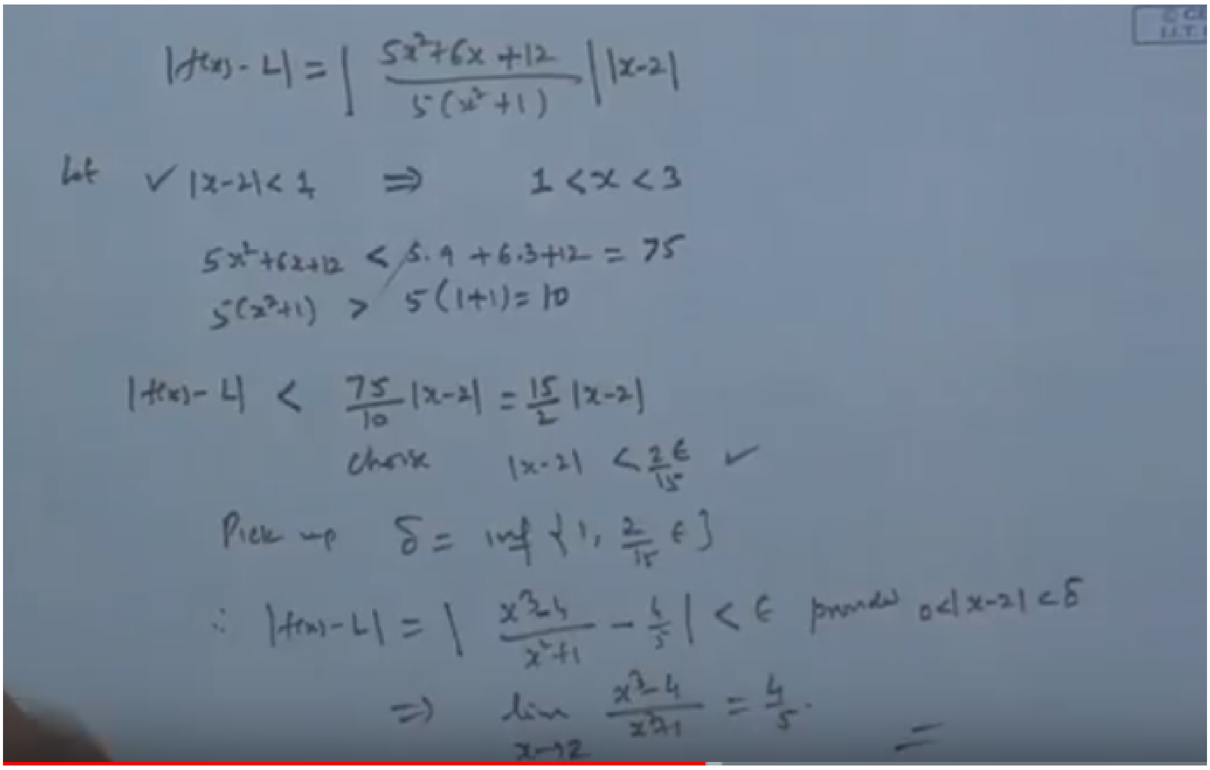

so if I separate out the seppings will come out to be mode of FX minus L, this is nothing but mode 5 X square plus 6 X plus 12 divided by 5 X square plus 1 into mode of X minus 2, after health there are no problem, now we want it to estimate the limit of this is in fact FX minus L must go to zero, when X is sufficient tending to 2 this is about problem so it means X minus 2 should be as small as we please and X may not be equal to 2 or so, so let us find the bond for this first bond for this requires the value of X upper and lower bound for X, so how to get the panel so let us take X minus 2 is support less than 1 let us take this one, restriction so if we restrict this 1 then what will be the X, X will lie between 3 and because X minus 2 is less than 1 so X is less than 2, or 2 minus X so again it is so it lies between X lies between 1 & 2 1 & 3, so this will be the bound for this is an X minus 2 right between minus 1 and plus 1 so minus 1 plus 2 will give this 1 and we get this, ok so this is the 1 all this. Okay? It’s good we've got two minus X less than 1 so 2 plus 1.

Okay? So that will be greater, so this one it means X is lying between this, so what will be the upper bound for this, 5x squared plus 6x plus 3 this will be less than X is 3 bounded so it is 5 into 9 is plus 6 into 3 plus 12, that is the upper value for this upper bound for this and that upper bound a we can either that is equal to what 75 and what is the 5 X square plus 1 because in the denominator so write down the lower bound of this, so it is greater than 5 into 1 plus 1 that is 10, it means this mode of FX minus L can we made less than 75 by 10 mode X minus 2, that is equal to what is nothing but the 15 by 2 mode X minus 2, so first one we have written like, this that is Delta 1 you can say, second bond we will start from here because this entire thing we wanted to be less than Epsilon, so if I choose mode X minus 2 is less than say epsilon 2 y 15, less than this so as soon as you write this thing then whole quantity will remain less than Epsilon, so now picked up the Delta as the infimum of 1 and this number, then with this delta so if we take this Delta, as this then this condition is satisfied, as well this condition is satisfied, therefore mode of FX minus L that is equal to what? Mode of X cube, the problem was X cube minus 4 divided by X square plus 1 minus 4 by 5, this thing can will be less than F signer provided the X minus 2 is less than Delta and greater than 0, because X may not be 2 but here of course it no problem we can also take 2, there no problem here ok, so we can choose therefore the limit of this ,X cube minus 4 over X square plus 1 as X tends to 2 is 4 by 5 and that is proved. so this way we can prove it, the that is our limit problems so this is the fourth ok.

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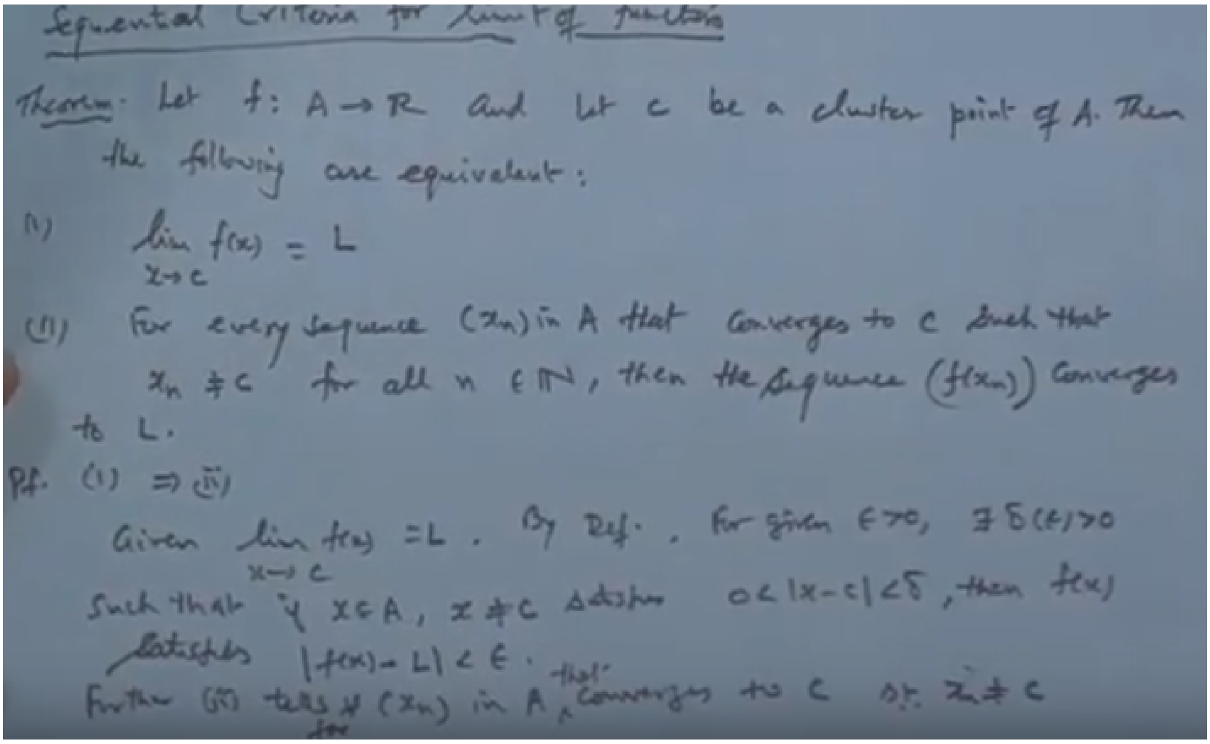

Now there is another way of defining the limit, that is the sequential definition of the limit, sequential criteria, sequence here criteria for limits of functions, limit of functions, CK so what is the sequence again this is the form of theorem, the theorem says, let F mapping from a to R and let's see and let's see be a cluster point. Cluster point of a, then the following are equivalent, the first days limit of F, FX when X tends to say C each L, second is the for every sequence for every sequence xn in a that converges to X, converges to C sorry such that, such that xn is not equal to C, for all n belongs to capital n natural number, then the sequence then the sequence, F of xn this sequence converges to L, converges to so both are equal in turn it means if the limit of FX when X tends to a see the L means for any given epsilon greater than zero such that FX miner less than F signer whenever mode X minus C is less than X is different from C, this is equivalent to say there will be a sequence in a which goes to C and then the corresponding image F xn will go to L, that's proof so first, first we will so first implies to, given limit exists, limit of FX when X tends to C is n, so by definition when the limit exists, so by definition what is that definition is that for a given epsilon greater than 0, by definition for given epsilon greater than 0, there exists a delta which depends of sign or positive below such that, such that for all X, such that if X belongs to it X is not equal to C and satisfy this condition, zero less than mode X minus C less than Delta, then FX satisfies the condition mode FX, FX minus L is less than Epsilon. Okay? This one, this is given, now further we wanted to show 1 implies 2, so 1 is given that we get this m uch information, now let us say the two, from two what is known is, we know the sequence xn that converges to C, so let us say given further from 2 so tells sequence xn in a converges, every second for every sequence in a that converges to C, for every sequence xn in here that converges to C, such that xn is not equal to C, so when xn converges to C it means for again for a given say small number say I take the Delta for a given Delta we can identify a capital M, such that when M is greater than equal to K, capital K or capital M then difference of xn minus C will remain less than Epsilon, so by definition.

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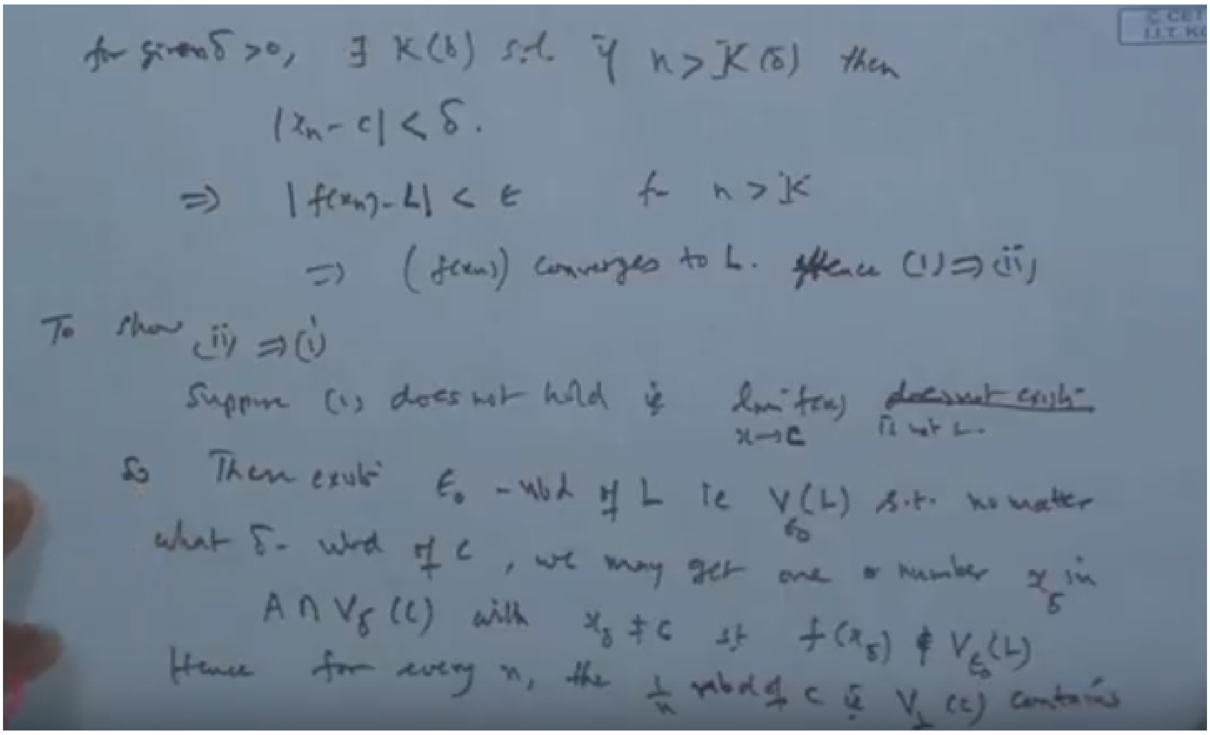

so for given so for given Delta, greater than 0 there exist a K depending on delta, such that if n is greater than K, n is greater than K which depends on Delta then mode of xn minus C will remain less than Delta, is it not that by definition when xn converges to C, this is by definition too, but for this particular xn the result of the previous say this FX tends to alpha limit says this condition, when all such X satisfy this condition the mode of FX minus L less than epsilon, so this X X n will also satisfy this condition, therefore from the previous mode of fxn minus C will also remain less than say F xn minus this is less than L, Epsilon is it not for such X, X n this will satisfy this condition. Okay? And this is true for word for all N greater than capital K, so it means the limit of FX and this is L, sorry limit of affection is nothing but L, so this shows the sequence F xn converges to that's one, second converse apart to so, so this implies so hence 1 implies 2, 2, so 2 implies 1, this we proved by contradiction, suppose two is true holds, but one is not true, suppose 1 does not hold, it means that is the limit of this function FX when X tends to L, X tends to C that's not exist does not exist, or does not go to L, is not L so for this limit is not equal to L. Okay?

It does not exist, or is not L. Okay? We can say like this, so if it's not L, or different from L it means what do you mean by that? It means that if 1 is not true then there exist, so, so there exist, epsilon nought neighbourhood of L, that is V L F final not, such that no matter, what Delta neighbourhood of Delta neighbourhood of FC, we made we pick up there at least one number we may get one number say X, depends on say Delta, in this set a intersection V Delta C, with X Delta is not equal to C, such that f of X Delta does not belongs to the neighbourhood epsilon not neighbourhood of L, that is the meaning of this, when the limit of this does not exist, if I go through again let's say the definition of the limit will be medic criteria you mean this one, first month when we say the limit of this exist is equivalent to this, it means for a given epsilon, that is when we take the epsilon neighbourhood of it correspondingly we can find a delta neighbourhood such that image of any point inside is, let us think that can support limit is not equal to L, does not exist or is not L, it means if we picked up the sum F now there will be at least some epsilon not neighbourhood of L will definitely exist, such that whatever the Delta neighbourhood we choose, whatever they there will definitely one point whose image will not fall inside It, that's the exact meaning of this, so we get this, does not work it means what that is for every hands for every n, for every the 1 by n neighbourhood of C, that is V 1 by n C, contains a number contains a number, X Delta this number X Delta or X n,

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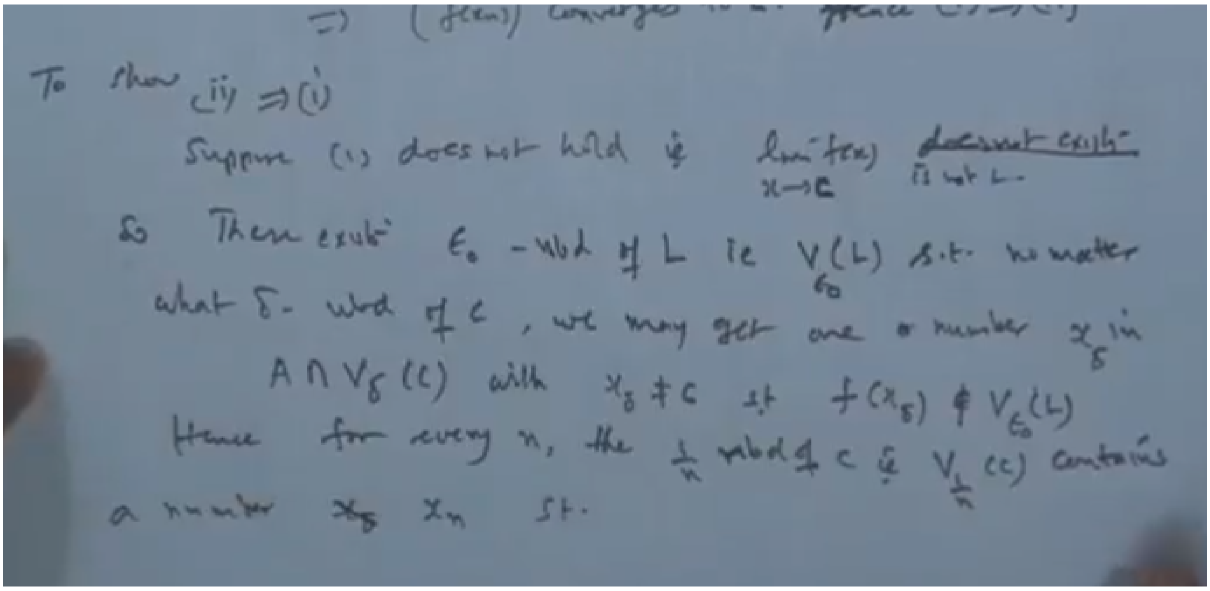

Contains a number X Delta that is X N, or we can say X F corresponding to this X n, such that such that,

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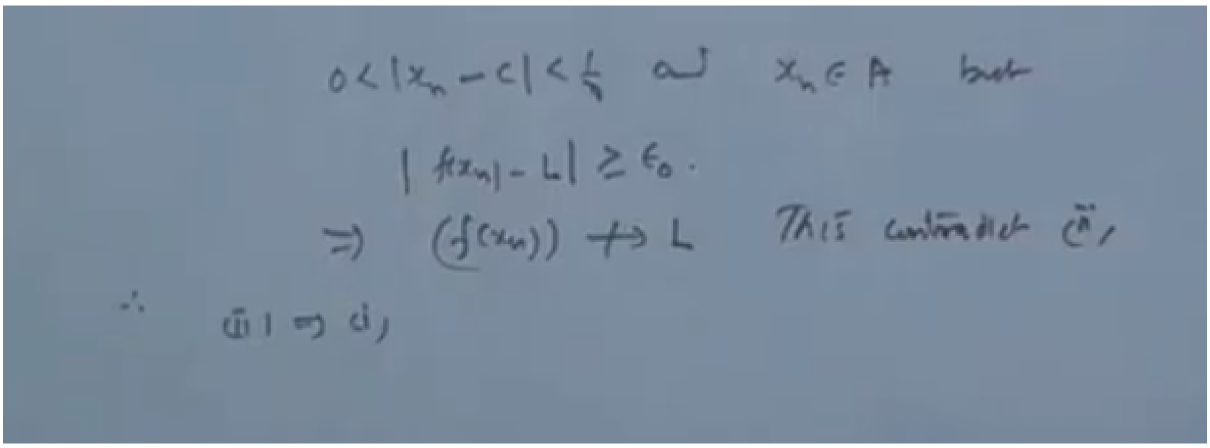

0 less than mode xn minus C less than 1 by N and X n belongs to a but mode of fxn minus L is greater than or equal to epsilon not, that's the mean is it not, so there does not what means for any there exists epsilon or not neighbourhood of F such that whatever the Delta neighbourhood of C we choose, there

will exist one point xn whose image will not fall within the epsilon neighbourhood of F so that's ,what this source the sequence xn is not this implies that F xn will not converge is it not so this implies that the sequence fxn will not converge to okay, but that contradiction, that is xn it does not converge to it but that contradiction, because with this contradicts the assumption, because ii ism son says that for every sequence xn n that converges to C such that for all n the sequence FN converges to L, this we are assuming and we wanted to prove one, so what we were we have instead of showing that one is to be are assuming two is to but one it does not hold, so one does not hold, that leads the contradiction of the two, therefore our assumption is wrong so if 2 is to 1 will differently will hold, so definitely 2 implies 1 so therefore 2 implies 1, so this completes the proof ok so that's what now.