Model 6

Lecture 33

Rearrangement Theorem and Test for Convergence of Series

Course on Introductory Course in Real Analysis

Okay in the last lecture we have discussed the concept of the absolute convergence series conditionally convergent series and also we have discussed the grouping of the series and rearrangement of the series.

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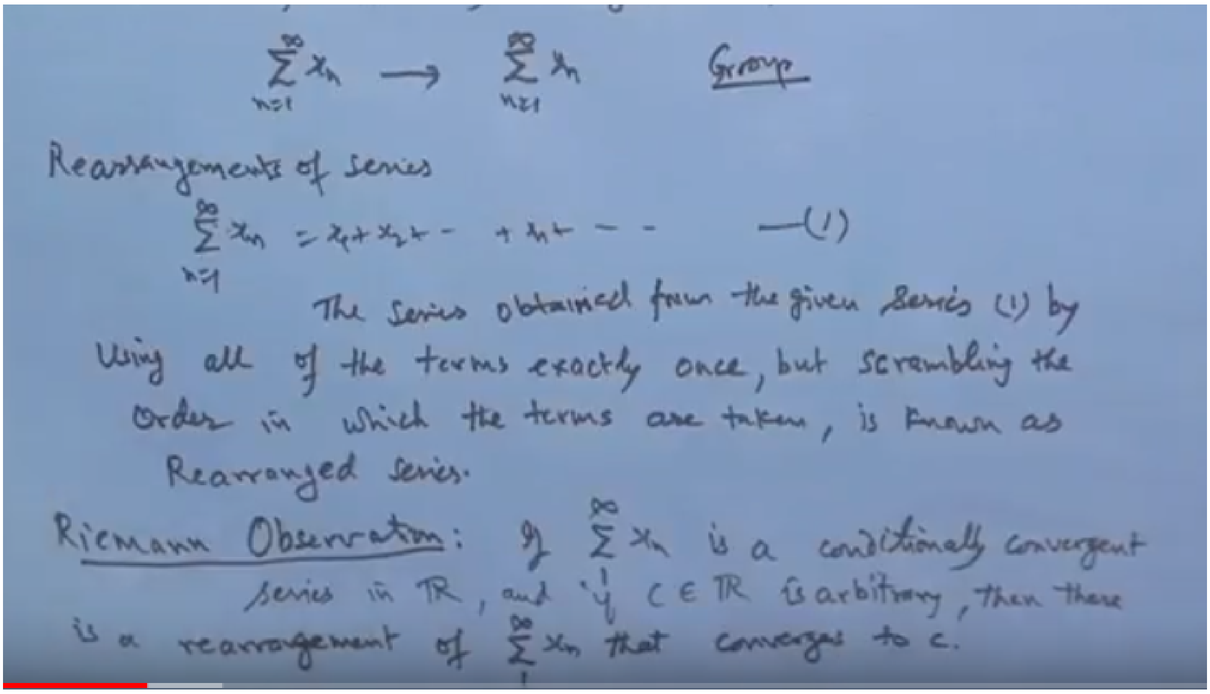

So grouping of the series we mean, if a series is given Sigma xn, of real numbers n, is say 1 to infinity, a series. now if with the help of this if we construct another series Sigma yn and is 1 to infinity, where the terms of the series, order of the terms of the series, is kept fixed, we are not disturbing the order the first element remain at the first place, second element remain in the second place, but what we are doing we are grouping the finite number of terms and then the new series obtained will be known as the series grouping of grouped series of the previous one and in that case we have also seen a result that, if a series Sigma xn is convergent and has the sum s then the corresponding group series will also be a convergent series and will have the same sum at the sum, it means by grouping the terms of the series the new series so obtained will not change his character, if the original series is convergent, the newly constructed series by grouping the elements will also be converging and the Sum will remain the same.

So this is the case of the grouping. okay but in case of the rearrangement of the series, rearrangement we have defined like this rearrangement of series, suppose a series is given Sigma X n n is 1 to infinity we are X 1 plus X 2 plus X N and so on and if we construct a series another series by it from the given series X1 such that, we are using all the terms of the series only once, we are using all the terms exactly once, but scrambling, the series obtained from the given series given series say 1, from the given series by obtained from the given series by using by using all of the terms exactly once, exactly once, but scrambling the order in which, in which the terms are taken are taken, so the series obtained from the given series 1, by using all of the terms exactly once but scrambling the order in which the terms are taken is known as rearrangement, rearranged series, in series. so in this case we are free to interchange the position of the terms and then taking the new series and then considering the new series so. Now as in the case of the earlier when grouping of the series does not change the nature of the series, but in case of the rearrangement of the series the nature is met changed even the some changes if the series is convergent having a sum s then if after rearranging the terms the series will remain convergent but the sum differ. So this was observed by Riemann and the Riemann basically, this is no Riemann observation what Riemann has observed what he says, the observer is, if the series Sigma xn n is 1 to infinity is a conditionally convergent series, convergent series, series conditionally convergent, in R set of real numbers, of course a series of real numbers and if C in any point real number belongs to R cell is arbitrary, is arbitrary then there then there is a rearrangement of the series Sigma xn, of the series that converges that converge is to see.

So this was the observation made by Riemann it means if a series is not absolutely convergent series but it is si conditionally convergent series having n finitely many positive and infinitely many negative terms then in that case if I realize the terms of the series and gets another series than such a series, can be built have a some different from the previous one. in fact if I if I want a sum to be C, then the an arrangement can be possible, so that the real series will converge to the value C. and this observation can be justified by the Riemann is justified like this, that's first the condition is the series should be conditionally convergent, second condition which is in Posey the there should be infinite number of positive terms infinite number of negative terms and then what he did is he first consider the first positive terms, whose sum, positive terms and the sum of that series of the positive terms does not converging to it does not exceed y greater than C. some of the positive terms greater than C, then later on he consider the the negative terms, which a greater than c and like this he has he is able to show that for any given C, one can make a rearrangement so that the series will converge to the same point C. so this was the observation. Now if such a series is given this is conditionally convergent but not absolutely convergent, then obviously the series will give problem when we interchange or when we task-shifting the position of the points, that is the terms are shifted or interchanged, then you won't get the unique sum. However this case is not there in absolutely convergent series. So this next is our source if a series is an absolutely convergent series then whatever the rearrangement we made, the series will remain convergent and will have the same sum as the original one.

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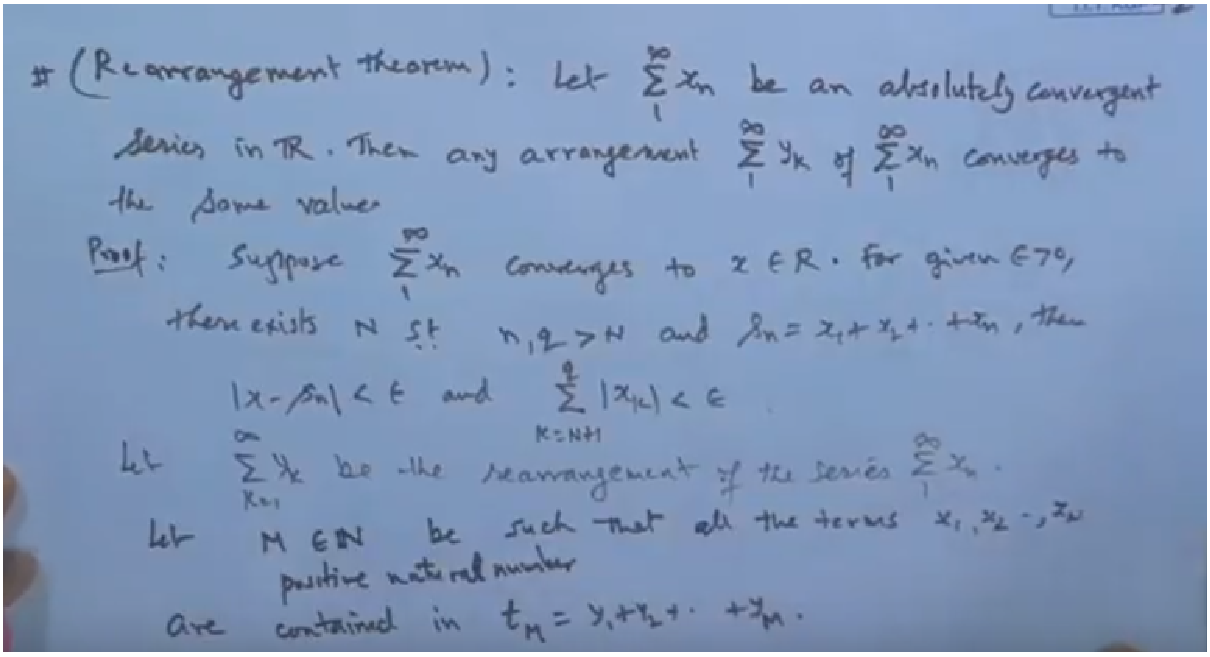

So this result is given in the form of theorem which is known as the Rearrangement theorem, rearrangement Theorem. The theorem says like this, let Sigma xn, 1 to infinity, be an absolutely convergent series, absolutely convergent series, in R, then any arrangement, any arrangement Sigma yK, 1 to infinity, of Sigma Xn, converges to the same value. See the proof of this, so this is very the testing result that if you are dealing with the absolutely convergent series then we need not to bother in that and the rearrangement of the series will give the difference sum, it will not give that same, it will give the same value as the previous one. so proof suppose, the series Sigma xn 1 to infinity, is convergent series and converges to the value say X, belongs to R, so by definition if the series is convergent then sequence of its partial sum will go to X when n is sufficiently large, so for a given epsilon greater than 0, there exists a positive integer capital N such that when nQ, N and Q both are greater than N and the SN be the sequence and we partial some, sequence of the power system say X1, plus X2, plus xn, sum of the first n terms of the series, then the X minus s n, X minus SN, is less than epsilon and Sigma of this mod of XK, when K varies from n plus 1 to Q, it remains less than epsilon.

That is if the series converges then by definition sequence of whisper system will go to 0, it means the remainder terms of the series will remain less than epsilon, so for any Q which is greater than K this is the basically the first remainder terms we had finite sum finite terms from the remainder R amending its remaining series in a bin remain less than Epsilon ok so this is 2. Now let us take the rearrangement of the series let Sigma by K, K is 1 to infinity be the rearrangement, of this series Sigma xn, with the rearrangement of the series Sigma xn, ok. Now if I choose let M, belongs to the positive natural number, this is the positive natural number, capital N and here is the capital N is just a some positive integer n. so set of this potential number so Capital m is a positive integer capital M be such t,hat all the terms of the all the terms, say X 1, X 2, xn are contained In, say are contained in say tm, some y1, Plus, y2 plus, yM, this is a rearranged series and what I am doing is I am taking the sum of the first m terms, since it is a rearrangement it means we are just changing the order of X1, X2, xn and the new series is so then but thus first M terms which we are choosing the involves this X1, X 2, Xn plus few more terms of course is there in that okay. So obviously it follows.

So obviously, when m is greater than n, so if m is greater than or equal to m then in that case the tm this sum, minus sn, because this sum will definitely involve x1, x2, xn so when you take the minus X1, X2, xn, will go out and this is will be is a sum of finite number terms X K if the sum of finite number of terms X k with k, x k with k, with k greater than capital N, because those X1, X2, xn will get cancelled and since M is greater than M and this sum contains all X1, X2, xn. so those term will vanish, will cancel and the remaining term will definitely start from N one.

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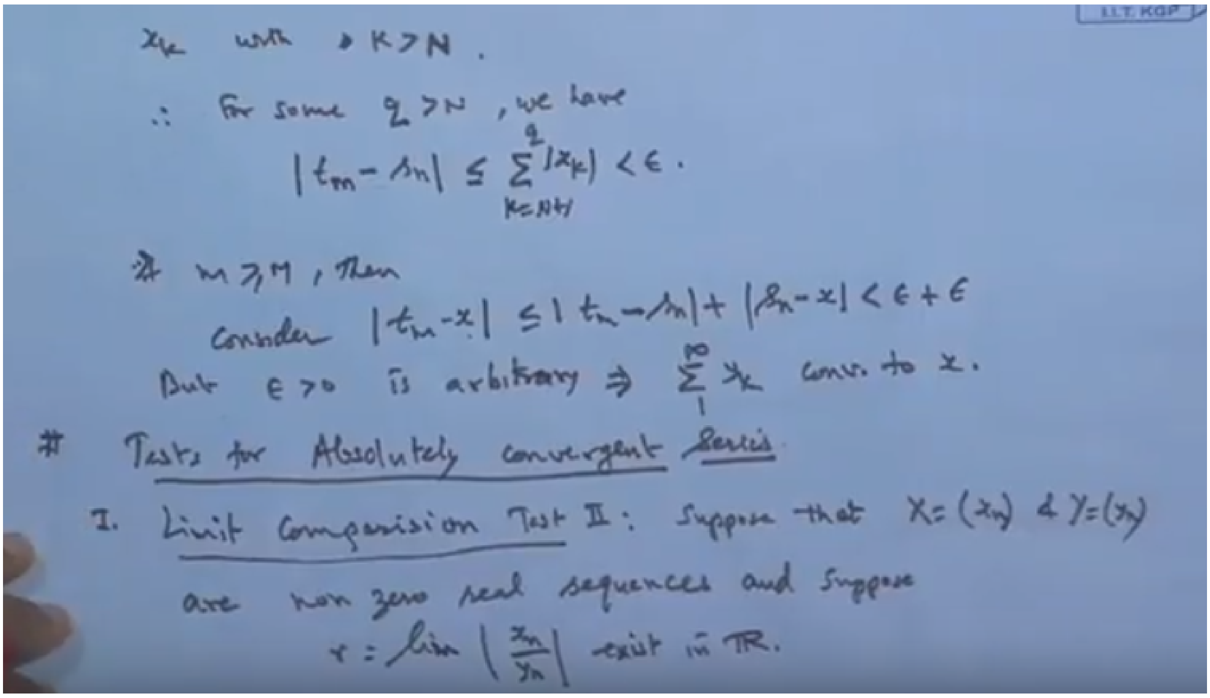

so this is XK when K is greater than n. like this okay so this will be there. Hence for some Q, therefore hence for some Q, which is greater than n, we have tm minus sn, TM minus sn mod of this, now this will remain less than equal to Sigma, K equal to N plus 1 to Q, mod of XK, because TM minus SN, will involve those terms, X K, when K is greater than n. so we can find Q, such that they store some of these terms will remain less than or equal to Sigma of this, but because the series Sigma xn is a convergent series. So this condition holds so this will remain less than epsilon. Okay so this will be less than now we wanted that series, the series Sigma yM is convergent, we want this series to be convergent converges to the sum X so let us find the TM minus X, X is that so consider. So if M is greater than equal to M, then consider mod of TM minus X, now this will be limit less than equal to mod of TM, minus SN plus mod of SN minus X, now TM minus SN is also already less than Epsilon, so this is less than epsilon and X is the sum of the series so SN will go to X, so this will remain as the channel for all M, M greater than equal to M. so this is true to SM, but epsilon is arbitrary, arbitrarily small. so once it has made this shows when epsilon tends to 0, this TM will go to X, therefore this implies the series converges Sigma yk, K is 1 to infinity converges to X. so thus proves the result that in case of the absolutely convergent series, the rearranged series, so obtain, will remain thus convergent and will have the same sum as the original one.

Okay so we are mostly interested in those series which are absolutely convergent. Because the nature of the series - if it is convergent then we need not to bother for the rearranged series, because it whatever the way you sum up the sum will remain the same. So let us go for the some few tests for the absolutely convergent series. Test for absolutely convergent series, absolutely convergent series. We have already seen so many tests for the convergence of series of real numbers and one condition for this is comparison test, we have seen, Nth root test we have seen, and then Cauchy convergence criteria, as also they are for the convergence of the series and like. so here we will simply state the few results without proof. Because the proof follows runs on the same lines as we have done earlier for a general case. okay so let's see the first result says which is the limit comparison test, comparison test first one we have seen, this is the second test, we are saying what it says suppose that, suppose that a series a sequence X equal to xn and y equal to yn, are nonzero real sequences, and suppose and suppose, limit of this exist, limit of mod xn over yn say is equal to R exist, In R.

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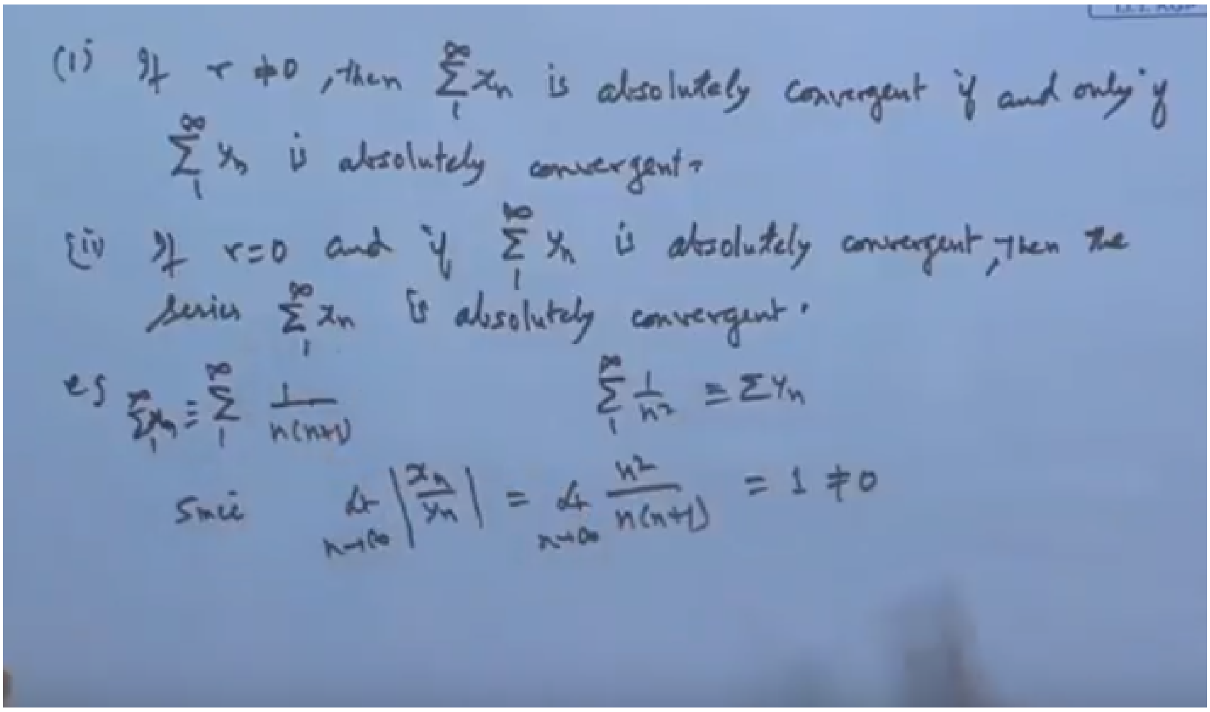

Then what is this result says, if R is different from 0, if R then the series Sigma xn, 1 to infinity is absolutely convergent, absolutely convergent, if and only if if and only if, the series Sigma yn is absolutely convergent. And second result says, if R is zero and if the series Sigma yn is absolutely convergent, convergent then the series Sigma xn, is absolutely convergent. so this is what this is are says is, suppose the two series are given, one is the Sigma X n, other one is the Sigma yn, the result says if Xn, yn both are sequence of real numbers and if this limit xn over yn, as n tends to infinity, limit of this exist and suppose if it is R, then the R if it is different from 0, then nature of this series Sigma xn and nature of this series will be the same, so for the absolute convergence is there, that is if a series Sigma xn is absolutely convergent then Sigma yn is absolutely convergent and by vise-a-versa. Now if R is 0 then in that case Sigma X yn is absolutely convergent will imply the Sigma xn is absolutely convergent. Okay but not the other way round not, the other way round, okay.

So just given series if we are able to construct the series yn, in such a way, so that the ratio limit of the ratio exists then one can identify whether the series is absolutely convergent or not. okay this is the one then another test is, I think examples, we have already seen, suppose I take a series Sigma 1 by n square say suppose, okay then 1 by n square and if we take another series say n, n plus 1say for example, if I take the Sigma 1 by n, n plus 1, say 1 to infinity. we want this series to be a testing this series all the terms are positive of course, then what we do construct the Sigma 1 by n square, this is equivalent to Sigma yn, this is equivalent to Sigma xn, 1 to infinity okay. Now this series nature of this series we know ,it is a convergent series because a Sigma 1 by n to the power P, now what happen if we take the xn, the same xn over yn, mod of this limit of this as n tends to infinity, what is this? this limit is nothing but what n square over N, n plus one, limit as n tends to infinity, so if I take an outside then we get basically the limit is one, different from zero. so here R is different from zero, therefore both the series will have the same nature so this series is absolutely convergent, therefore this is a certificate okay so that way we can find. Similarly for the R is zero we can get it.

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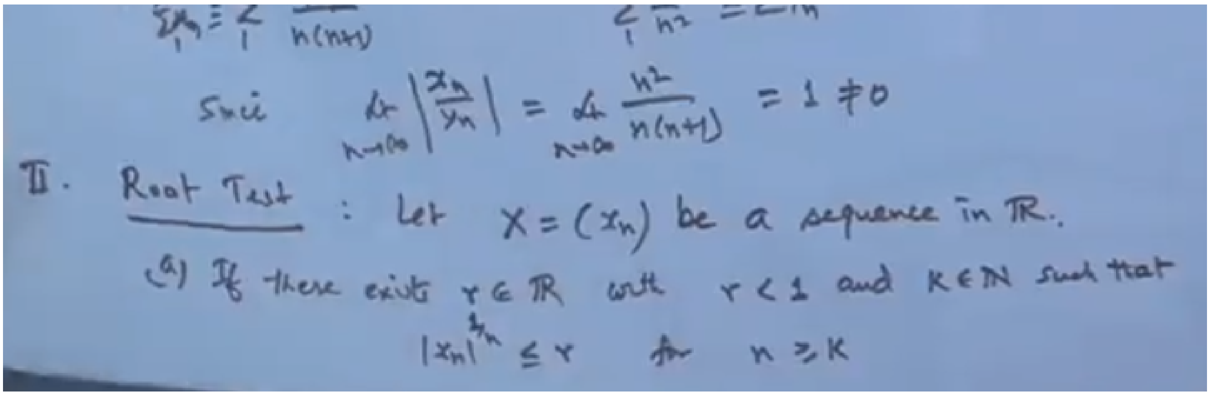

the second test which is root and ratio test, second test, which is the root test Let X which is say xn be a sequence in R, then first, if there exists R, if there in the set of real number, with R less than 1 and a positive integer K belongs to set of natural number n, such that such that, mod of xn to the power 1 by n, mod of xn to the power 1 by n, this mod is less than equal to R for N greater than equal to K, may be the few term this condition may not be satisfied, but after a certain stage the mod xn to the power 1 by n remains less than that number R, which is less than 1, then the series.

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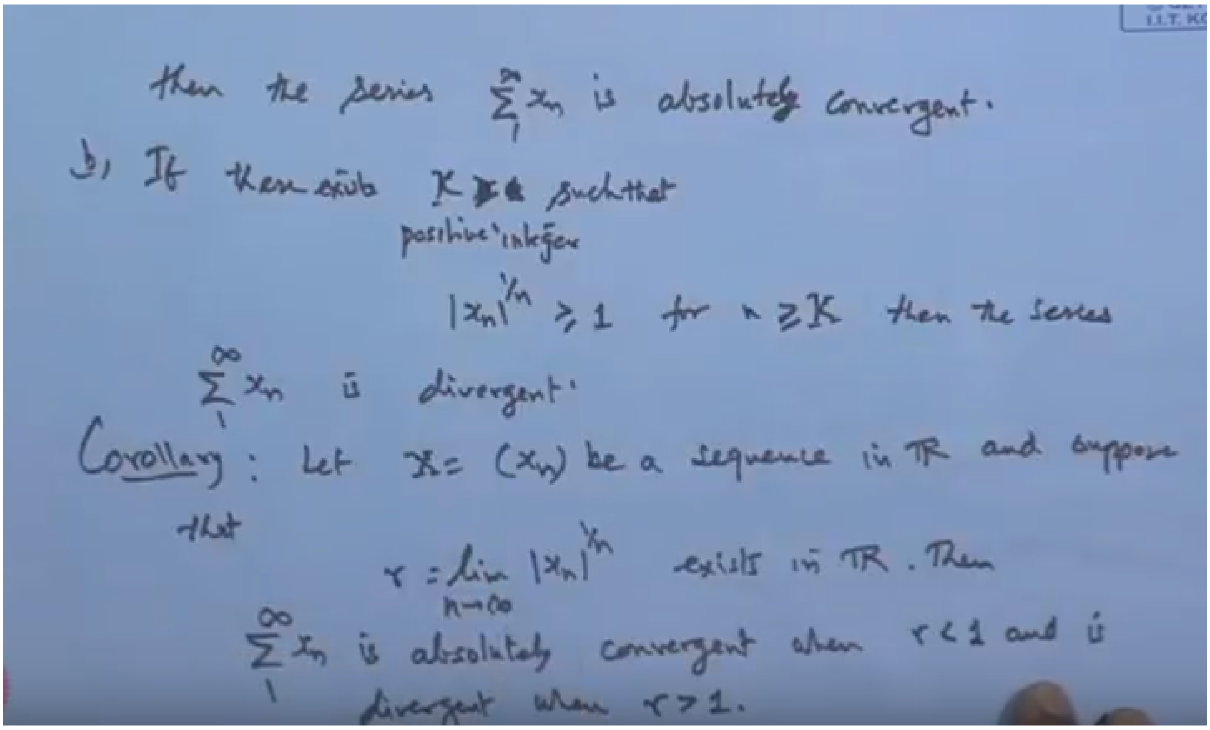

then the series Sigma xn 1 to infinity is absolutely absolutely convergent absolutely convergent this is one and second what if suppose if there exists K if there exists a positive integer K greater than 1, K greater than 1, belongs to n of course, K belongs to n positive integer, I will say positive integer K, positive integer K, belongs to n, such that may be equal also there is no problem, ok, belongs to n such that or you can remove this there exists a positive integer K, such that mod of xn, power 1 by n, is greater than equal to 1, after this n, greater than or equal to K. then the series Sigma xn, is diverging is divergent. So again this is the Annette root test is the peddler to our root test for the general Sigma Xn when the when go mod of xn to the power 1 by n R if lying between 0 and 1 then this is convergent greater than 1, then diverges is it not so mod xn is greater than or a logistical less than 1.

So again this proof will be the same so we are just dropping. now since as a corollary of this in earlier case also we have seen the limping, instead of choosing because this inequality to identify such an R, is a difficult one, so what we do we wanted to avoid this part, so instead of this we can take the limiting value and as a corollary, we can say of this result is, let X which is xn, be a sequence in R, sequence in R and suppose that and suppose that, the limit of this xn mod xn, to the power 1 by n as n tends to infinity exists and equal to R, in r existence then the series Sigma Xn, is absolutely convergent, absolutely convergent, when R is strictly less than 1 and is divergent and it's divergent when R is greater than 1. so for R is equal to 1, we cannot say anything about it because if we take the Sigma 1 by n, then R is 1 series diverges, if I take Sigma 1 by n square, then also R is 1 but the series converges so for R equal to 1, conclusion cannot be drawn.