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Lecture: 32

Absolutely and Conditionally Convergent Series

Course

On

Introductory course in real analysis

In this lecture, we will discuss few results, related to absolute early and conditionally convergent series, are very much helpful, now there are some few more tests which we will discuss after the concept of the absolute convergence and conditional convergent.

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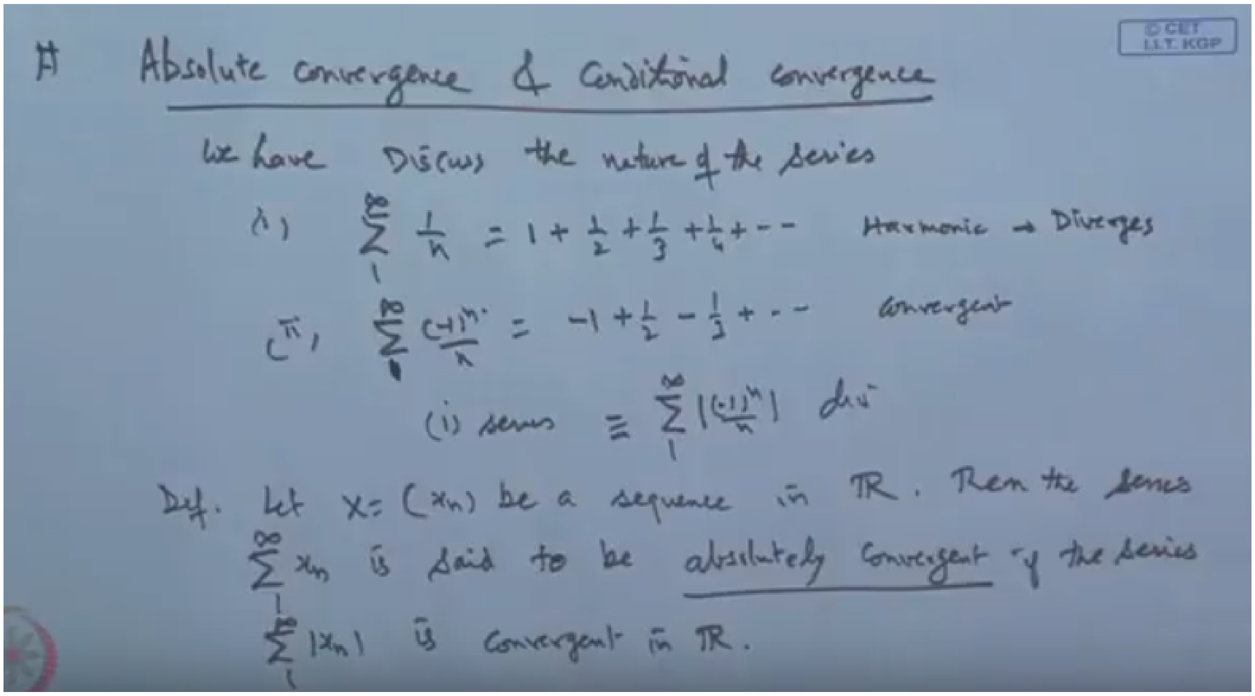

So let's take this, absolute convergence and conditional convergence. Now we have seen the two type of the series, we have discussed the nature of these two series, series, one is Sigma 1 by n, n is 1 to infinity and second one is Sigma 1 to infinity, minus 1 to the power n now what is the difference between these two, if we look the first series all the terms are positive and in fact a strictly greater than 0, they are non negative strictly positive, is strictly positive real number and this is a harmonic series, which we have already shown is a diverging one, diverges. While in the second case the terms are alternately positive negative, so if we take start with this, 1 to infinity or may be 0, I will take 0 1 to infinity let us take here we can say it's, Ok. Minus 1 plus ½, minus 1 by 3 and so on like this so the terms are alternately positive negatives and this is a alternating series and we have also seen that this series is a convergent one. Because if you remember we have this find out the even number of terms and the odd sum of the odd number of terms and then the difference of these two is bounded, by 1 upon 2 to the power 2 n plus 1 which goes to 0 so limit of even number terms and/or number terms are coming to be the same and the limit exists and in fact the sum we can identify the sum for this series, so it is a convergent, now these two series gives you an idea of the two different types of convergence, one is whether when the original series is convergent but when you replace this series terms of the series by its absolute value because the basically 1 by n is the absolute value of this because this first part is nothing but what this, first part one series first series that is the Sigma is nothing but the absolute term of this thing, equal to this so the original series is convergent but when you take its absolute some of its absolute term it diverges, so this shows that the series the convergence can be taken up into two types, one is the odd convergence which is we call it as a conditional convergence, another one is the absolute convergence so we define the absolute convergence of the series as follows. Let X which is X n be a sequence in R, in the real numbers set of real number we say the series comes, Ok. Then this series, then the series, Sigma xn say 1 to infinity is said to be, absolutely convergent, absolutely convergent, if the series of its F through terms that is Sigma of mod xn, and is 1 to infinity is convergent, is convergent in R, so one more thing when we say the series is convergent in our it means the sum of this series must be a real number, or the limit of the sequence of s and must exist and should be a real number, the point must be real number there are this if suppose I take the set say minus 1 say 0 to 1 open interval and if I say the sequence 1 by n then this sequence is not convergent in 0 1 why? Because the limiting value is coming to be 0 which is away from this set, so for the convergence when we say this limit of the sequence of partial sum must exist and it should be the point in the set we are the domain, where we are considering so when you say it's convergent in all means the sum must be real number, okay? So a series sigma action is said to be absolute convergent, if the series of its corresponding series of its absolute term is convergent or not, Okay?

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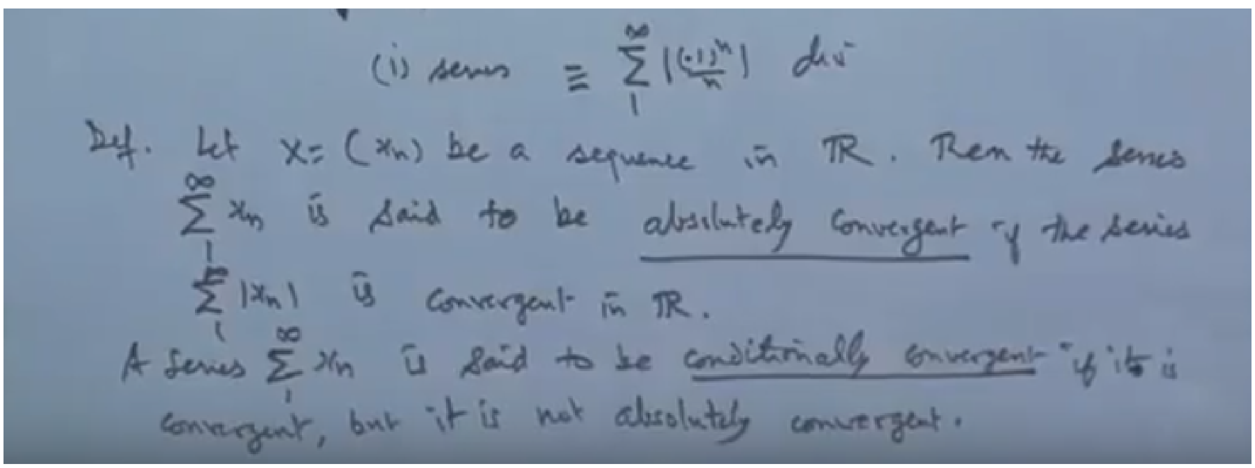

And a series is said to be a series, a series is said to be conditionally convergent a series Sigma xn 1 to infinity is said to be conditionally, conditionally, convergent if it is convergent, if it is convergent but, but it is not absolutely convergent, so that's what in so such type of basically a case occurs when the series is having at some positive some negative terms. because in that case only we can talk about the conditional convergent and the absolute column, if all the terms of the series are non-negative and if the series convergent then obviously absolute converging condition conversion is the same, because there no difference at all, so we say the results which is valid for the non-negative, series of the non negative terms.

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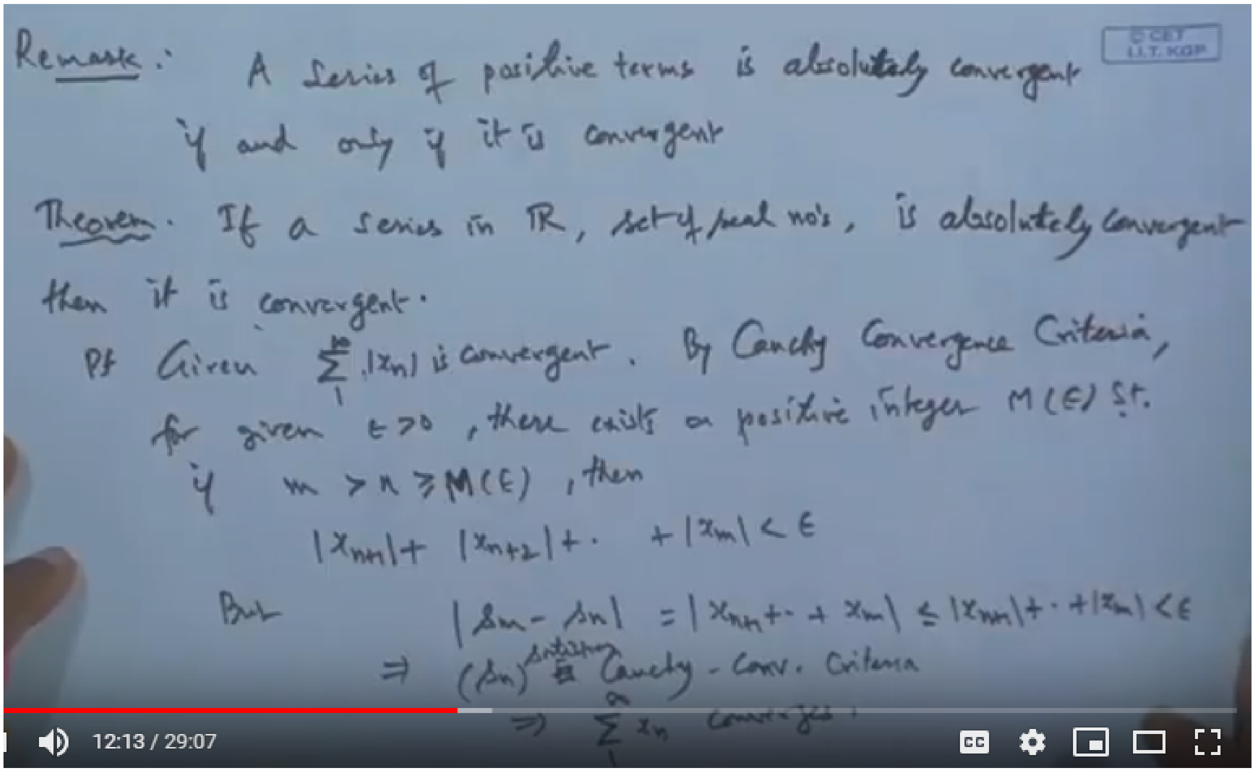

So as a remark you can say, a series of positive terms, a series of positive terms is absolutely convergent, is absolutely, absolutely convergent, if and only if it is convergent it is convergent, because when the non negative terms of there ,there is no question over the conditional case occurs, Okay? In this sequence we have few results, which will help in further study first resort is if a series in R is set of real numbers, we will denote by or real numbers if a series in R is absolutely, absolutely convergent then, convergent then, then it is convergent. I think proof is simple when the series in R is absolutely convergent so what is the meaning of this is? The series of its non negative terms is convergent, that is when we replace the terms why it's a corresponding absolute terms then the corresponding series converges, so given that give me Sigma of mod xn 1 to infinity convergent, even this series is convergent, Ok? So by definition by Cauchy criteria, so by Cauchy convergence criteria, we can say that for a given epsilon greater than 0 there exists a positive integer Capital M say which depends on epsilon, such that, such that if we take m and n both are greater than or equal to capital N, capital M sorry, because we have choosing Capital M which depends on epsilon capital A, then SM minus SN should be less than Epsilon, then mod of xn plus 1, plus mod xn plus 2, up to mod xn is less than Epsilon. Okay? But basically what? This is nothing but, and this will be but what is the mod of SM minus, SN this mod of SM minus SN is mod of xn plus 1, up to X M which is less than equal to mod of xn plus 1 and so on up to xn, by triangle inequality and but this is less than Epsilon so this is less than Epsilon, so this implies shows the sequence SN satisfy the Cauchy convergence criteria, is Cauchy hence it is converging Cauchy is satisfying or set Cauchy sequences and satisfies, Cauchy convergence criteria, therefore the limit of SN will exist therefore, the series Sigma xn 1 to infinity converges, Ok? That's what so is because epsilon is an arbitrary thing, so we can say this is convergent. Now when we have the series Sigma xn, then a terms of the series is fixed up now, that the first term, second term, third term and onward, now without changing the position of the terms if I regroup the terms, then the new series so tend, the question is whether this series will retain the same character as the earlier one? If the earlier one is convergent whether by regrouping the terms of the series without changing their order, the new series whether it remained convergent or not, the answer is yes. If a series is convergent and if you don't change the order of these terms that is the first term remaining on the first position third term remains the third position but we regroup it may be first three terms we are combining, then another five terms we are combining like this way then the new series so obtained will be convergent, if the original series is convergent and not only this it will have the same sum, so that this is called the grouping of the series.

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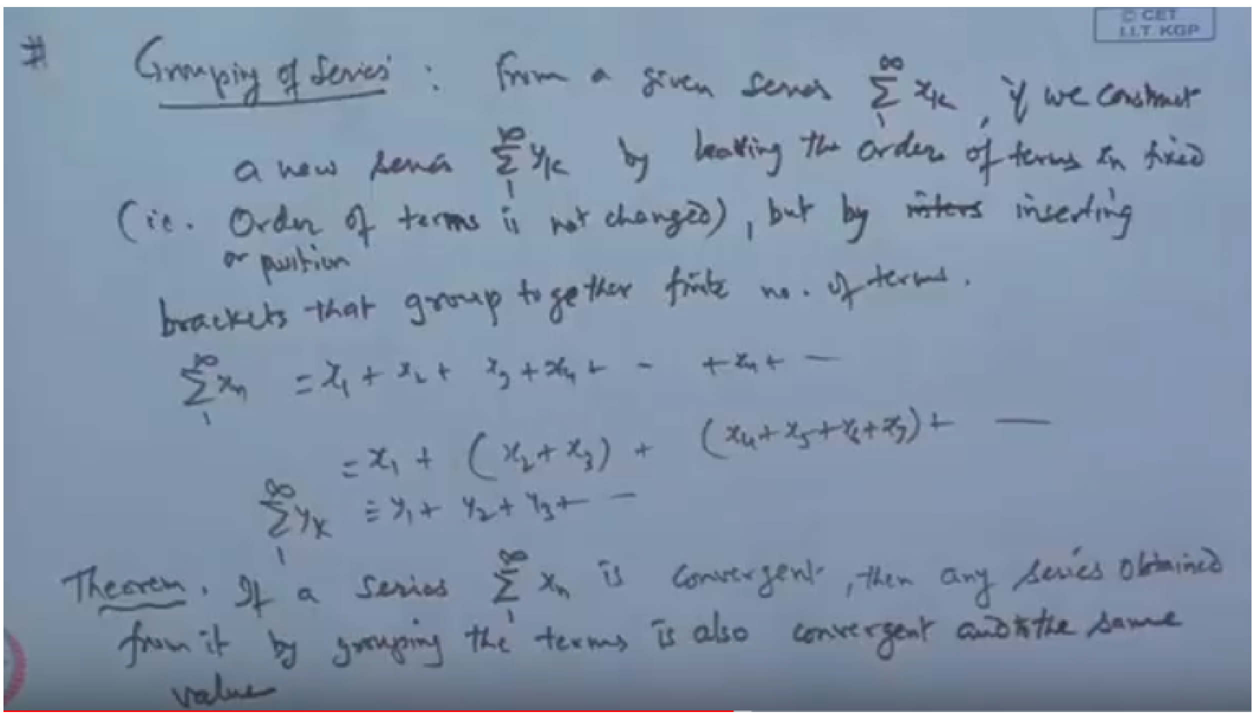

So what is the grouping of series? Grouping of the series we mean, suppose a series is given, Okay? From a given series, Sigma XK K is 1 to infinity if we Construct, if we construct a new series, sigma by k, k is 1 to infinity by leaving the order of the terms x and fixed, by leaving the order of the terms xn fixed, that is, the order of the terms or the position, or position, of the terms is not changed so if initially is obtained by leaving the order of this term order of the term X and fixed but, but by you inserting the brackets, by combining by combine by inserting, inserting, by inserting brackets, that groups, that group together, together finite number of terms, finite number of terms, then such a see we call it that grouping, suppose I say we see this x1, plus x2, plus x3, plus x4, plus xn and so on this is our original series, 1 to infinity and now what we do is we construct a series like this x1 say plus, x2 plus, x3, say x4, x5, x6, x7 like this way so this new series which obtained say equivalent to the y1 plus y2 which is equivalent to y 1 plus y 2, by 3 and so on so this series Sigma by K K is 1 to infinity this is called the grouped series of corresponding to xn, initially obtained. Now the nature of these two series will remain the same, so that's the result is the result says, if a series, Sigma xn 1 to infinity is convergent, is convergent then any series obtained, obtained any series obtained, Okay? From it by grouping from it, by grouping, by grouping the terms is also convergent, is also convergent and to the same value, same value. It means the sum will not change, Okay? Let's see the proof of, proof is also straight forward simple, what is given is a series is given to be convergent.

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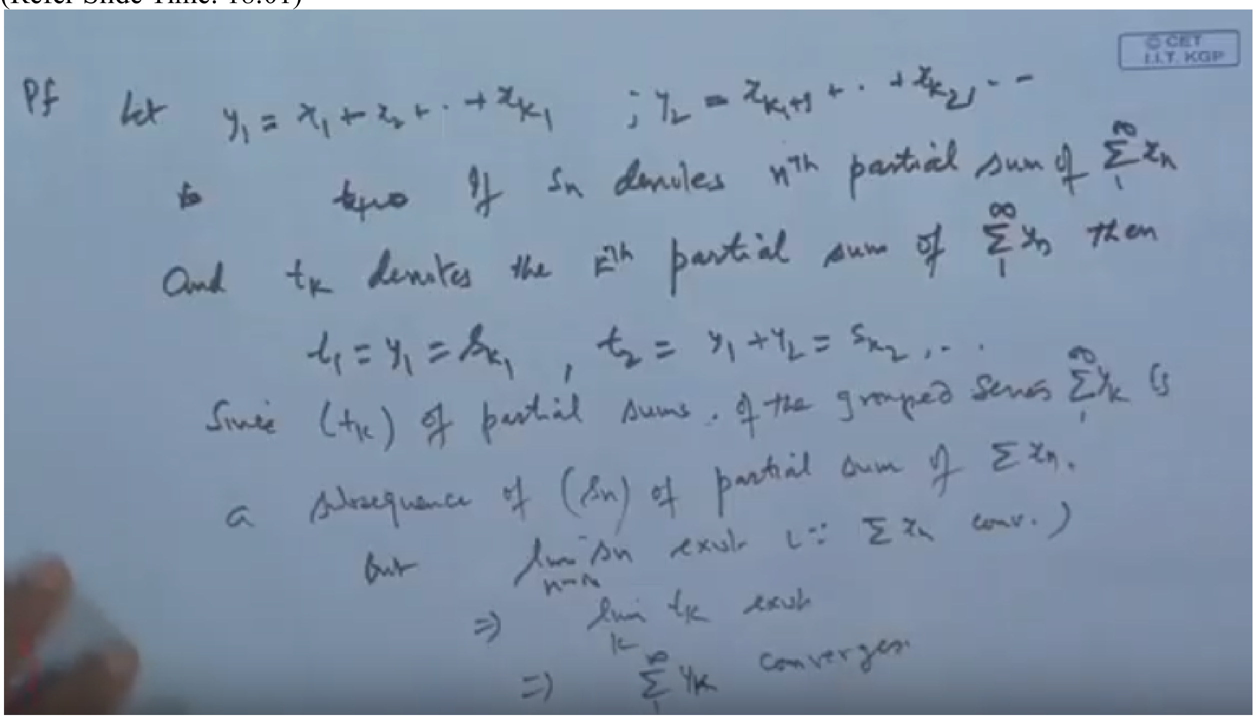

Let us construct let y1 is the one term which is grouped, up to say k1 terms y2 is another term which of the new series which is group from the original series, by choosing these terms, up to k2, k1 it start from k1 plus 1 to K 2 and like this continue, so what happen is that basically so we get the T1 the original series and let so, if suppose if SN denotes the Nth partial sum, of the series Sigma xn 1 to infinity and TK denotes the Kth, denotes the Kth partial sum of the series Sigma by N, 1 to infinity then, what we see here is T 1 that is the first term , first term is what up to here y 1 that is the by 1 so Y 1 is nothing but what? This is the first K 1 sum of the original series so it is the sk1, then T 2, T 2 is the second term means this, now what is this term this is starting from XK plus 1 and so on, so basically when you are choosing this T 2 partial sum then this is equal to 1 by 1 plus by 2, this is first in, okay. So Y 1 minus, y 2 that is equal to by 1 by 1 is distant or up to K 1 and then this is up to K 2, K 1 is say another term so we can say this is equal to Y 1 plus by 2 if I take this second term 2 terms first two terms sum, then what we get is y 1 plus, y 2 this is equal to SK to, Okay? Up to a K by 1 this term Plus this term, that is second term of this new series and continues, so what the first term is the partial sum of the original series second term is also the partial sum of the original series and the original series convergent, so this sequence of the partial sum will converge, therefore this sequence will also converse, so since, the sequence TK of partial sums, Sums of the group series of the group series Sigma by K 1 to infinity it's a sub sequence is a subsequence of the sequence s n of s n SN is the first interval, Okay? The subsequence of this series and this is partial sum of s of partial sum of the series Sigma X K, X N and what this series converges to this partial sum is convergent so therefore, but limit of s n as n tends to exist because the series is convergent, therefore, this implies the limit of T K over K will exist and his implies Sigma Y can by K 1 to infinity converges and that's the proof, Ok? So what we have seen is here that if a series is given which is a convergent series and if I regrouped the terms of the series without changing their sum, then the new series so obtained will also be convergent now let's think a converse part, suppose n series is given whose nature is not known but we are regrouping the series, we are regrouping the series and getting a new series which is suppose a convergent. Now the question is whether the original series is convergent or not? The answer is not necessary to be true.

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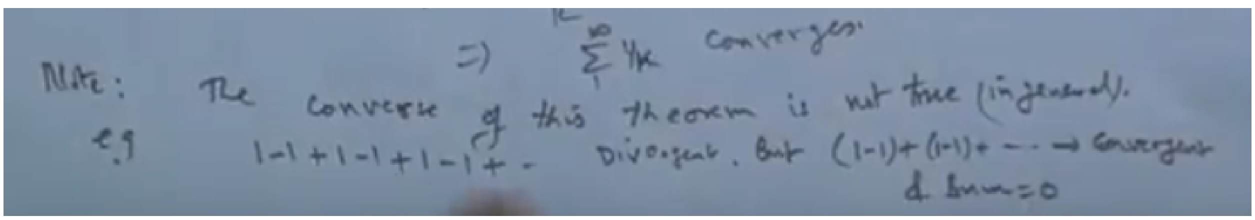

So the converse is the note or is a remark is say the converse of this theorem, the converse of this theorem, of this theorem is not true in general, in general, Okay? For example, if I take this series for example, if we take this series 1 minus 1, plus 1 minus 1, plus 1 minus 1 and so on and if I regroup this procedure diverging series this is a diverging, Ok? But if I regroup the series 1 minus 1, plus 1 minus 1 and so on then it is convergent, it is convergent and converges to what and the sum is 0, and sum is 0, so what we see here the original series is when we regroup from the whole inner we are getting a convergent series but the original series basically is not a convergent series, so the converse of part is not true in general, Ok?

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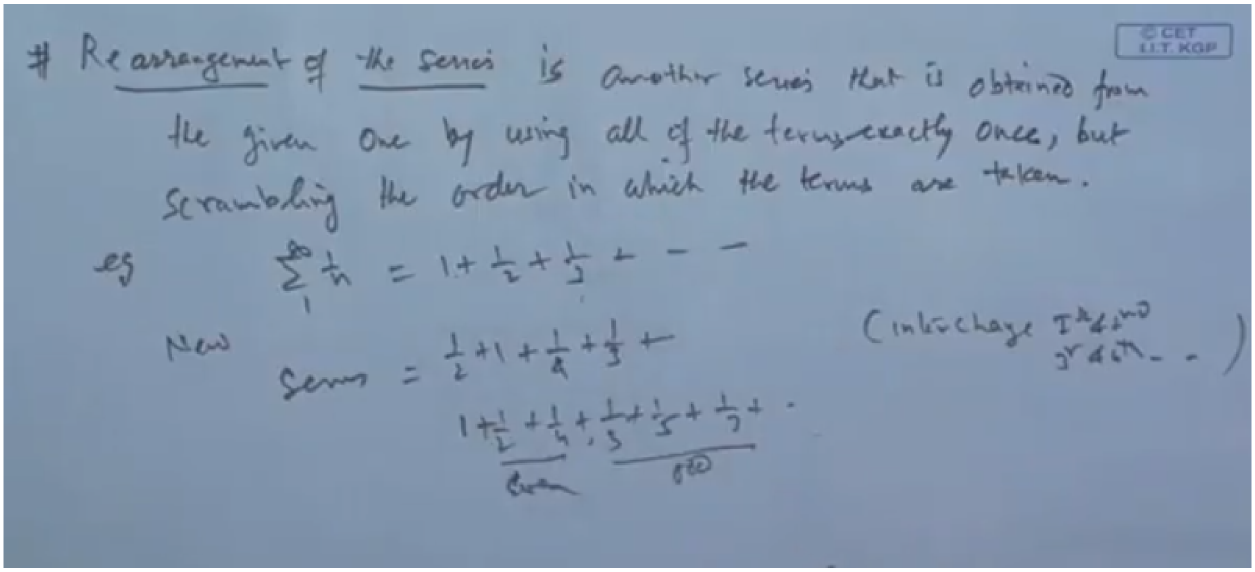

So that's a very interesting example now another concept is also rearrangement of the series, rearrangement of the series, now in case of the grouping of the series we are not changing the order of the terms that is the place of the term is fixed but in case of the rearrangement of the series we are free to take up, to ship the element from first place to ninth place, seventh place to 100 place, like this so if I keep on shifting the position of the Element, we are getting infinitely new sequences new series, from the original one now in this process the series new series soft and whether that series will also retain the same character as the original one if the original is convergent and by rearranging that series, if we are getting a new series whether the new series is so convergent.

So the answer is in not true that a form series which is given to be convergent series and if I rearrange the say terms of the series then in general you cannot say the new series often will be a convergent and will have the assignment and in fact this is a very good result, very good result, and very important result which is given say by one of the famine mathematician Riemann, what he said in case of a alternating convergent series if I regroup the terms of the series, then we can get any real number which is the sum of that series, that is for any given real number one can have a rearrangement of the series one can have a rearrangement according to that given real number so that the series will converge in converge to the real number, so rearrangement of the term is very so clear? What is rearrangement let me just talk it another series the arrangement of a series each is, another series, another series that is obtained, that is obtained from the given one, given one by using, by using all of the terms, all of the terms exactly once.

But scrambling, scrambling the order in which the terms are taken are taken, are taken for example, if we take a harmonic series, this is our harmonic series Sigma 1 by N 1 to infinity 1 plus 1, by 2, 1 by 3 and so on the new series, suppose I take that rearrangement of the services in touchingly the first and second term, so half plus one, third and fourth ton, one fourth and one third, like this so interchanging, interchanging in the first and second term, first and second, third and fourth and so on we get new series so this is the new series, which is the rearrangement of which is obtained by rearranging the ton, or maybe another series if I take like this 1 plus ½, 1 by 4, 1 by 3, 1 by 5, 1 by 7 and so on. Means first the even terms we are talking first event, even terms 1, then 1 by 2, then 1 after one BRN 2 even terms than 3 odd terms and like this, or terms even terms and so this another one, now this will give a different series, but the range series will have a different nature, so that we will discuss next one.

Thank you very much.