Model 6

Lecture – 34

Ratio and Integral Test for Convergence of Series

Course

on

Introductory Course in Real Analysis

Now, then in next says or so, this is the root test, ratio test. Third test is ratio test.

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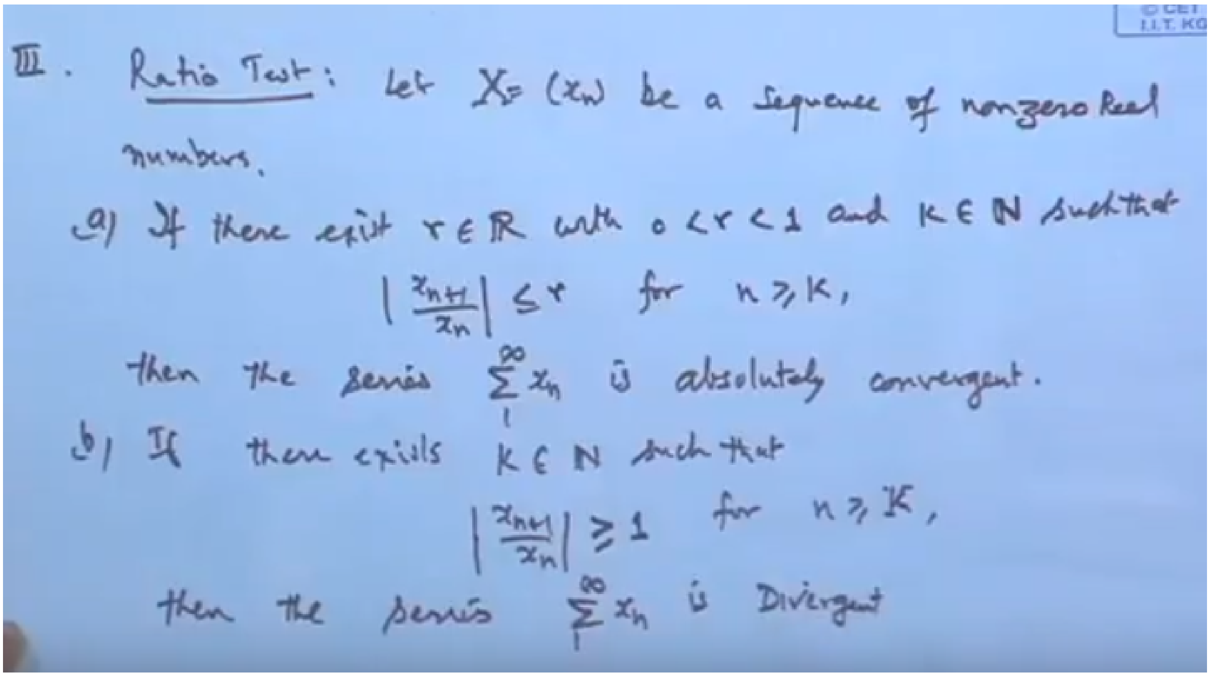

Let x, which is xn, be a sequence of nonzero real number, real numbers. Then the first set, if there exists an R, if they are exist, there exist r, belongs to the set of real number, capital R, with 0 less than R, less than 1 and k belongs to n, set of natural number and such that, mode of xn plus 1, by x n, this is less than equal to r, for n greater than equal to k, then the series, then the series, Sigma xn, 1 to infinity, is absolutely convergent. And Part B says, if there exists, if there Exists, k belongs to the set of natural number n, such that, mode of xn plus 1 by X n, is greater than or equal to 1, for n greater than equal to K onward, then the series, Sigma xn, 1 to infinity is, divergent.

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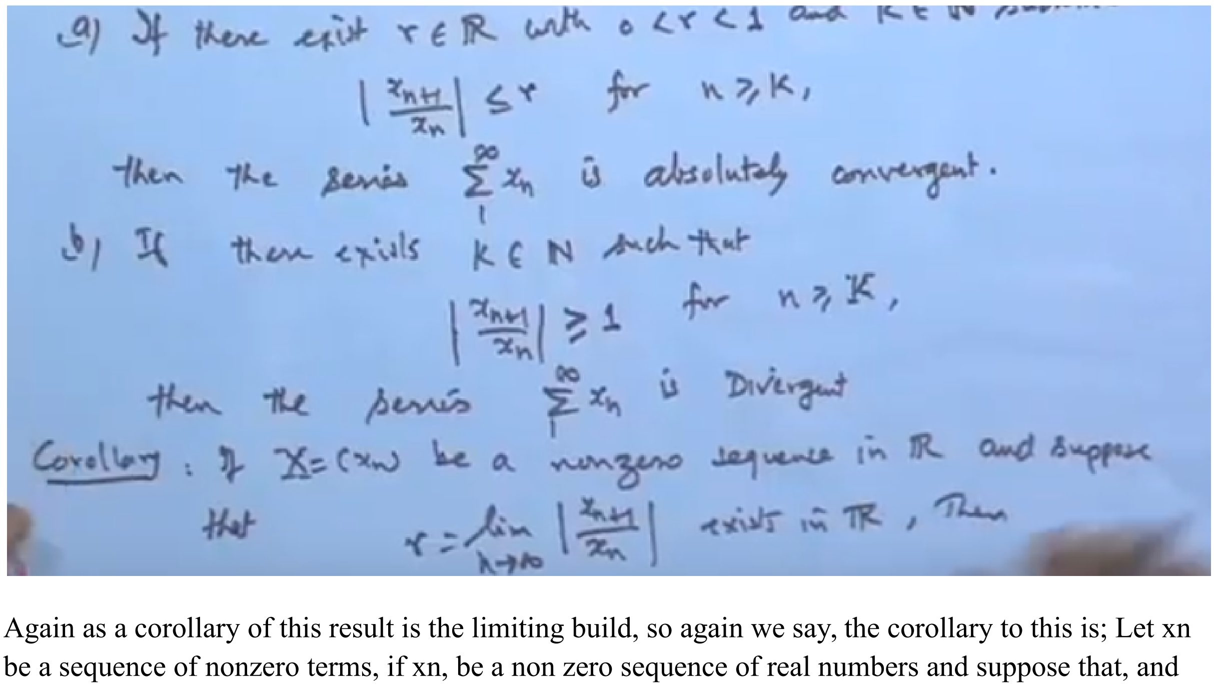

Again as a corollary of this result is the limiting build, so again we say, the corollary to this is; Let xn be a sequence of nonzero terms, if xn, be a non zero sequence of real numbers and suppose that, and suppose that, the limit exists, limit of mode xn plus 1, over X n, as n tends to infinity, exists and say R.

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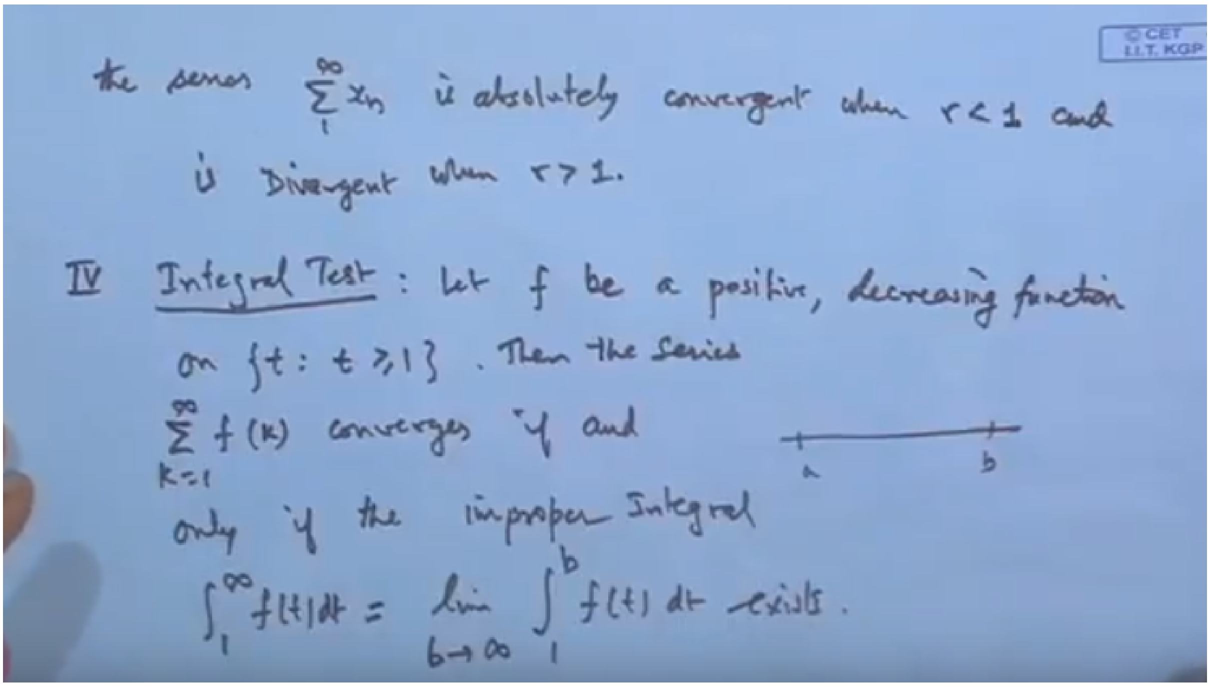

Then the series Sigma xn, Sigma xn, 1 to infinity, in absolutely convergent, absolutely convergent, when r is strictly less than 1 and is, in each divergent, when r is strictly greater than one. Again for r is equal to one, test fails. Okay? So now if we look this thing, then for r equal to 1, the testing fails. Okay?

Now there is another test, which is known as the,' Integer test', and which is very powerful test. But of course, it requires a knowledge of the Riemann integral function, but here we will issue f is an element in the Riemann integral function. Riemann integral function we mean, that suppose a B, is an interval, okay? AB is an interval, say here, when we divide this B, into the partition and then finding the sum of this, then A to FBA can be written as the limit of the sum of this. So this we will discuss it when we go for the Riemann interior chapter, on Riemann integration. But let us assume, the f is an element of the Riemann class of Riemann integral functions, where the integral test. So we are interested in this test right now. So third test is or fourth, is the integral test. Which is? Very powerful test, in it, Okay? So here we assume, let F be a positive now one more, thing is, concept of the improper integral. So here, let F is an element, let us take one. Let f be a positive, decreasing function, on the set T, where the t, is greater than or equal to 1. Then the series c d, sigma FK, case is, 1 to infinity, converges, if and only if, the improper integral, improper integral, 1 to infinity, ft dt, if the improper integral, which is limit, 1 to b, ft dt, v tends to infinity, exists. Okay?

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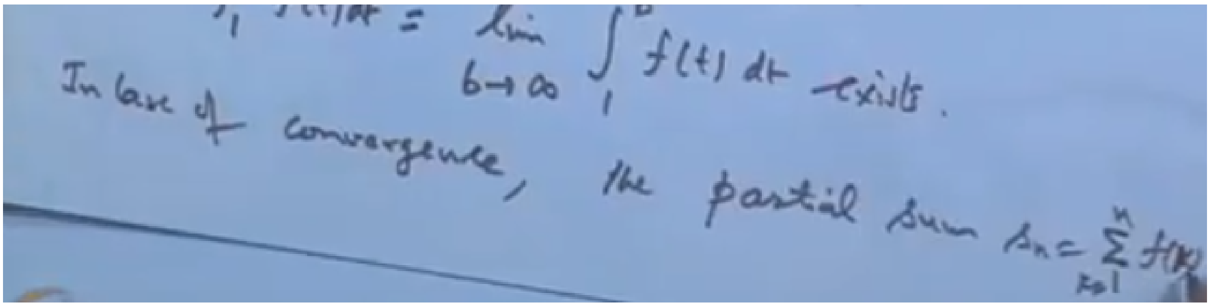

In case of the convergence, convergence, the partial sum, SN of the series, sigma f k, K is 1 to n, the partial sum is K is 1 to N, K is 1 to n, FK and this is K, K, okay

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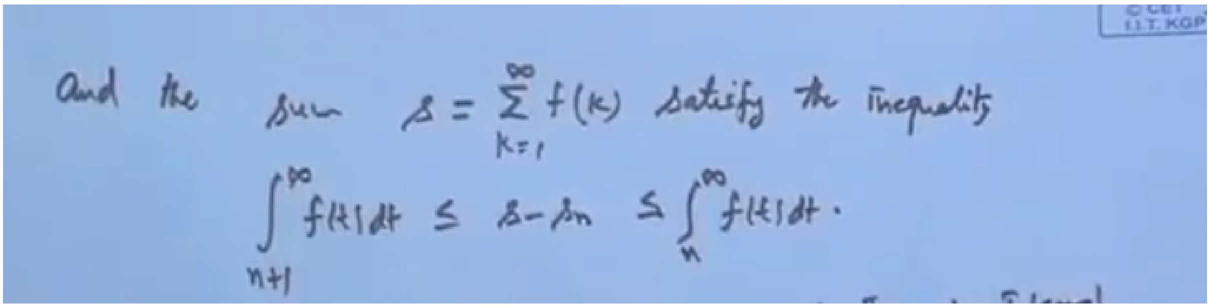

and the integral, and partial sum of this. And the some s, and the sum s, which is sigma, f k, KH, 1 to infinity, satisfies the estimates, the iniquity, integral n plus 1 to infinity, ft dt, ft dt, which is less than equal to s minus SN, which is less than equal to, n to Infinity, n to infinity ft dt.

So what this is, that if F be a positive decreasing functions, on this interval ,on this set, T, where T is greater than equal to 1, the series Sigma FK, converges if and only the corresponding improper integral, exists. So basically the integral says, test, it connects the convergence of the series, infinite series, with a improper integral. So here we are having the two terms, one, is the improper integrals and other one is the, that, of course, that Riemann integral functions.

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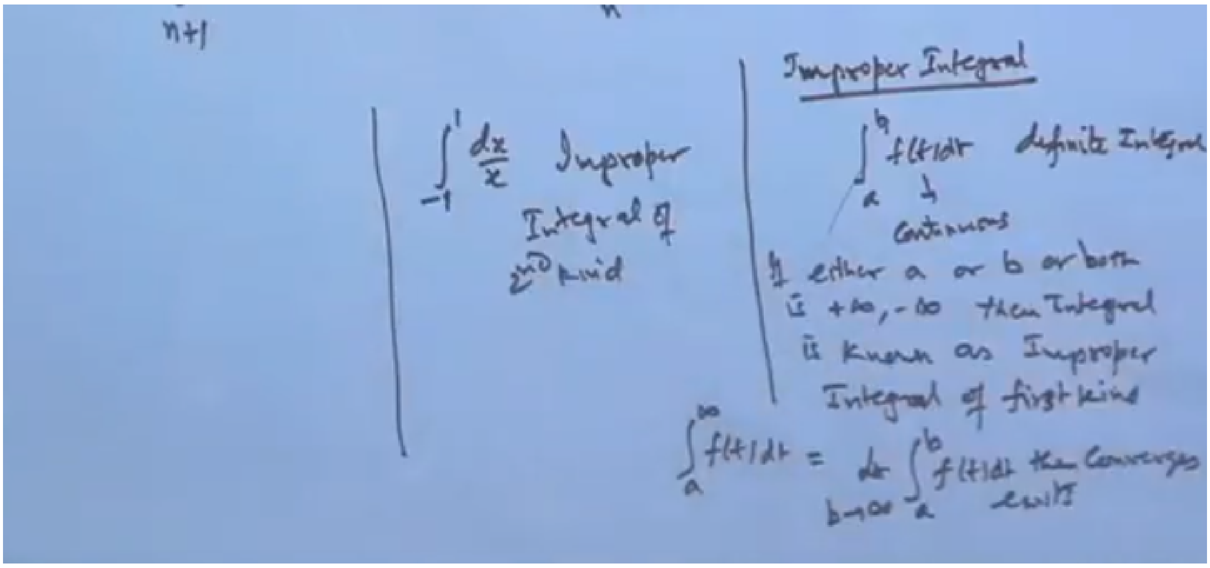

Now what is this improper integral? Let me, before going for the proof; let me see the first, the improper integral, improper integral. Improper integral is, if suppose the function f, which is defined over a certain interval, say a B, then a to b, ft dt. If F is a continuous function, then this integral, is known as the definite integral, which we know, definite integral. Is it not? If F is a continuous function, then A to B of T DT, represents the definite integral, and is known as the definite integral and it represents the area bounded by the upper bounded, by the function, y equal to ft, and below y, x axis and a to b, to 1 itself, If one of the limits, is infinity or minus infinity, then such integral, we call it as an improper integral of the first kind, first kind. If either A or B or maybe both, one of the limit, either A or B, is plus infinity, minus infinity, or maybe both, then the integral is known as, improper integral of first kind, of first kind.

Now once we have, one of the limit is improper integral, that is integral, a to infinity, ft dt. So this is improper integral, of the first kind. Now whether this integral converges, or diverges, depends on the limit of this. What we do is, we consider a to b, ft dt. And then we take the limit, v tends to Infinity. Now if this limit exists, then we say this improper integral converges. Now if this limit does not exist, then we say this improper integral, diverges. Similarly minus infinity to alpha, we can in a similar way, we can say, define the convergence or the divergence of this. The improper integral, of the second kind here, If a and b, ft dt, is given, but f is not continuous. Suppose it has a point of discontinuity, over the interval a b, then such an integral, is not defined basically, over the whole interval a b. Say for example, if I take the, Improper integral a 2 or 0, minus 1 to 1, DX by X. Now this integer, function FX is 1 by X, is not defined at X equal to 0. So it is an improper integral, of second kind, of second kind. Clear? So these two types of improper integrals, one is the importance of first kind; another one is the improper integrals second kind. Now here what we are assuming is, improper integral, of the first kind.

So if a series is given and the function f is such, which of decreasing nature, over the interval 1 to infinity, then the nature of the series sigma f k, sigma FK converges, is converged, if the corresponding improper integral, will exit. Clear? And of course vise versa, corresponding. So this relates the convergence of the series, with the improper integral. And that is why it is known as the integral test. It is very powerful test. One can drive the earlier test, with the help of this integral test.

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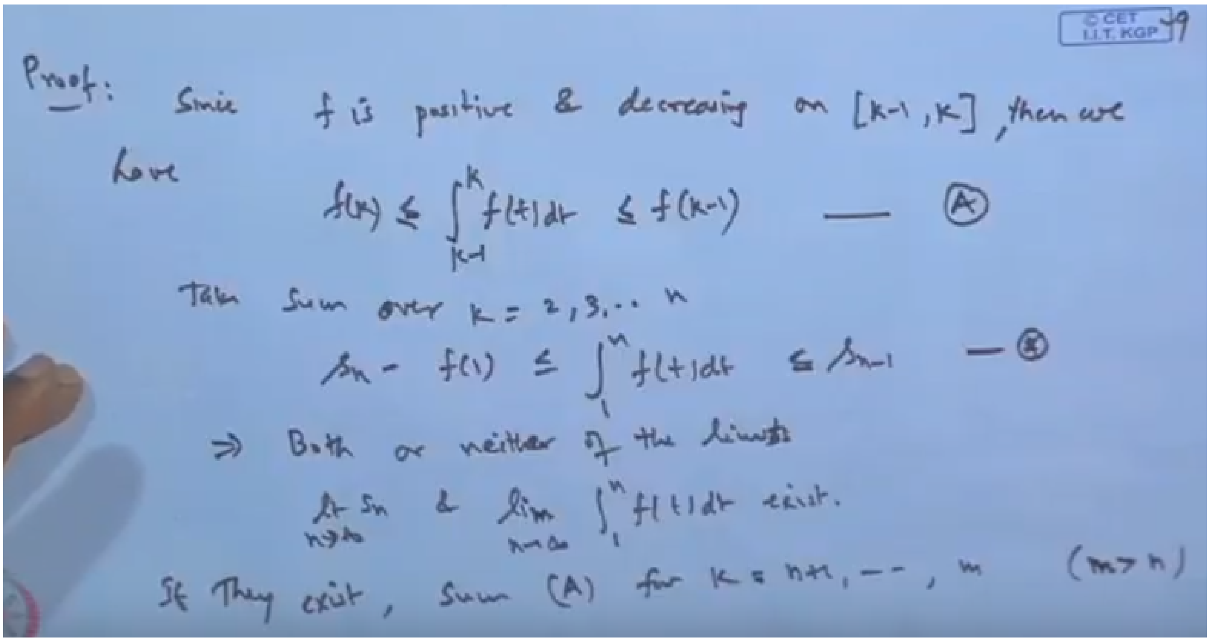

So let us see the proof of this result. So we go to proof of this. Okay? Now, if given that function is decreasing. Since F is positive and it is of decreasing nature and decreasing, on the interval, over the interval T is greater than equal to over the set. So we can take the interval, say k minus 1, k. This is interval; function is decreasing, over this interval, k k minus 1. Then we have, then we have, then we have, obviously this relation, that f of k, is less than equal to, integral k minus 1, to k, ft dt, which is less than or equal to, or equal to, f of k minus 1. Since the function is decreasing, the last test value will be attained at this point. k minus 1 and lowest value will be attended at the point FK. What is this value? This is the area represented by this. So, basically the functional value, between K minus 1to K, so this will be the portion, in between these two. Okay?
Now let us take the sum, so take the sum of this, take sum over K, over K, when varies from 2,3 say up to n. Then what happens? When you take the sum of this k, then it is sigma k equal to one to n, is the Sn. So we can say SN minus the first term, f1. Like this. And this is, k equal to 2 3, so 1 to n, ft dt, which is less than equal to. When you write the sum of this, it is s n minus 1. Okay? So we get this. Now, from here, say star, what we say? The nature of this, limit of this integral, will depend on the limit of SN and vise versa. If the limit of sn exist, the limit of this will also exist. If limit of this exists, limit of SN will also exist. So this shows, that this implies, the both or neither of the limits exist, both all neither of the limit, of the limit, limits, sm, as n tends to infinity and, and limit as n tends to infinity, 1 to n, ft dt, both will exist or both will not exist, that is one thing. Now if they exist, if they exist, then add them. So then add, this, say this part, from here again, let it be a, then sum is a, for k is equal to, n plus 1, to m, Where m is greater than n. Sum, sum up this. So when you sum up this thing, what happens? This is k equal to m2, so S M, minus sn,

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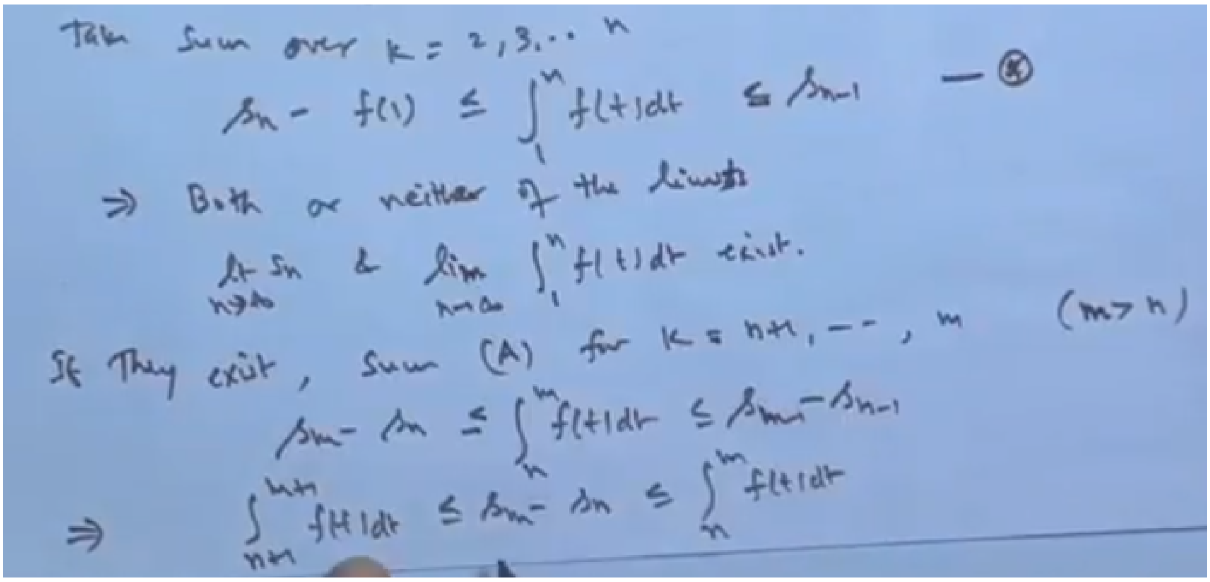

so this will give s m minus sn, which is less than equal to, n 2 m, into m and ft dt ? Which is less than, equal to? Again you sum up this thing. That will be equal to, S, M minus 1, minus s, n minus 1. So that will be there. Okay? now from here. So this implies that. Now if I take n integral of this. Say suppose, I replace, n by, n plus 1, M by, n plus 1, then what we get? N plus 1, to m plus 1, ft dt. This is nothing, but what? When you write m equal to m plus 1, n equal to n plus 1, this is less than equal to, s m minus Sn. But s m minus Sn is also, less than equal to this integer n,m, ft dt. Okay? Now clear?

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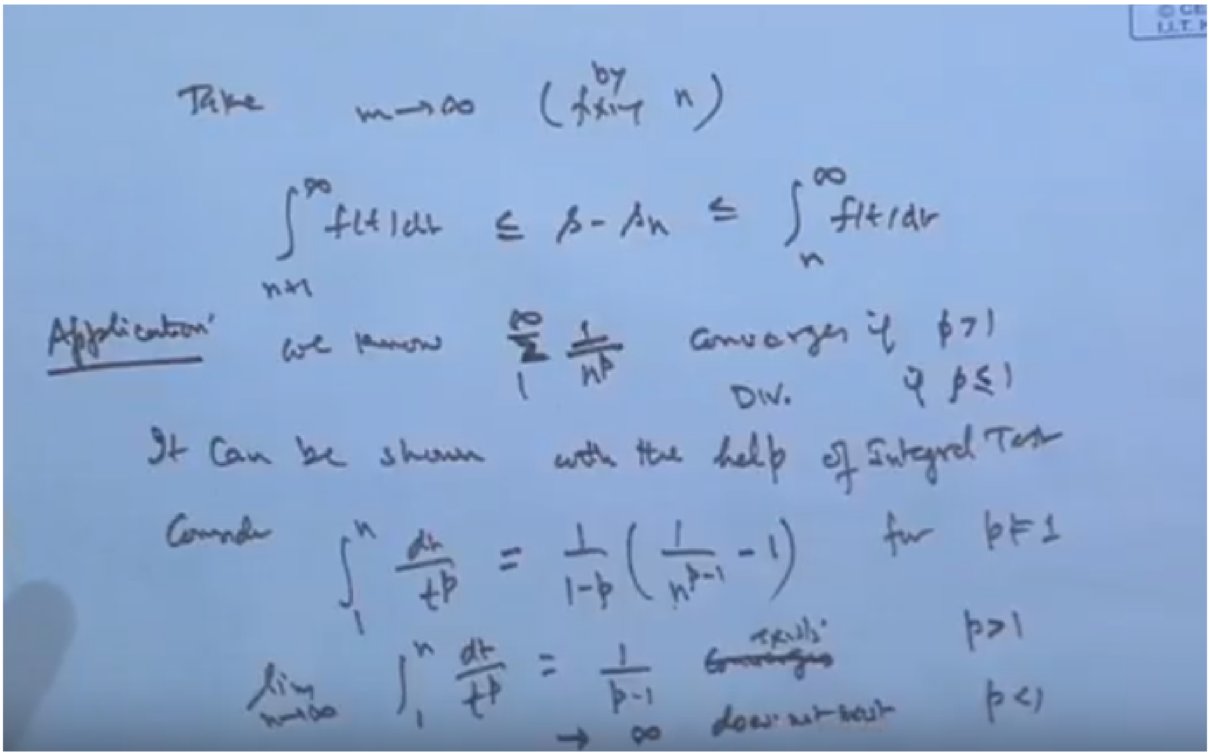

Now from here take the limit as n tends to infinity. So take the limit, where M tends to infinity, fix n, by fixing n. N we are not touching. Then what happens? When you are packing the limit, you are getting, n plus 1 to infinity, ft dt, ft dt is less than, when m is sufficient, this will go to s. So s minus

SN, this is less than equal to n to infinity, ft dt. And that is what, the result says. The result is this. Is it not? So we got the result for it. Okay? So this proves that it.

Now let us see the application part. Suppose I take how to apply, of this result. Let us take this series. We know, the series Sigma, 1 by N, to the power P, 1 to infinity, converges, if P is strictly greater than one, diverges, if p is less than or equal to 1 because, equal to 1, it is, a harmonic series. Now this can be proved, it can be shown, with the help of integral test. How? Let us consider, that what the integral test says, if you look the integral test, the integral test is this. That if you want to find the nature of the series, write down the corresponding integral, ft, ft, and then if the limit of this exists, then corresponding integral, will converge. So this will be taken like this. So consider integral 1 to n, DT, over T to the power P. Consider this integral because, we want, the Sigma, of this thing. So ft is, 1 by T, So 1 by NP. And then this will give, 1 minus 1 by p, 1 over N, to the power p, minus 1, minus 1, for p different form one. Now if p is greater than 1, if p is strictly greater than 1, what happens? The limit of this, as n tends to infinity, 1 to n, DT, over T to the power P, this Will, when p is greater than 1, this will remain positive, power is positive, it will go to 0, and we are getting basically 1 over P minus 1, a definite number. So it converges, exists, instead of this, this exists. Okay? And if P is less than 1, this will go in the numerator, so it will diverge, so it will go to infinity. That is, does not exist. It means the corresponding series.

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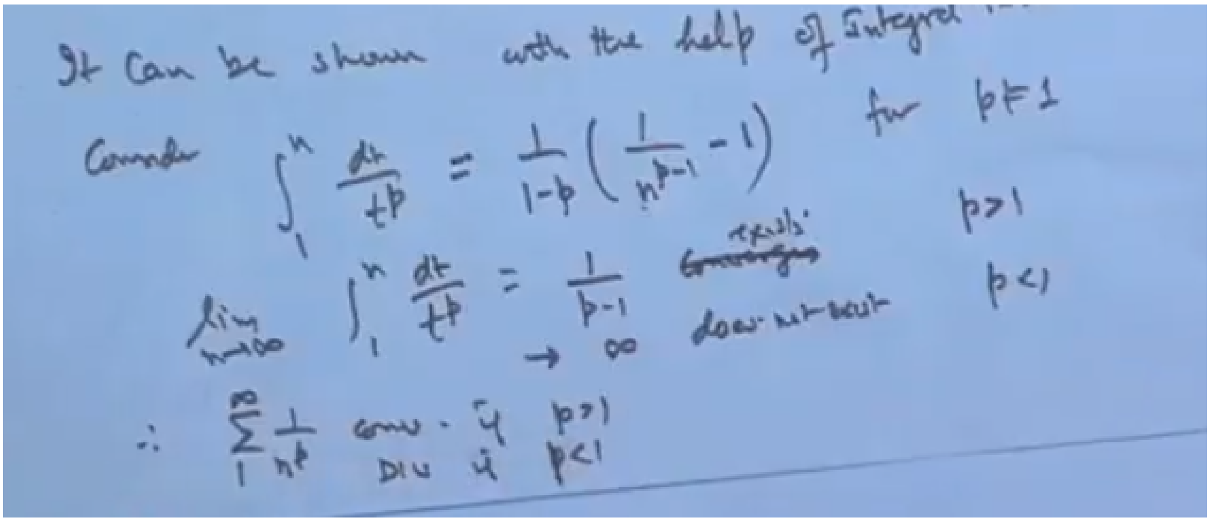

Therefore, the series Sigma 1 by T to the power P, instead of T, we can write, n to the power P, n to the power P, and is 1 to infinity, converges, if p is greater than one, diverges, if p is less than 1. And for p equals 1, obviously it is a harmonic series, so we can get this result. Okay? So that is what, we are getting. Now there are few more tests, which are known as the, Raabe's Test', and of course, so Raabe's test, we will take up next time. Then okay.

Thank you very much.