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Lecture: 35

Raabe’s test for convergence of series

Course on introductory in real analysis

So in the last lecture we have discussed several tests, for judging the absolute convergence of the series of real numbers, and those sets mainly we discussed the true test, comparison test, ratio test, integral tests, now even we see that this tests are good enough to just the series to be convergent absolutely convergent or not, but still there is a case of failure.

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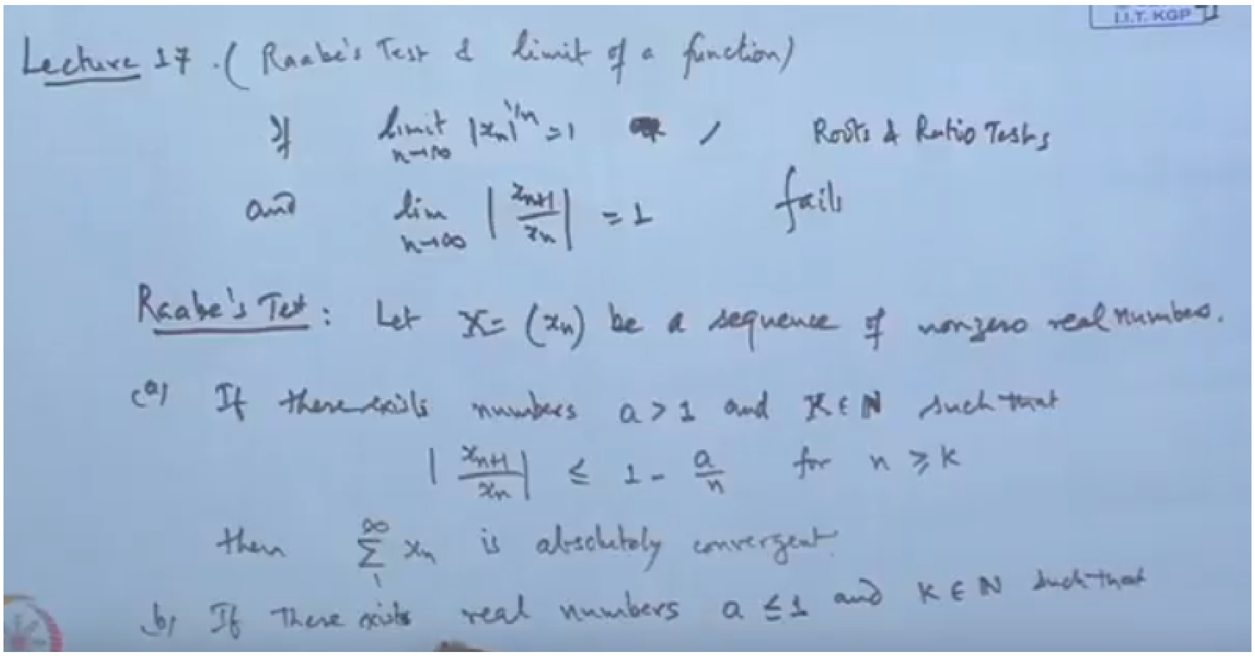

When the limit of xn to the power yn, by a limit of X mod xn to the power 1 by n as n tends to infinity, if this limit comes out to be 1 or limit n similarly, and limit of this mod xn plus 1 over X and ratio test as n tends to 1 comes out to be 1, then both Cauchy and both in the true test and ratio test fails. Root, root test and ratio test fails, we are unable to decide whether the given series is absolutely convergent or divergent, or that it may be convergent may be divergent depending on this, so the further test which will help in this direction is given by DA Beach, which you know is the DA Beach test and that can be applied only when we see that this test fails, we cannot also, but initially when we start with that we always considered step by step, go step by step and then if required then we compile the higher order higher level tests.

So Raabe’s test says, that is let X which is xn be a sequence of nonzero real number, be a sequence of nonzero real numbers, the first one if there exists, if there exist numbers E, which is greater than one and, and K which belongs to and the set of natural number, set of natural number, K is a positive integer, such that, such that mod of xn plus 1 divided by xn under mode is less than equal to 1 minus a by n, for all N greater than equal to K, then the Raabe’s test says, the series Sigma xn and is 1 to infinity converges is absolutely converges, absolutely convergent. Now on the other hand if there exists, if there exists real number, there exist real number exists,

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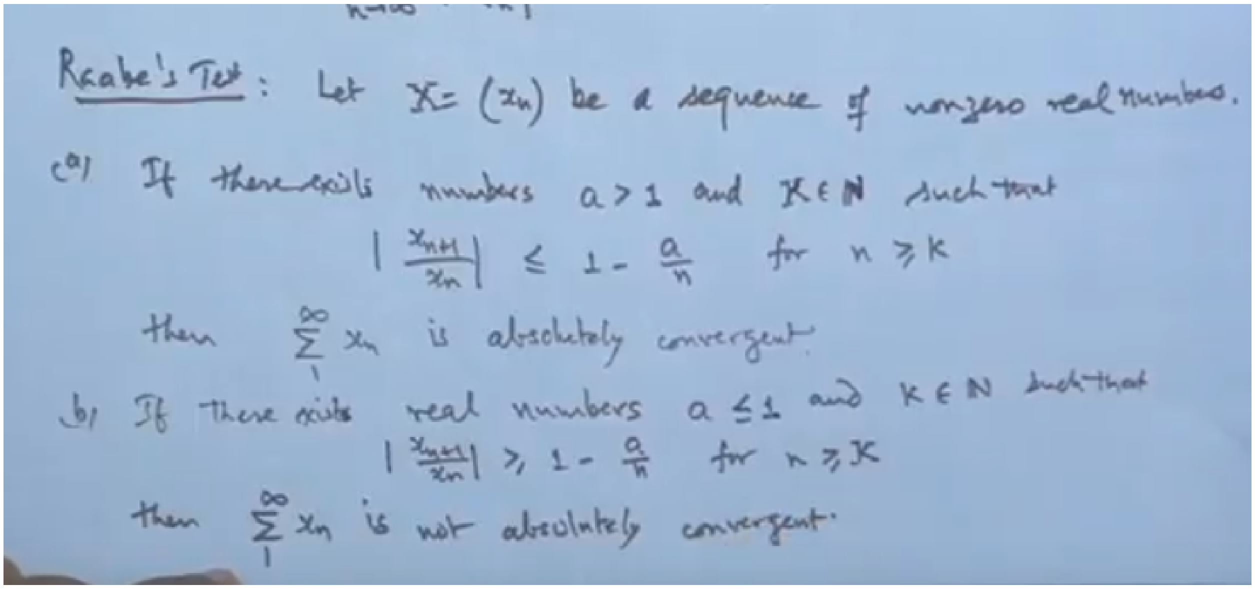

real numbers, numbers a which are less than or equal to 1 and K is a positive integer such that mod of xn plus 1, divided by xn under mod, if it is greater than equal to 1 minus a by n, for all N greater than equal to K. Then the series Sigma xn 1 to infinity is not absolutely convergent, it means it will be a diverging sequence, divergent sequence, absolutely that**.** The proof of this follows as it is, suppose let us give this number, suppose this inequality 1, this is 2.

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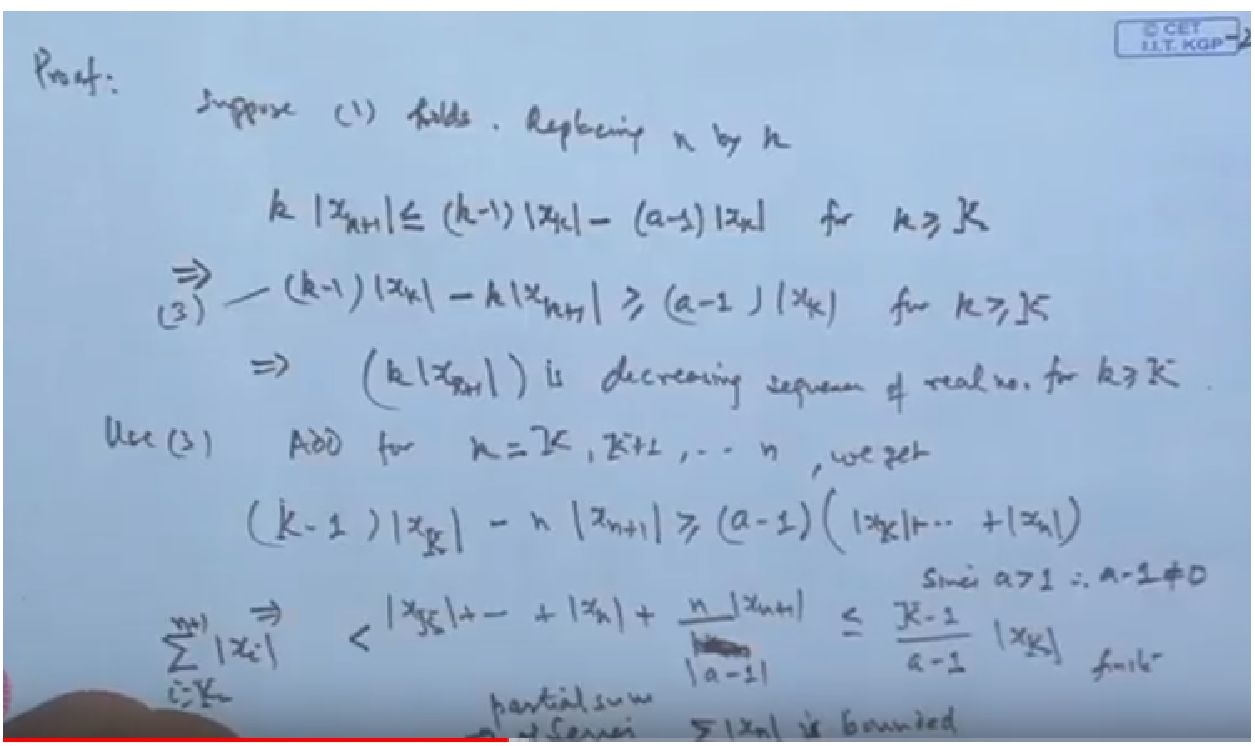

Suppose 1 holds, then we have to saw the series is absolutely convergent, so one holds means mod of xn plus 1 by xn is less than equal to 1 minus a by n, there exists a number a greater than 1 such that this is true for all N greater than equal to K, so let us replace n by K, is small K suppose then what we get is here, that K into replace n by small k it's nothing, and then we multiply this k into mod XK plus 1, minus less than equal to K minus 1, into mod XK, minus a minus 1 mod XK, this is 2 for all K greater than equal to capital K, what I did is I just use this in place of n let us say small k, so when you multiply by this K you are getting a K times mod XK plus 1 is less than equal to, one minus a into mod xn, this can be manipulated like this, it can be written as K minus 1 mod XK minus n, because mod XK get canceled from here and basically you are getting this K minus a into mod a, mod XK, Okay?

So this way we are doing, now again reorganize it, so this implies that K minus 1 into mod XK, minus K times of mod XK plus 1, this term is greater than equal to a minus 1, into mod XK and this is true for all K greater than equal to capital K, now a is strictly greater than 1, in that case 1 first case, so again so this is a positive quantity, this is also a positive quantity because we are choosing xn is a sequence of the nonzero real number, so mod of XK will be positive quantity, so what is so this shows, that K times of mod XK plus 1, basically is what is a decreasing sequence, so what we get is so this implies that the sequence K into mod XK plus 1, XK plus 1 this sequence is a decreasing sequence of real numbers, of real number for K greater than equal to capital K, Ok?

Now if I add this one let it be this quantity K minus 1 let it be 3, this third, Ok? Now if we add this from Use three and add for small k is capital K, capital K plus 1, up to say n, so when you add this will automatically telescopically, few term will get cancelled, and finally we get capital K minus 1 into mod of X capital K, minus n mod of xn plus 1 is greater than equal to a minus 1, into mod of X capital k, plus mod of xn, Okay? Now since a is greater than 1, so a minus 1 is nonzero, so a minus 1 is different from 0, we can divide by this so this implies that a mod of X capital K, plus up to mod xn, then plus n by mod xn plus 1, sorry, a n times mod xn plus 1 divided by a minus 1, divided by this a minus 1, this a minus 1, this term you are taking here, Ok? And divided by a minus 1, is less than equal to capital K minus 1 divided by a minus 1, mod of XK, Okay. Now K is any numbers is it not? K minus is a fixed number, K is also fixed so this is basically a finite quantity this is finite, Ok? Bounded. Now here if I look this one n is greater than mod a minus 1, because once it's fixed then n can be taken so large so that n over m minus 1 is greater than 1, so basically this entire thing is greater than the series Sigma mod X I, I is capital K – n plus one is greater than this, so what does shows, that this series when n is sufficiently large, this is bounded series so this implies that the series, series Sigma mod xn,

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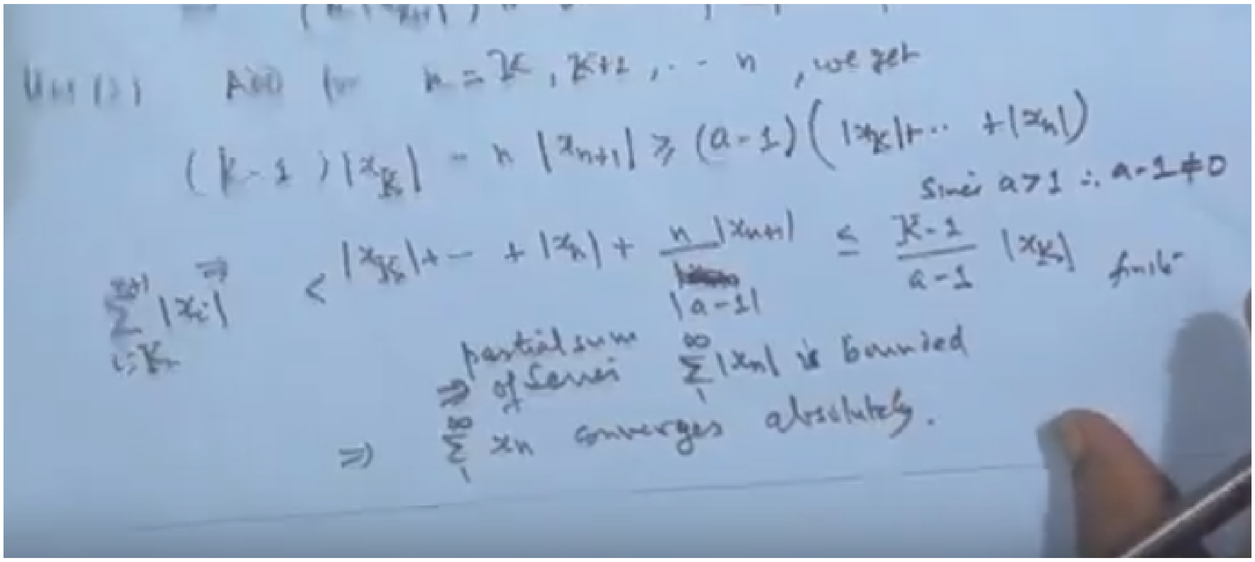

this series is a bounded, partial sum of the series, partial some of the series, this is bounded, Ok? The partial sum of this series bounded 1 to infinity and the series and the terms of this series, K into X 1 is a decreasing sequence, of real numbers is it not? So this is a bounded sequence and monotonically decreasing sequence of real number, because once it's decreasing mod XK minus 1 is also decreasing, K times, Ok? So this is a bounded sequence, hence every bounded sequence is a convergent one monotonic, Okay? So this shows that the sequence this implies the series xn 1 to infinity converges absolutely, absolutely.

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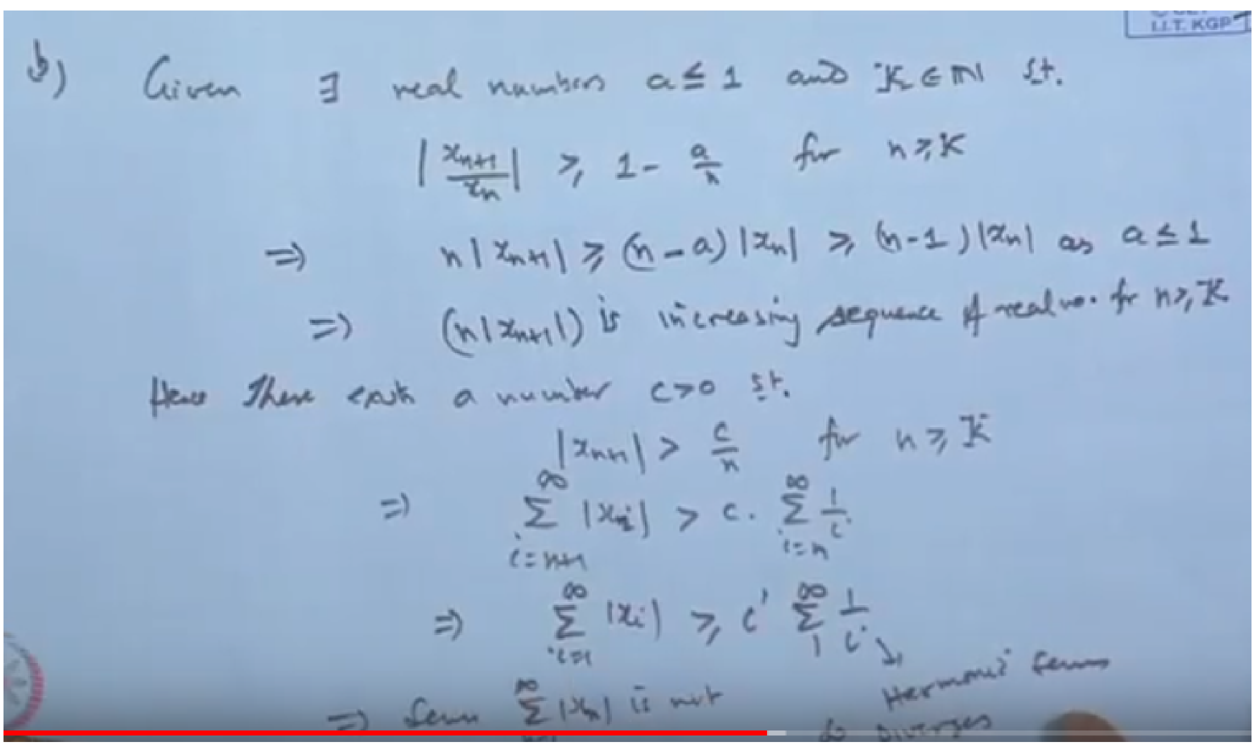

So this is the proof for this part 1, this is part a, proof for Part B, now in this case it's we given that this relation, relation given that there exists real numbers, less than or equal to 1, and K a positive integer such that, mod of xn plus 1 y xn is greater than equal to 1 minus a by n, for all N greater than equal to K, so what is so this shows, that if we multiply this n into mod xn plus 1 this is greater than equal to n minus a, into mod X n, minus M into mod xn, but what is but a is less than equal to 1, so minus a is greater than equal to minus 1, so it is greater than equal to n minus 1, mod of xn, as a is less than equal to 1, so n of mod xn is greater than this implies the sequence n into mod xn plus 1, this sequence is an increasing sequence, increasing sequence, of real numbers for all N greater than equal to capital K. for all N greater than equal to capital K, So once it is increasing so if we take any C, then a n can be obtained so that is greater than equal to c, so 1 and there exist a number C, so there exists a number C, get other than 0 such that this mod xn plus 1 is greater than C by n, Ok? And this is true for all N, greater than equal to capital K, therefore, the Sigma of this X, I is n plus 1 to infinity is greater than C times Sigma of 1 by I, I is n plus n to infinity, now this is a harmonic series, this is a harmonic series, so what we do is that if I take I equal to 1 to infinity, that is Sigma I is equal to 1 to infinity mod of X I is greater than equal to say same terms we are adding here so sum C replaced by C dash is greater than equal to Sigma 1 to infinity 1 by I, is it not?

Some few terms we are adding here so that can be written as some C dash, or maybe plus something, but this is the harmonic series, harmonic series so once it is harmonic it means it will be a diverging one, so diverges so once this diverges this series will be diverging one, so this implies that series Sigma mod xn n is 1 to infinity is not absolutely convergent, is not convergent. Therefore original series will not converge absolutely, so that's the reason for this, Ok?

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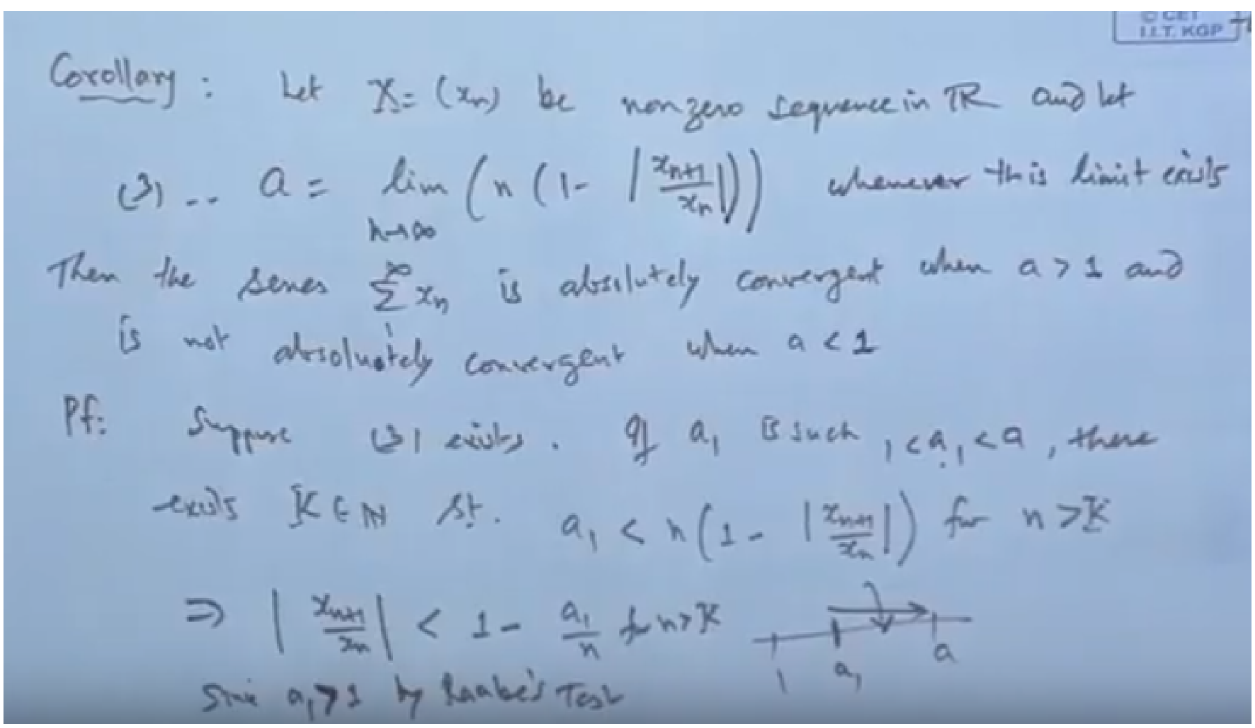

Now the equivalent is the corollary of this because it's a difficult to get this ratio, so in terms of the limit we can easily get this, test which is applicable, and so as a corollary we can say, let X which is xn, be nonzero sequence in all, sequence in R set of real number and let a is the limit of this, limit of N 1 minus mod xn plus 1 by xn, mod of this as n tends to infinity suppose a, whenever this limit exists, exist, Okay? So this is, then the series Sigma xn 1 to infinity, this is absolutely convergent, convergent when a is greater than 1 and is not, and is not absolutely convergent, convergent, when A is strictly less than one, for a equal to one, again test fails, we can unable to get so we have to apply some other tricks for that, Okay?

Is less than, the proof runs is history suppose the limit in this and let it be this three, suppose third exist limit exists, in three so it means the limit of this is a where is strictly greater than one, so we can identify, so there exist, so if a 1 is a number a one is such that, such which lies between a 1 is less than a 1 less than a, for this a 1 there exist, some K a positive integer K, such that such that a 1 is less than n times of 1 minus mod xn plus 1 by xn, for all N greater than capital K, this is better because this limit is a, Ok ? And is a strictly greater than 1 so if I choose a number in between 1 and a it means there will be a sequence of the term of this series which will exceed by a 1 then only the limit will go to a, Ok?

There are this you can become limiting is we are going to a, this is 1 here is a, that limit is going tending to this, so what I am doing is I am choosing 1 a 1 here, Ok? So an N can be obtained a K can be obtained such that when you take all N greater than K, all the terms of the sequence will fall here, that is they are greater than a 1, so if it is so then what happens to this, therefore, the series when this converges so we get from here, that therefore this implies what? n plus xn plus 1 over X n this will be when you transfer it what you are getting is, is less than 1 minus a 1 by n, Ok? But a 1 is what? And this is true for all N greater than K, but even is strictly greater than 1 so why Raabe's test, since even in systole greater than 1, so by Raabe's test,

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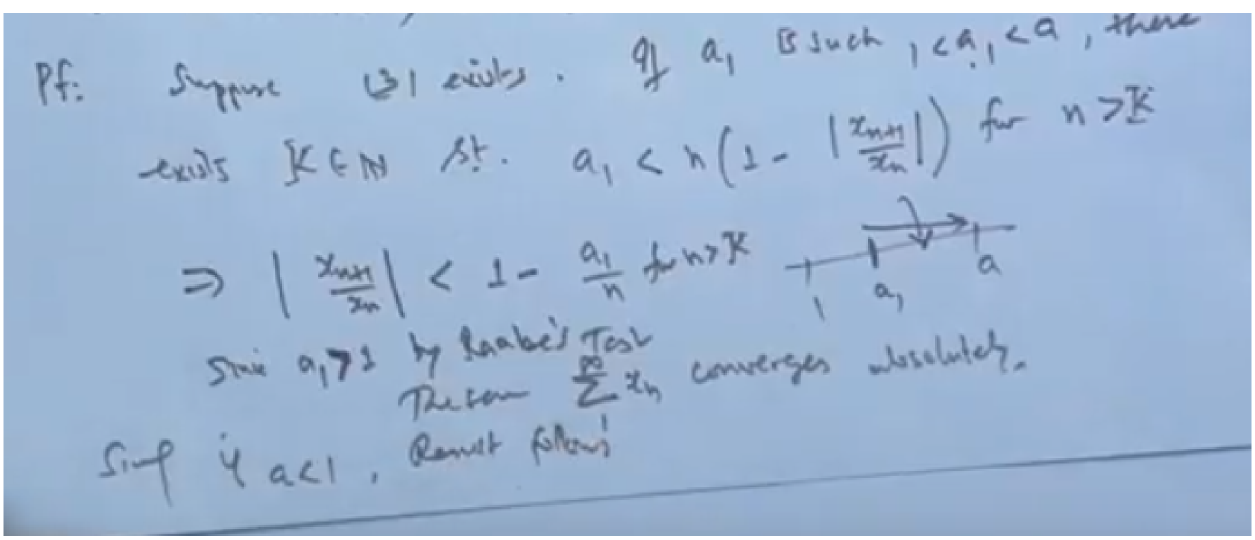

the series Sigma xn 1 to infinity converges absolutely. Similarly, for the other similarly, if a is less than 1 the result follows, in a similar way you can prove that this result is true, Ok?

Now we have seen the corollary and the result, now this is Corollary recall basically the result is very strong, that is this ratio this type of thing is stronger than this one, which can be seen easily by the following example.

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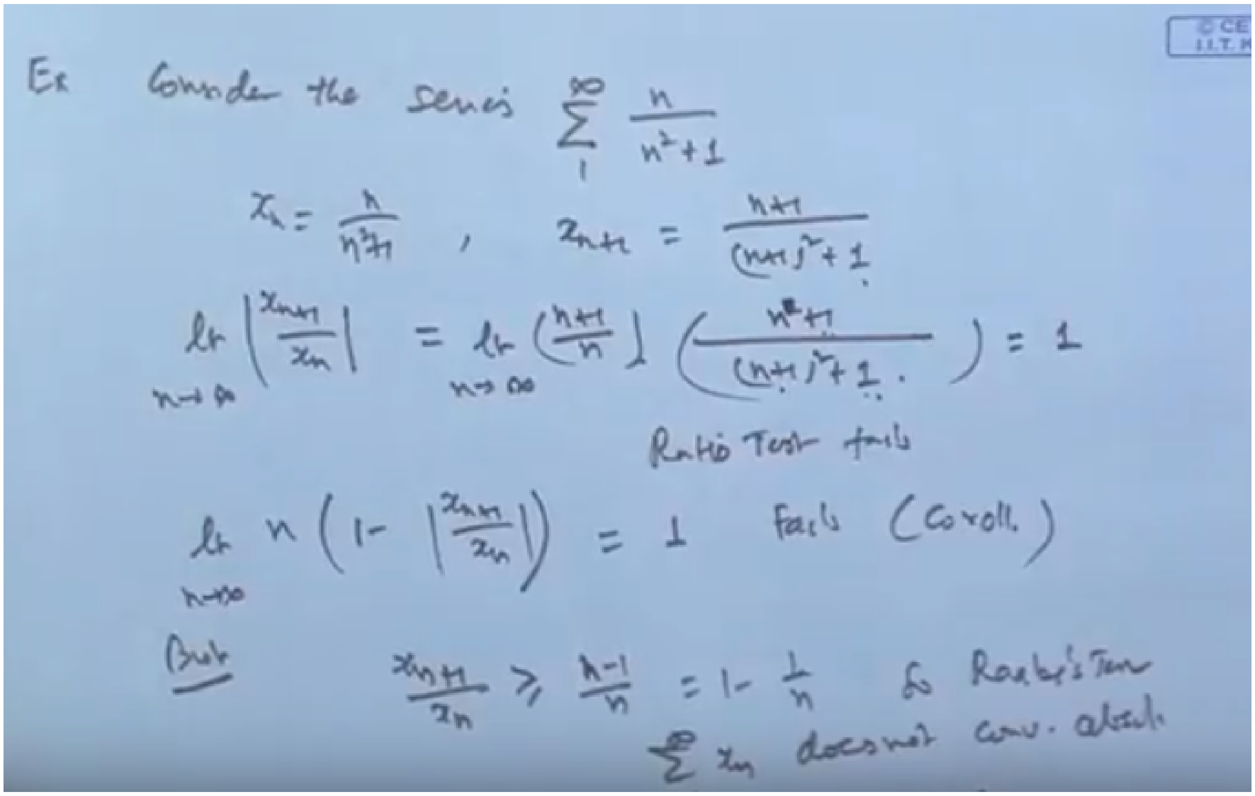

Suppose I take this problem let us consider, or consider the series Sigma, n over n square plus 1, test for the converging absolute convergence all the terms are positive test for the convergence of the series, now if we apply the results here suppose I take here the xn is n over n square plus 1, so xn plus 1 becomes n plus 1, n plus 1 whole square plus 1 so when you take the xn plus 1 y xn mod of this, then what happens to the xn plus 1, the this comes out to what? And plus 1 by n this top and then n square plus 1 n plus 1 whole square plus 1 and then limit of this as n tends to infinity, limit of this as n tends to infinity, now limit of this is 1 again when you divide by n square the limit will come out to be what?

This is 1 and then this limit xn plus 1 by xn, xn plus 1 by xn n over n plus 1 it is 1 or not let us see n square plus 1 so 1, then 1 and plus 1, 2 so it is n plus 1 the calculation limit of this is, xn is n over n plus 1, so if you take this n plus 1, n plus 1, n plus 1 divided by n plus 1 square, divided by n plus 1 so we get from here is n plus 1 is square so what we are getting is 1, divided by this n plus 1 is square this is sorry, that's why it's a mistake, this is n plus 1 not n, Ok? So divided by n plus 1 is square here so this limit is 1, this limit is 1, product will be 1, so the ratio test fails, fails. Now if I take the Raabe’s test in into column rise of the line minus xn plus 1 over X n this thing, and then take the limit of this, as n tends to infinity, then again you will see this limit comes out to you on, Okay?

We can just solve it and get this here; we get the limit to be 1, so again fails, Ok? We are unable to there but, this is by corollary, but if we look the xn plus 1, by xn what you are getting, xn plus 1 by xn, this is nothing but what is greater than or equal to n minus 1 by n, is greater than equal to n minus 1 by n, if you just see, the ratio of this and find out the term, you will get this is greater than equal to that is 1 minus 1 by n, Okay? And this is posing a 1, so this diverges, so by Raabe's test, the series Sigma xn does not converge absolutely, so this shows that though the Raabe's test is difficult to find out this type of ratios, and regain and get the duration, but, it is much stronger than the corollaries of this Raabe's test, Okay? So but practically huge, practical the corollary is much more easy to apply rather than to this, Okay? So that's what so these are, all about this convergence or absolute convergence of the series.