Modeel 6

Lecture – 37

Limit of Functions and Cluster Point

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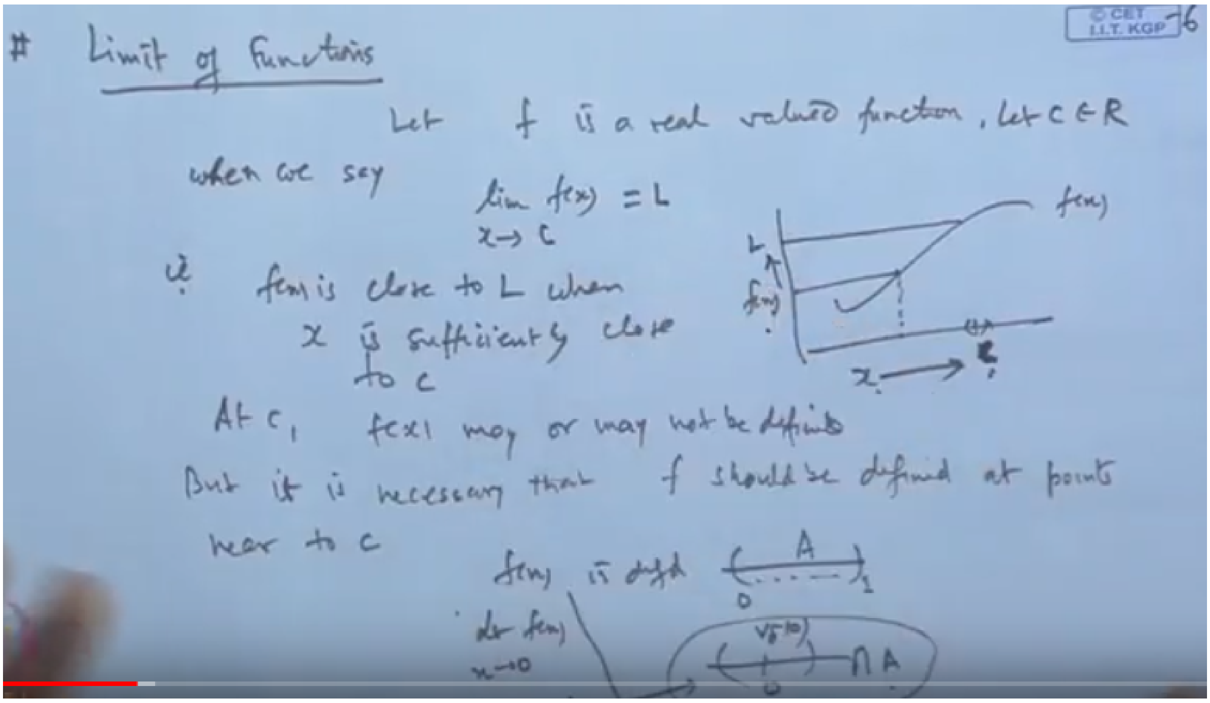

So next topic you say, limit of functions. First, what is the idea of this? Suppose f is a real-valued function. Let f is a real-valued function. That is, FX, when X belongs to R, is a, an FX is also real. When we say, when we say, the limit of this FX, when X tends to a sum, number C, where C is, any real number, say R, where the. Okay? When we say the limit of this is L. The meaning of this is, that, if this is function FX, I am assuming to be continuous function of course, and this is our C, sorry, C and X, is a point, which converges to C. Then correspondingly the functional values f of X, is tending to L, is tending to L. When X approaches to C, the limiting value of this FX, is L. At the Point C, the function may or may not be defined; function may or may not be defined. But still the limiting value of this FX, we say it is L. It means the difference between L and FX, is very very small, come, when X is sufficiently close to C. It means that is, the FX is close to L, when X is, sufficiently close to C. That is the meaning of the limiting behaviour of the function FX, where at the Point C. At the Point C, the function at C, the function f X, may or may not be defined. It is not necessarily defined. But what is used a important part is, what it is necessary, necessary, that the point, at the necessary part is, that F should be defined, F should be defined, F should be defined, at points, near to C. Then only, limiting, limit of the FX has some meaning. If the function is not defined, at the point closure to C, then we cannot approach, the value of FX, when F is approaching to C. Because there are so many point, where the function is not defined, we cannot find out the F X. So the important part or necessary part is, that in order to get the limit of the function at a point C, the necessary condition is or necessary part is the function must be defined, at all points, which is very very close to C. That is the important point.

Second point which we get, that at the point C, the function, the C, may or may not be available in the set a, may or may not be available in the set a. Still, we can find the limiting value of FX, when X approaches to C. Suppose I say, the function FX, say FX is defined, over the interval, say 0 to 1, and we find, the limit of the FX, when X tends to 0. So though the function is not defined at the, so the zero point is not available in the set a, but still this has a meaning, because, the function is defined, at every point, in the interval 0. So what it says is, the Point C, may or may not be point of a, but even if a, the limiting of the FX is possible. We can ignore that point, that whether it is in C or not. What we are interested, that we will find out the neighbourhood of the point a, neighbourhood of the 0, this is the 0, neighbourhood of the 0, with a suitable radius delta, and then find the intersection with a. In this, all the point in this set, consider the functional value at this point and take the limit, when X approaches to 0. If this limit exists and equal to L, we will say, the limit of the function exists. Okay? So that is what. So note we can write it, not like this.

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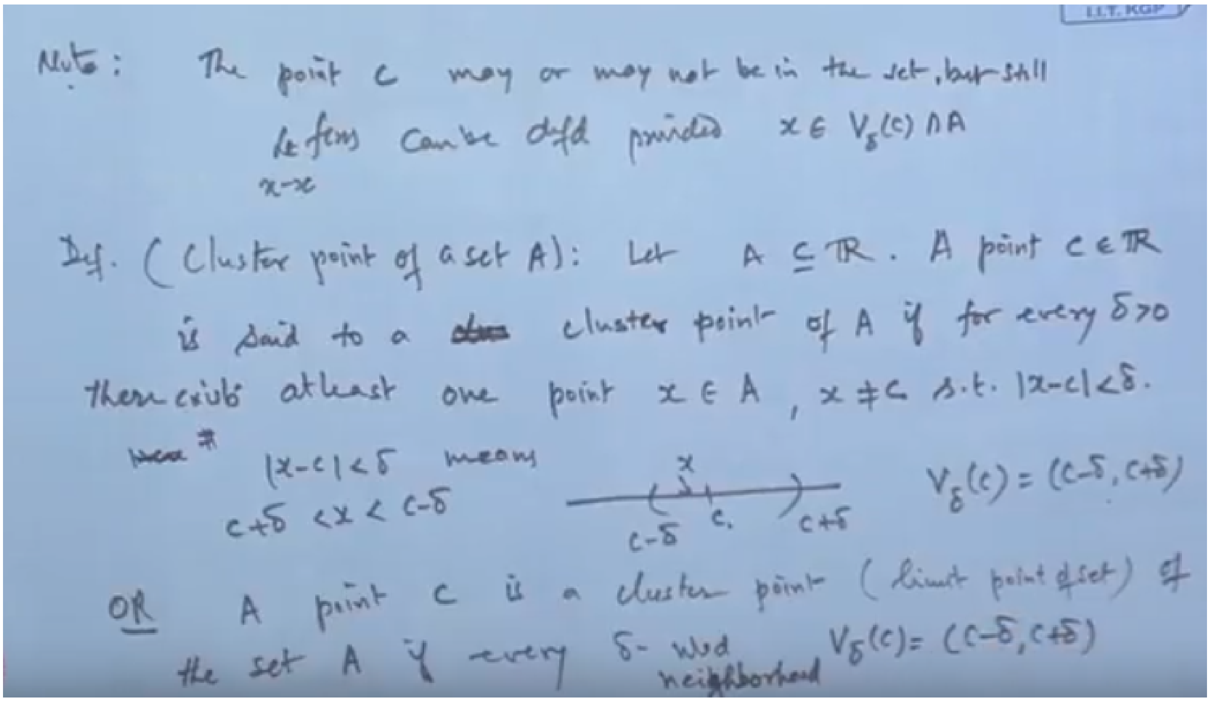

The point C ,the Point C. Okay? May or may not be in the, the Point C, may or may not be in the set, be in the set, but still, but still, still the limit of FX, when X tends to C, can be defined, provided,, provided, X tends provided, X belongs to the neighbourhood of C, intersection A. Okay? Function is defined, for this. Okay? So that is the one.

So before going for this limit, let us see the concept of the, Clustar point, because, always, this limiting value, that we will consider C as a Cluster point, for this set. Okay? So let us see the definition. Cluster point, cluster point, of a set A. Let A, be a non-empty subset, of R, be a non-empty subset of R. A point, C, a real number in R, is said to be, said to be, a cluster point, in C, cluster point, said to be a cluster point of A, If for every epsilon, every Delta, greater than 0, there exists at least, there exists at least, then exists at least, one point X belongs to A, which is different from C. Such that mode of X minus C, is less than Delta. Okay? So what is the meaning is, that a point C is called the Cluster point, of a set A if for every Delta greater than 0, there exist at least one point X, such that X minus C is less than Delta, where X is. So this meaning is, X minus C, mode of this, less than Delta. Obviously means, means, that X lies between C minus Delta and C plus Delta. It means, there exist a neighbourhood, around the point C, it is a neighbourhood, this we denoted by B Delta, C. A neighbourhood, which is, C minus Delta, C plus Delta. Okay?

So neighbourhood can be obtained, this. In which, at least there exists one point X, different from C, differently from C, C and this is true, always for each delta. Okay? Then we say C is the limit. So C is the cluster point of this set A, if when we picked up the neighbourhood of Delta, then at least one point X, will belongs to this, different from C. In other words you can say, a point or we can also define terms of the neighbourhood, a Point C, a Point C, is a cluster point, cluster point is also called the limit point. Okay? It is cluster point or maybe the limit point of the set. Cluster point of the set A, of the set A, if every delta neighbourhood, if every delta neighbourhood, NBD, neighbourhood, every delta neighbourhood, n e i g h, neighbourhood, neighbourhood, neighbourhood, NBD, we could use NBD, neighbourhood, V Delta, C. That is, C minus Delta, C plus Delta, every neighbourhood of this, of C contains, contains,

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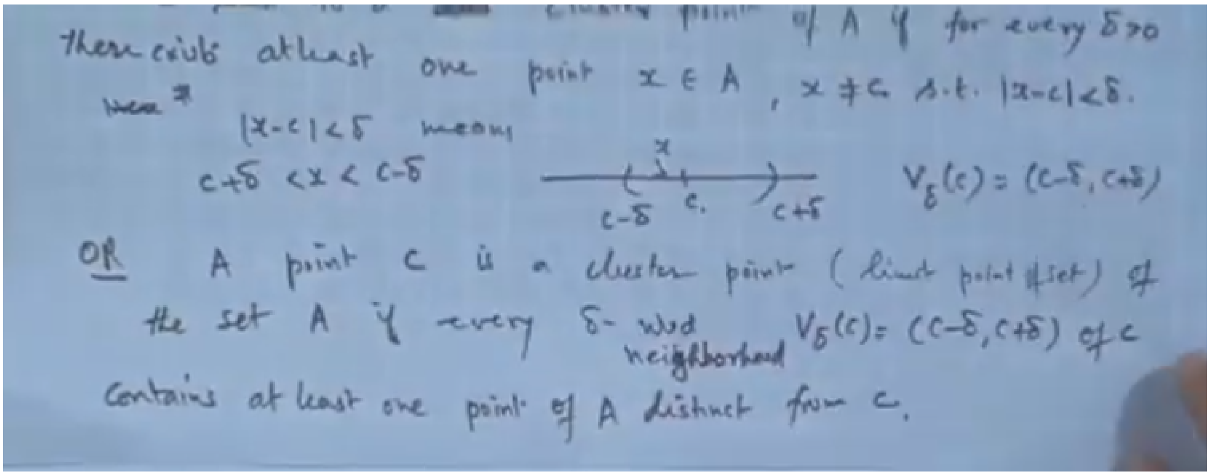

every neighbourhood contains, at least one point, at least one point of A, distinct from C. Okay? Then we say, C is the Cluster point of this. Okay?

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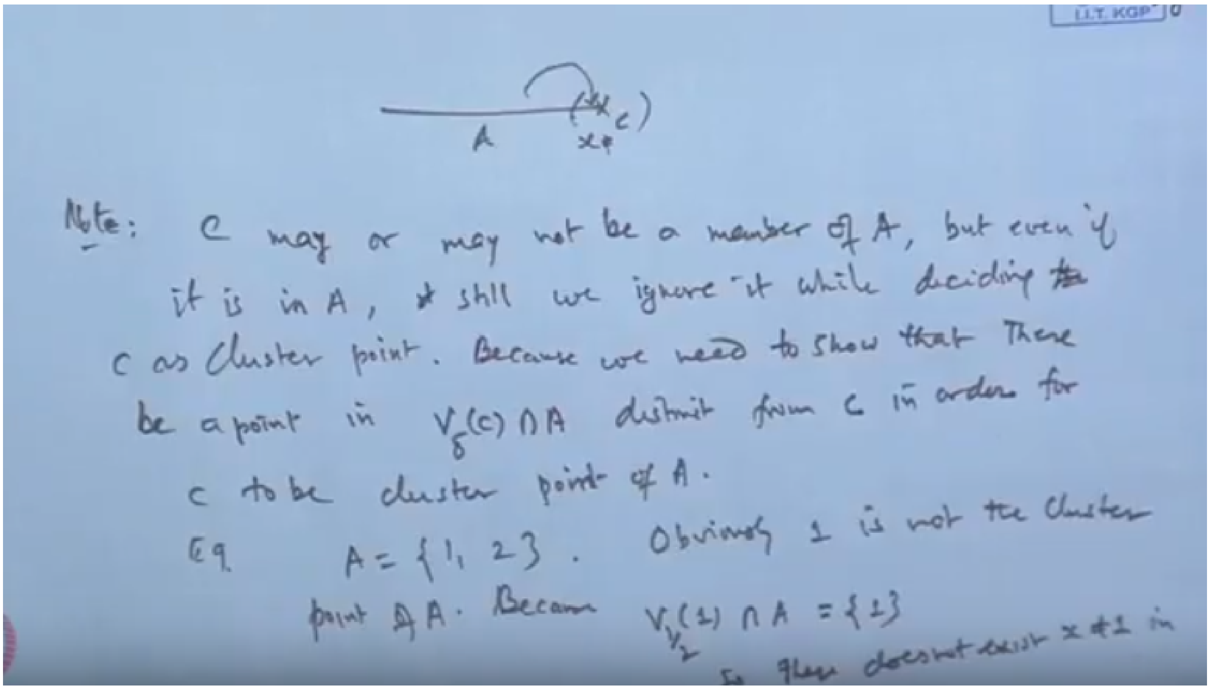

So basically, this is our suppose set A. C will be somewhere here. Now this will be the cluster, if we picked up a neighbourhood around the P, then there must be at some point X, of A, different from C, available. Whatever, howsoever, small radius we choose, then, then. Then we say C is the Cluster point. The point C may or may not belongs to A. So that is the important part is, C may or may not belong, may not, be a member of, be a member of A. If it is in the set A, then also we can ignore it, then. Even if, but, even if, it is a member of A, it is in A, it is in A but still we can ignore ,then still we ignore it, while deciding, while deciding the Cluster point, while deciding, deciding C as a cluster point. What is more important is, that be picked up, we simply picked up, whether it weakest point is. Okay? What we see here, we recall that is, because, we need, we need, to show that there is or there be a point, there be a point, in the Delta neighbourhood of C, intersection A, that is important part ,distinct from C, in order, in order for C, to be a cluster point, cluster point of A, that is important. For example, suppose I say A is a set, having two elements 1 2, then 1 is not a cluster point, obviously, 1 is not a cluster point. Why? Cluster point of A, because, what we see, if I take, that neighbourhood of the 1, which centres at, half, with radius half and find out the intersection with A. Then this set contains only the single point 1. So there is no other point, so, so there does not exist, X which is different from 1,

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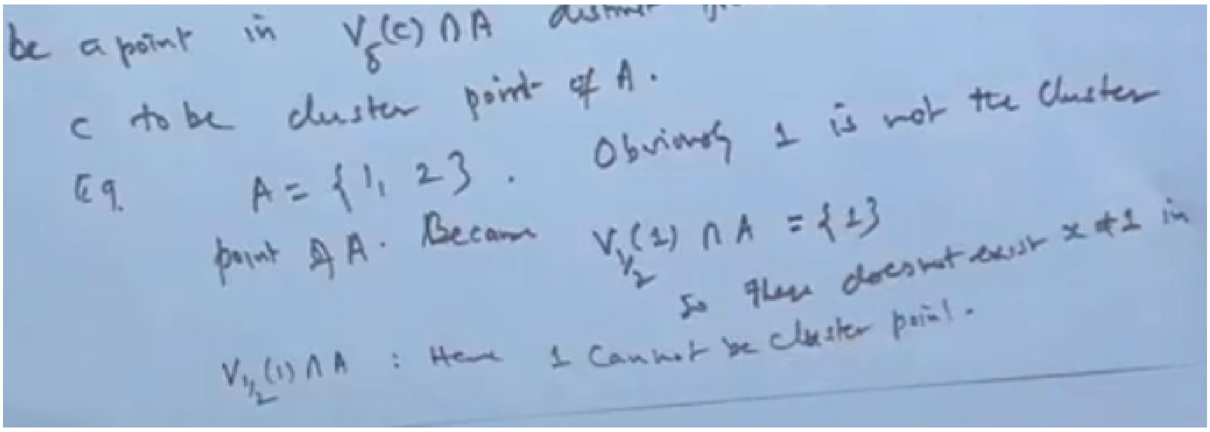

in this set, in this, in this collection, V, 1 by 2, 1, intersection A. Hence 1 cannot be cluster point. Similarly two cannot be cluster point on this. Okay? So this way we can identify for them, okay. Now there is a relation and we can also define the cluster point, in terms of the sequences.

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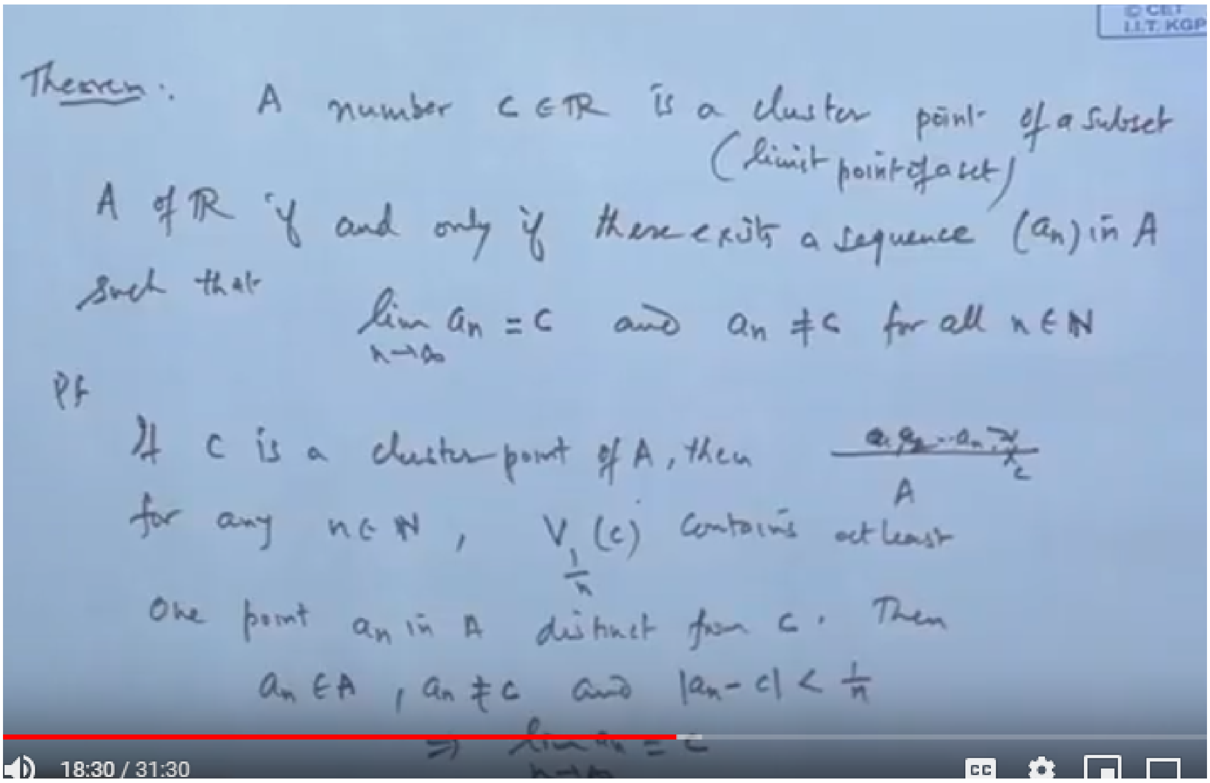

So this result is, which connects the, concept of the convergence of the sequence and this cluster point. A number C, belongs to R, is a cluster point, cluster point, of a subset, of a subset, cluster point or limit point, same. Okay? Limit point, of a set. Cluster point of a subset A of R, if and only if, if and only if, and only if, there exists a sequence, a sequence a n, in capital A, such that, such that, limit of n, as n tends to infinity, is C and an's are different from C, for all n, belongs to capital N, set of natural number. So that is the important part is. Okay? proof, of this. It means, if C is a cluster point of a subset a of R, then there will be a sequence, available in a, if this is our C and here is a, so there will be a sequence, X 1, or a1 a 1, a 2, a n and so on, which will converse to C, which will go to C. But an's will be different from C. That is, if I draw a neighbourhood around the point C, then you get the an's different from C, for any neighbourhood we get at least one an's available in this. So let us see the proof. If C is a cluster point of A, then by definition, if it is here, then by any Delta neighbourhood of C, must includes the some point, which is different from C.

So Delta I am choosing to be n. Then for, one by n, then for any n, belongs to n, the 1 by n Delta neighbourhood, of C, the 1 by n Delta, 1 by n neighbourhood of C, Delta becomes 1 by n, contains at least, one point n, which is different, distinct in, a n and A, distinct from C, this by definition, of the, cluster point? So what we get? Then we get an's, these are in A, an's are different from C and the distance of a n minus C, is less than 1 by n, and this is true for each n, there will be some answer. So this implies, the limit of n, as n tends to infinity is C. Okay?

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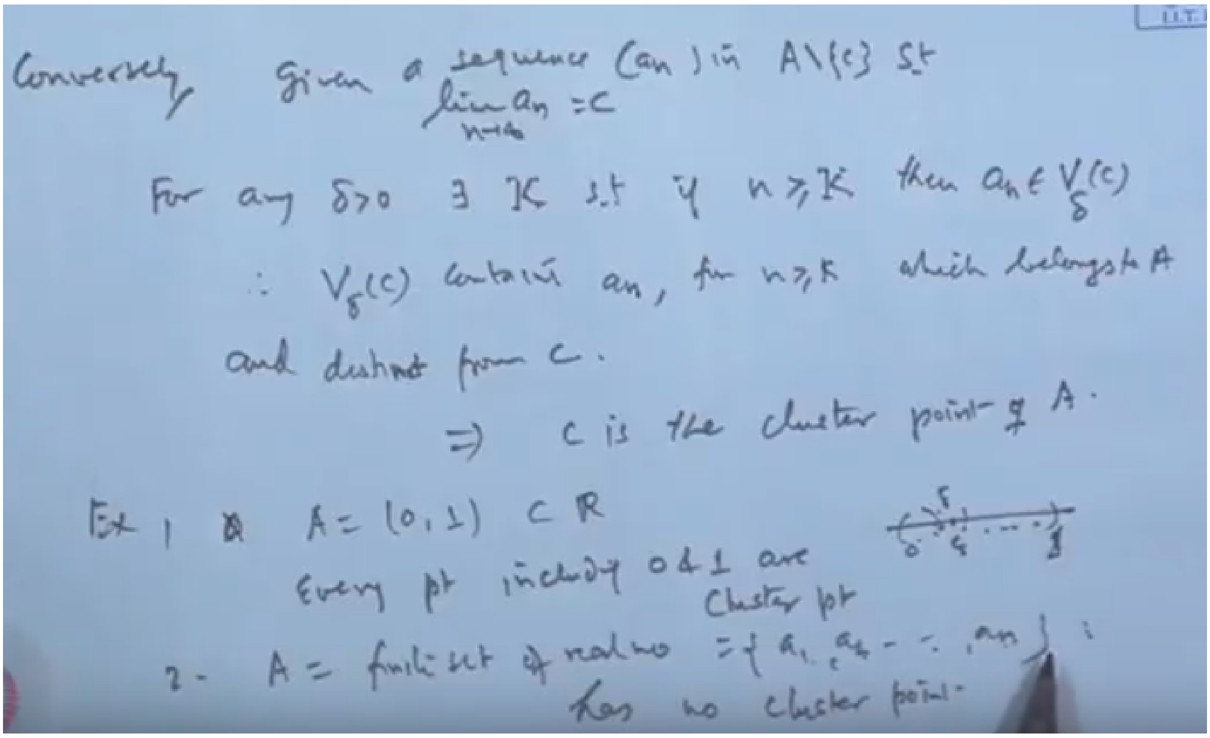

Conversely, conversely, if suppose the limit of an, is given to be C, given that limit of an’s, as n tends to infinity C, if there exists sequence a n, say such that limit is a n. Okay? With the. So conversely, give a sequence a n, given a sequence a n, in, the set A minus this C, such that, limit of this a n, is C, this is given. So by definition of the limit, then for any Delta, greater than 0, there exist a K , positive integer K, such that if n is greater than equal to K, then an’s, will belongs to the neighbourhood of C, minted, delta, neighbourhood of C. That is by definition of this. Okay? For any Delta, we can find out this, therefore, after certain stage, all the terms of the sequence belong to this Delta neighbourhood C. Therefore Delta neighbourhood, therefore, that V Delta C, Delta neighbourhood of C, contains the point, an’s, for all N, greater than C and different from this which belongs to A and distinct from C, distinct from C. So this implies that C is the cluster point, of cluster point of A. Is it not? So C is the cluster point, hopefully, that is what. It means we can, in terms of the sequence also; we can define the cluster point. Okay? Neighbourhood or cluster point.

Now let us see few examples. Suppose X, I take, open interval or A be the, open interval 0 1, which is subset of R. Okay? R. Now this is a complete interval, 0 to 1. Every point will be the cluster point. Why? Because you when you take any point here, C, any point of this, say I take C here, then one can identify any neighbourhood around the point Delta neighbourhood, if I choose. Then one can identify the points, of the set, which is distinct from C, so C becomes the cluster point. So and this is true for any C. C is arbitrary in between 0 1? So every point in the, this is a cluster point. As well as, so every point, including 0 and 1 are cluster point. 0 1 is also cluster point for this, cluster point. But 0 and 1, both are not available in A. So similarly, similarly close interval 0 1 we can choose like this. Say finite set, of real numbers, then finite set of real number means, suppose a 1, a 2, a n, these are the sets, elements of the set. But it does not fit in any limit point, has no cluster point. Why cluster point? Because, if suppose some point is a cluster point, say any, any point whether it in between or outside. If suppose say a 2, is the cluster point, then there is a gap between a 2 and a 1, a 2 and a 3, so if we draw a neighbourhood around the point a2, you won't get a point, other than a 2, inside this. Basically no point will be available. So it cannot be a cluster point. If a point is outside of the set, again there is a gap between a n and this and we cannot get the neighbourhood, we cannot get a point, in the neighbourhood Delta, which is distinct from this. Okay? So, point of the set. So this is not a given, So finite set does not have a limit point, no Cluster point.

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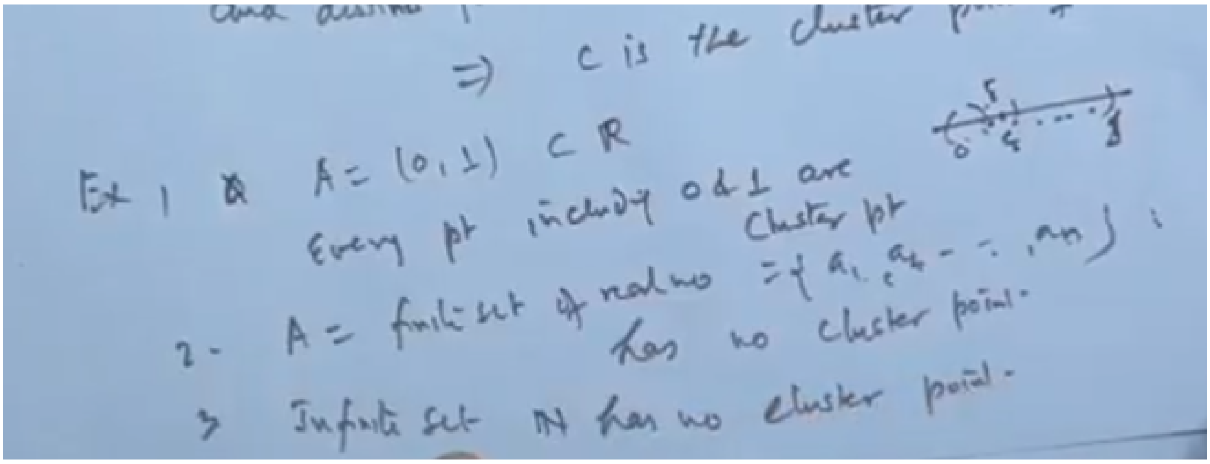

Similarly infinite set, infinite set, capital N, of natural number, has no cluster point. The same reason, is the same,

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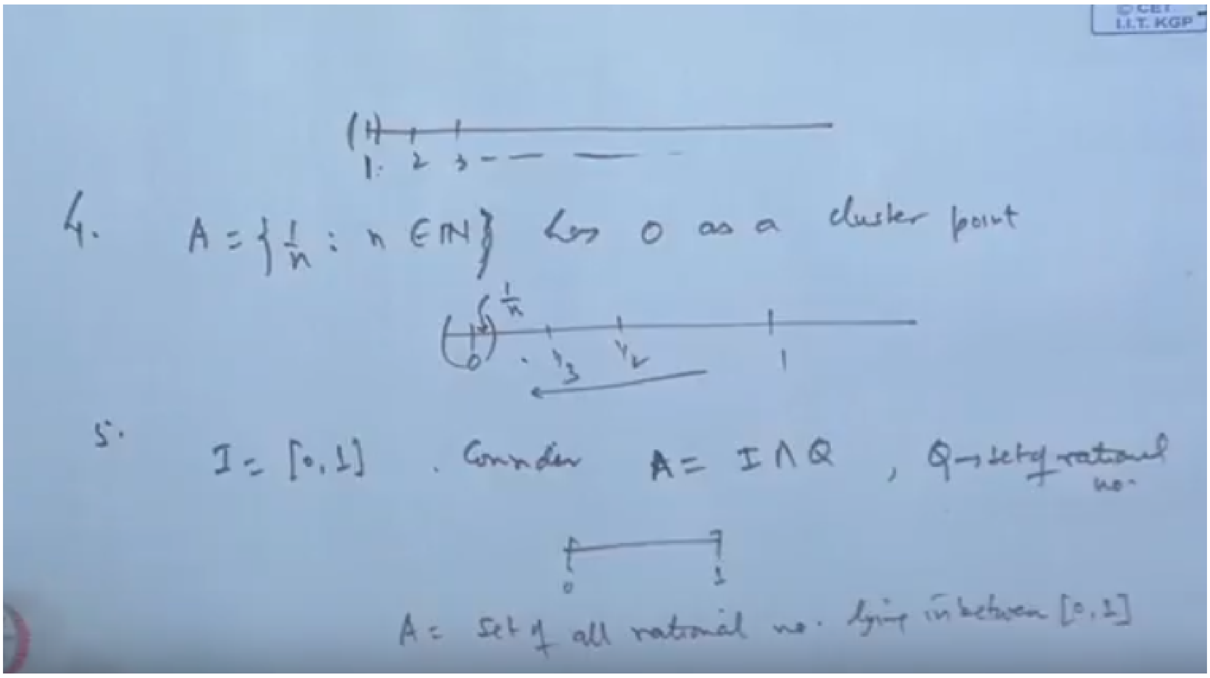

because this is our natural number, one, two, three and so on. Now we cannot take the, we cannot get any neighbourhood around the point, which contains the point, other, of the set, other than that point. Suppose I take 1. And I say, 1 is the cluster point, say. Then if I draw a neighbourhood like this, now this neighbourhood, does not contain, a point of the set, other than 1. Because basically no point is available, so 1 cannot be a Cluster point. Similarly, 2, 3, also. This infinite set and has no Cluster point then fourth example; the set N. If I take the set one by n, where, N is a natural number. Now this has, zero as a cluster point. Why? Because, this is our zero, her is 1, then ½, 1/3 and like this. So, the sequence, decreasing. Okay? decreasing, sequence. And it crusted, around the point zero. Means, if I draw any neighbourhood around the point zero, there are so many points will be available here, after a certain stage, which are close to zero and different from zero. So, zero becomes the cluster point for it. Then if we take the I, an interval, close interval 0 1 and consider a set A, which is I cross, intersection and Q, Q is the set of rational number, set of rational numbers. It means, the interval 0 1, is to digit and then from there, you are finding the intersection with Q. So A is the set of all rational numbers, lying in between 0 1, this closed set 0 1. Okay? Then every point in I, is a cluster point of a 5.

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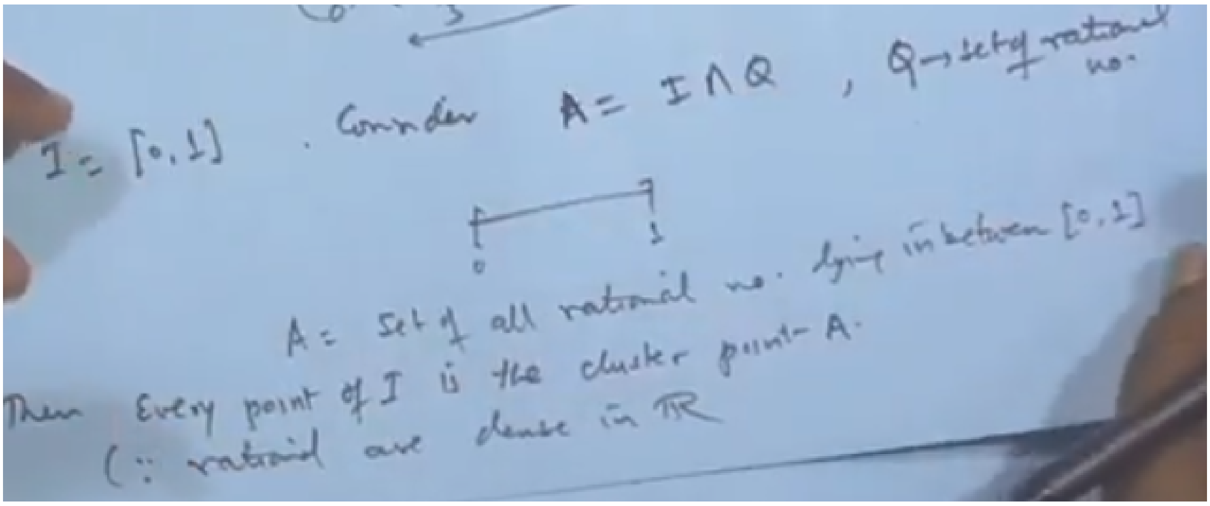

Then every point I, then became, then every point of I, is the cluster point of this set A. Every point in I or I is a cluster point of this. Why? This is because of the result, which we have seen, that between any two rational number, they are always a rational number, infinite there rational number, can be find at. in between the two rational number.

So if I picked up, any point here, say 0, then draw the neighbourhood. Now since between 0 & 1, rational numbers are dense, this is the reason is, because rationales' are dense, in real. Closure of rational number is the entire real line. Okay? So once it is rational dense so if you picked up any number, you will always get in vicinity, another rational number and, which is distinct from that number. So every point of 0 1, becomes, the closed cluster point for this setting. Okay? So that is what.

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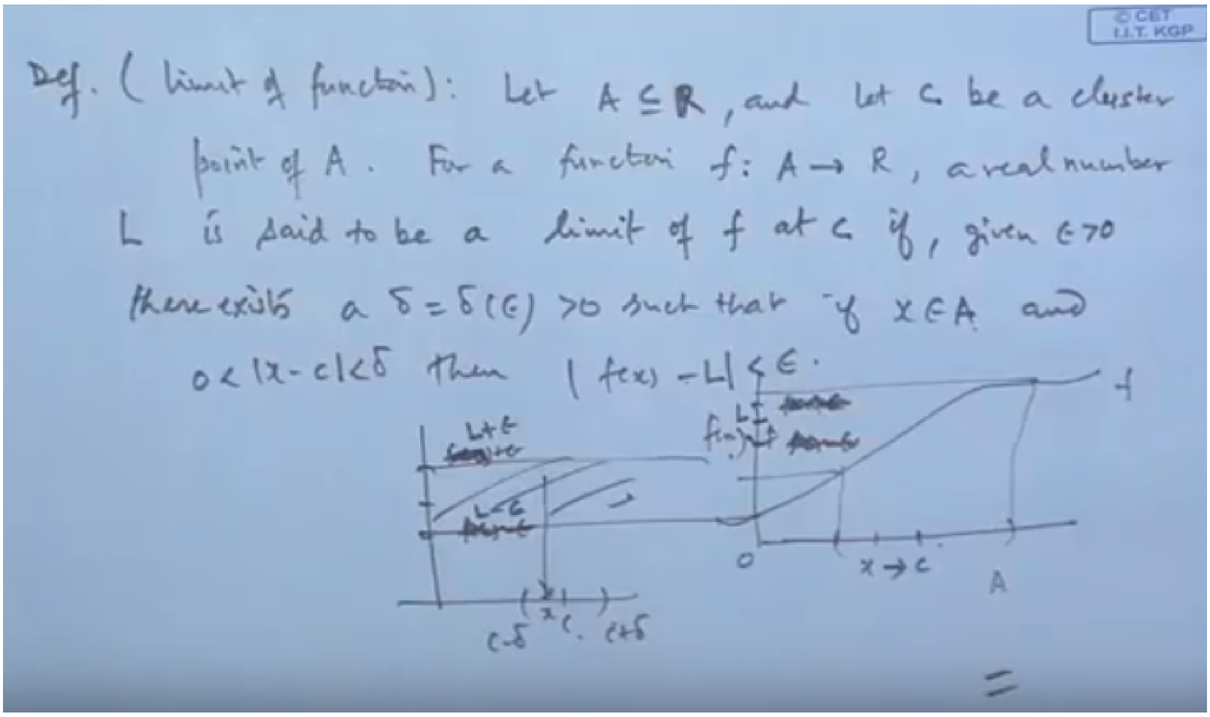

Now we come to the concept of the functions, cluster, now, this, important term. Definitions limit. Let A of limit of the function that is input. Let A be a, subset of non entry subset of R, subset of R and let C be a cluster point, Cluster point of A. For a function F, F, which is a function from A to R, A to R, a real number, capital L, is said to be, is said to be, a limit of the function, limit of F, limit of f, said to be a limit of F, at C, if given any epsilon, greater than 0, there exists, there exists, a delta, which depends on epsilon of course, Delta dependent and positive, there exists a delta greater than 0, such that, if X belongs to A and 0 less than mode X, less, C less than Delta, then, mode of FX, minus L, is less than Epsilon. That is what is. Okay?

So what is the meaning of this is, that a, this is our C, is the cluster point, C is the cluster point of a set. Suppose this is the set A. Okay? And C is a cluster point for this set. A function f, is defined from A to R. So this is our function f, which is defined, carrying A to R, this is R suppose, here every real the function is defined, so we get this, this is FX, range of this. So what it says is, that if we and we say, that L, this L, is the limit of the function FX, at the Point C, that is if I take X and X tends to C, the corresponding function FX, this goes to L, limit of FX is L, means. That if we take any epsilon, greater than zero, then, there exists a delta, it such that, this, for all X belongs to this and FX minus L, is less than Epsilon. So if I take a epsilon neighbourhood around the point L, that is here, FX minus epsilon, FX plus Epsilon, if we choose this. That is this is our FX minus epsilon, FX plus epsilon, here is C, C minus Delta, C plus delta. So for the given epsilon, greater than zero, for the given epsilon, greater than 0 ,there exists a delta, such that, such that, whenever we take any point X, different from C, in this neighbourhood, the image of this will fall within this range, within the L minus epsilon, to L plus, sorry, this is, L minus epsilon, L plus epsilon and L minus epsilon. This will be so. With for within this range, then we say that this limit of the FX is L. Okay?

Thank you very much. Thanks.