Model: 6

Lecture: 31

Comparison test for series

Course

On

Introductory course in real analysis

Okay. So, last time we have discussed the various tests, for the convergence of the series, now in continuation we will do few more tests and discuss few more tests like a comparison test and limit comparison test and after that we will come to the conditional and absolute convergence of the series.

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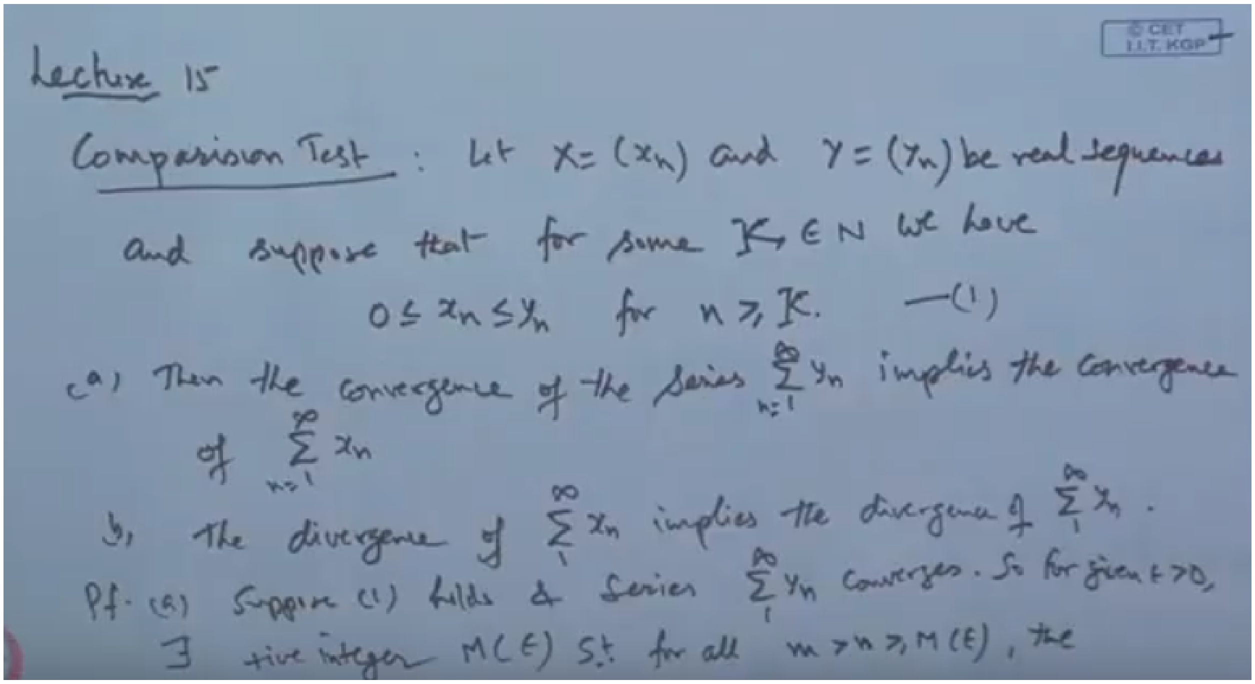

So first stage to each a in continuation we have another test age which is known as the comparison test, comparison test, the test says, Let X which is X N and Y, say Y be real sequences, sequences and suppose, and suppose, for suppose that for some K, for some K which is I of course a positive integer, we have this inequality zero is less than equal to xn less than equal to Y n and this is valid for N greater than equal to K. it may be true for all ends also if not possible then at least after certain stage if this inequality holds or then, the series the convergence of the series, then the convergence of the series, Sigma by N and is of course 1 to infinity, converges of the series implies, the convergence of, of the series Sigma xn 1 to infinity, 1 to infinity and the divergence of the series of the series, Sigma xn n is 1 to infinity implies, the divergence of the series Sigma Y n, in fact this time we that we have discussed already in case of the sequences, so similar results we have for a sequence of non-negative series we have sequence of the non negative terms, all terms are non negative so what the comparison test says if we are having a series say Sigma n of each terms is non-negative then the to discuss the convergence of the series to find out the nature of the series Sigma xn if we are able to identify some sequence by n for which this less and 0 is less than equal to xn less than equal to yn holds for either for all n, all may be after certain stage, then the convergence of this by n will implies the convergence of the series Sigma, except it means that with the help of xn if we find out a suitable yn for which the convergence of the series Sigma yn natural is known then one can easily establish the convergence so by, and in case if this reason is true but if the series is divergent Sigma xn then the divergence of by hand will be there, it means if we are interested to know the one and we expect or something that that series is diverging then we have to find this race we have this is diverging then this will diverge, so this is the main motto for this comparison test, the proof of course is very straightforward, suppose we have this series converges given suppose, the ration hold one holds and the series, Sigma yn 1 to infinity converges, this is given, Okay? So by definition of the convergence a series is said to be convergent if and only if it is Cauchy, so for a given epsilon greater than zero, so for given epsilon greater than 0 there exists a positive integer say Capital M, which will depend on epsilon, such that for all m, n for all M which is greater than say n and greater of than equal to M of course all MN is greater than equal to M as n, I am choosing m to be larger than n, for all MN greater than equal to M,

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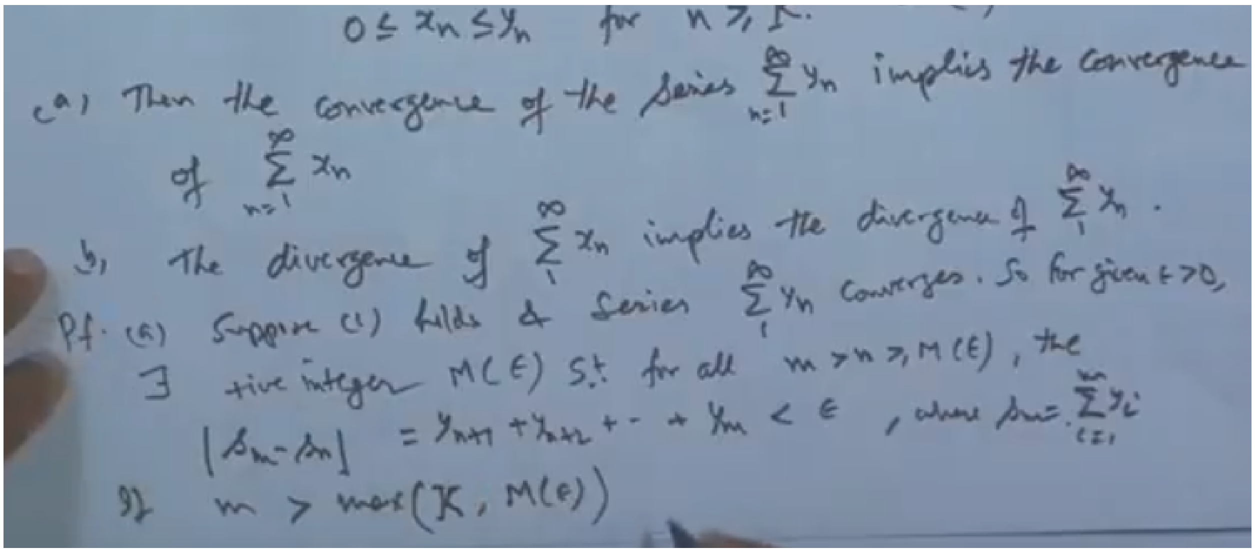

The condition, the condition that YN plus 1, y n plus 2, up to say ym is less than epsilon because basically, this condition is what? This is nothing but the SM minus SN, where, SM is the Sigma yn a y I, I is 1 to M. So basically this is the Cauchy convergence criteria, so a series because it is given to be convergent so by means of Cauchy convergence criteria, for any epsilon one can identify a positive integer, so that this is a source. Okay? Now let us choose m, to be greater than the maximum value of this K, for which this result is true, this inequality is valid as well as the capital M which we have already got it because of this convergence, so if I choose m to be greater than this, then in that case the result this thing because YX n is less than equal to yn so this part will give that xn plus 1 by n xn plus 2 etcetera.

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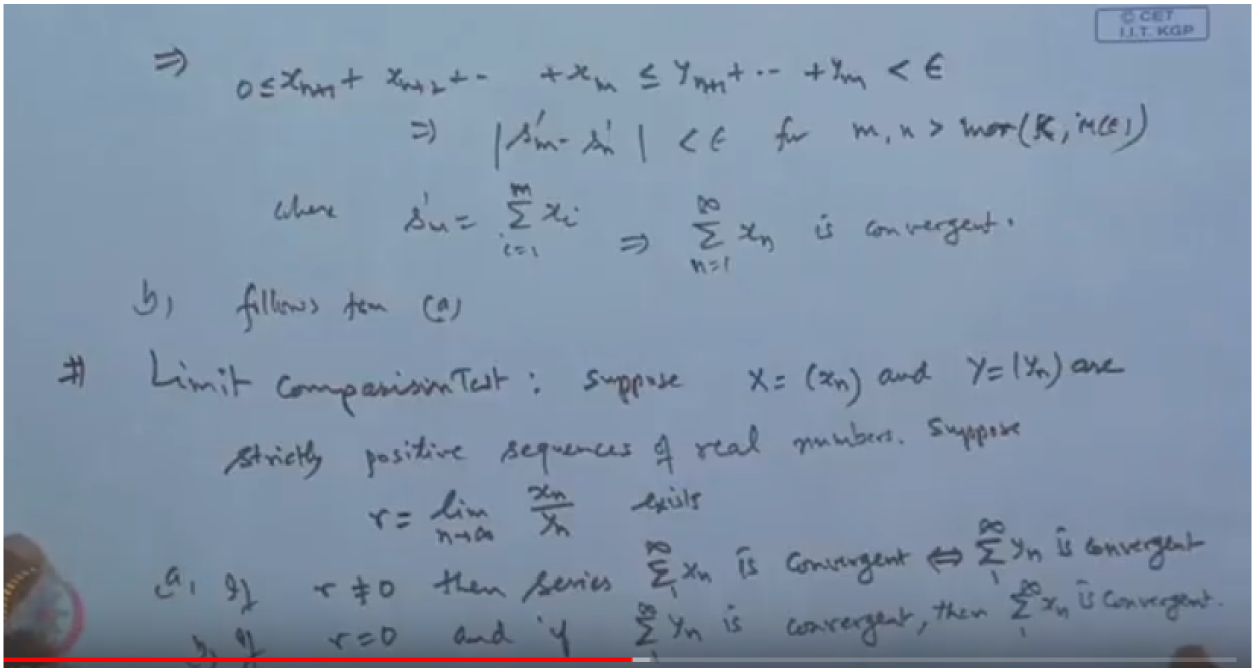

This will imply that X n plus 1, X n plus 2, and so on up to say X M, this will remain less than by n plus 1, by n plus 2, ym because xn is because M is greater than the number so it satisfy both the conditions, Ok? And then this will be less than epsilon, this is true for all M greater than the maximum of candy and this is greater than 0 because all terms are positive so it will be greater than or at the most equal to 0, so what this shows? This shows that this sequence s, dash M minus s dash and this remains less than epsilon R for all m n N greater than equal to capital N, capital and say K, maximum of K and epsilon, this some, some integer, Okay? K in Epsilon, and we are s dash R is what? As X dash stands for Sigma, X I, I is 1 to M so this shows the sequence X s this N is a Cauchy sequence, there it is convergent, so this implies the series Sigma xn 1 to infinity is convergent, Ok? So the proof is with, the statement be follows from the first it follows from what the he says the series diverges then Sigma BN diverges, suppose the series given Sigma xn to be a divergent series, and let this series Sigma yn converges, but the station 1 also holds, so it means if a series y n is convergent then according to the part a xn must be convergent, Sigma xn must be convergent but it contradicts the result given thing because the given is Sigma xn is divergent, therefore, a contradiction is because of our assumption this series converges. So wrong, so B follows from immediately, Okay?

Nothing to prove, now another test is, all let us first take the test then another test is which we call it as a limit comparison test, in fact this is the modified version of this here, because we it's very difficult to get this identity, which is valid for all N or after a certain stage, it's very, very difficult for a sequence xn and yn to arbitrary sequence, are there so it's not possible always to get such an inequality, therefore this test comparison may not be very much helpful, unless you know this inequality, so what is it we have slightly it is modified and a simpler form is given which is known as the limit comparison test, the test says suppose that, suppose X which is X N and Y which is y n Are strictly positive sequences, of real Numbers, of real numbers and suppose, and suppose the limit of this xn divided by by y n over, n when n tends to infinity exist. So for this limit exist say is equal to R, say is equal to R, this limit X then what it says is if R is different from 0 means when the limit of xn over yn is different from 0 then the series, then the series Sigma xn 1 to infinity convert is convergent, if and only if, if and only if, Sigma yn is convergent, is convergent, it means the behavior of the series Sigma by N and Sigma xn are parallel that if this limit xn over yn exists and it different from 0 then both the seen it will have the same nature, if this series is convergent this has to be convergent, Okay? And second part is, if R is equal to 0 of course the divergent part here is not being some, divergence we cannot say if it is 1 is diverging other will also diverge but obviously, if we have this sort alienation that that also gives a result for a divergence of the series but here we can only take the claim, about the convergent part of it, that is so far the convergence is concerned both these series will have the same nature. Same means convergent then come and if R is zero and if, and if Sigma by n is convergent when is convergent then Sigma xn 1 to infinity, is convergent. So there is certain limitation in this, limitation is that we are unable to test the divergence part when we cannot claim anything about a divergent, but obviously for the convergent which we are interested more to judge, the convergent with the help of the given series, Ok.

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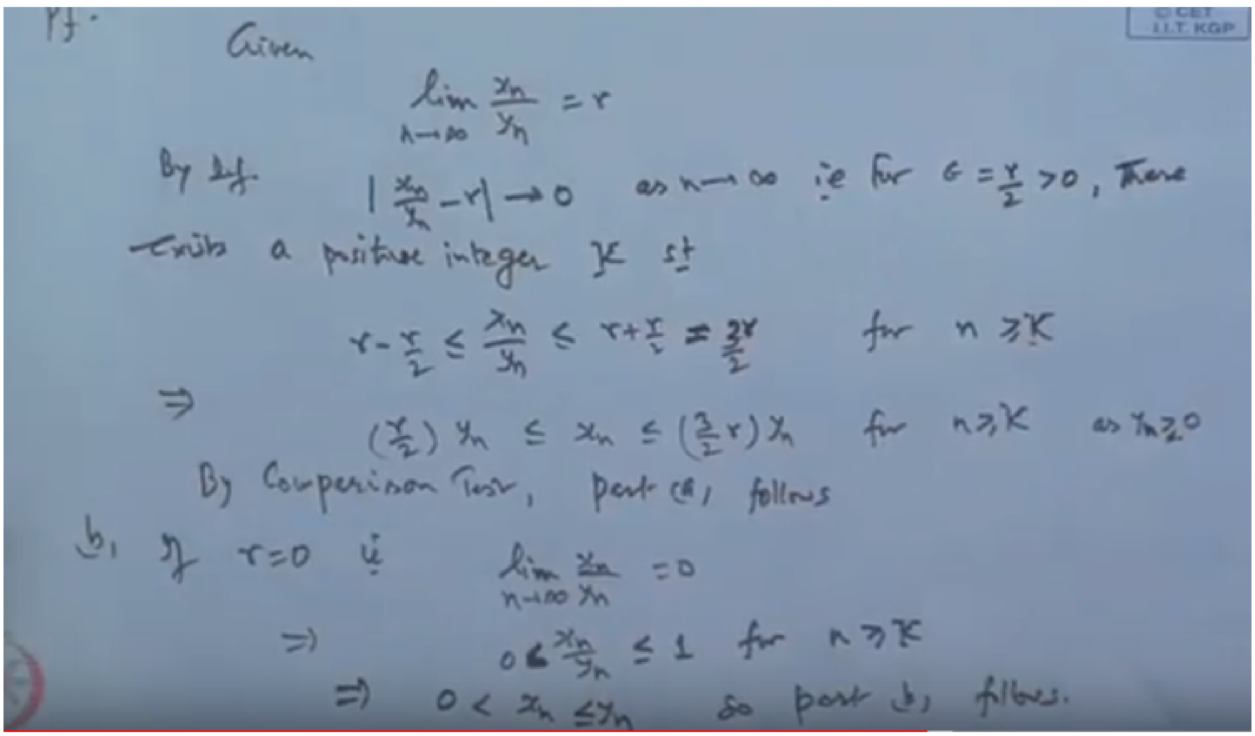

So let us see the proof of this, now what is proof is, given that, given the limit xn over yn, as n tends to infinity is say R, which is different from 0, of course, now once the limit is given then following any epsilon greater than 0 by definition, what happen? By definition this mod of xn, over by n minus R this will go to 0 as n is sufficiently large, or for a epsilon greater than 0 that is for a given epsilon is forgiven epsilon say equal to R I am taking r y2, greater than 0 there exist, there exist, there exists a integer, a positive integer, say K such that xn over yn, this will lie between what 2 1 R minus epsilon L plus Epsilon, so, here there are minus epsilon means R by 2 and then here is also R plus epsilon R by 2 and which is obviously less than 2 R, this will be less than or equal to 2 R this hardly matters even this 3 by 2 will work. Ok? So we get the 3 by 2 so this is equal to 3 by 2, Ok? 3 by 2 R, into now if we let like this and this is true for all N greater than equal to K, is it not? This is true for all N greater than equal to K so this happens, now from here; can we say that R y 2, into yn is less than equal to the X n which is less than equal to 3 by 2 R into yn, Ok? Now apply the comparison test and this is true for n greater than equal to K, because the by n are positive, therefore when we are multiplying this it is not going to change the inequality, as y n are greater than or equal to 0, of course greater than 0 otherwise this will equal to 0 will help problem, so yn is greater than 0. Now apply the comparison say so by comparison test we can get, comparison test, this is comparison o and here also it so not io comparison test, Okay? So by comparison test, we can say the lizard a follows the part A follows, Okay?

That's what, now it gets second is Part B if R is 0 RS 0, means that is limit of this xn over by n as n tends to infinity is 0, now since X N and by n both are positive non negative, so the issue a cannot be negative so clearly from here it implies that xn over yn will always be greater than or equal to 0, in fact it is strictly greater than 0 because they are non-negative to on a strictly, strictly positive sequences, of real numbers instead of positive sequence of none is 0 even so xn over yn will be is strictly greater than 0. Now since the limiting value is 0 it means the terms or keep on decreasing, and decreases to 0 so after a certain stage this ratio will remain less than 1, so once it is less than 1 so you can say this is less than equal to 1, for N greater than equal to K, this is too because limit is 0 means it keeps on in decreasing and decreases to 0 so after certain distance the ratio will remain less than or equal to 1 and it is always greater than 0, so now what happens if we apply this if suppose Sigma by n is convergent, so from here you multiply by by L so this implies that, xn X 0 is less than xn less than equal to yn, so if Sigma by n convergent Sigma xn will come, so Part B follows, so this is the two results.

Which will help, in getting the nature of this series whether it is convergent or not, only what we have to do we have to suitably identify YN Okay? And then inequality or may be the limit problem, limit third I advise that limit is a much better way of judging the, convergence of the series. So how to identify Y M? so that limit of xn over yn will exist and then nature of the yn will decide the nature of the action.

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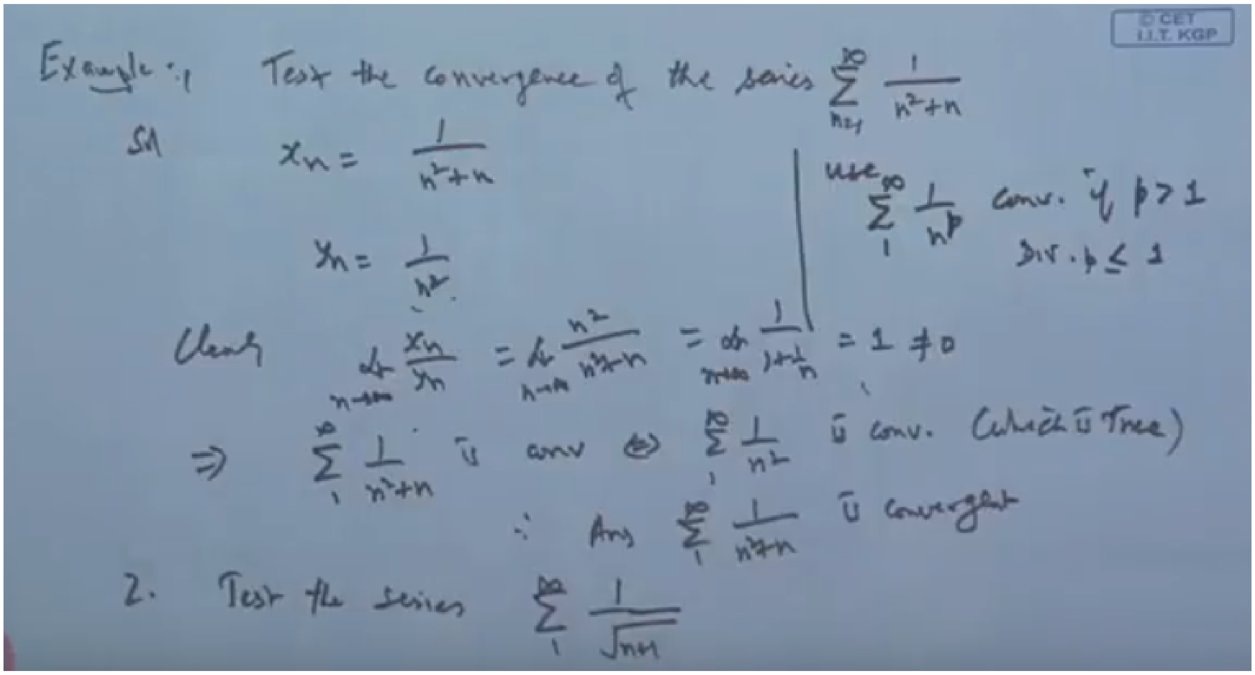

Okay, let's see the few examples which you will help in getting this thing clear, suppose I wanted to test, the convergence of this series, of the series, Sigma and is 1 to infinity, 1 over N square plus n, now when we say the convergence of this at does not mean that we are only, we have to test only the convergence part convergence of the series means we have to see whether this series is a convergent series or diverges, but the way of writing is test the convergent means it includes both whether the series is convergent or divergent, Ok? So let's see this is the sequence of non-negative real numbers, strictly of course positive because n is 1 to infinity, therefore we can apply the ratio comparison limit test, but what comparison limit test says there must be the sequence xn and yn, this sequence if we want to test this series Sigma n you have to identify by in such a way so that the limit of this exists.

In order to identify this so here xn is given to be 1 over N square plus n, now normally when we say the yn to be identified, we normally use this result. Sigma we know Sigma 1 by n to the power P when n is 1 to infinity, this is Convergent, converges if P is strictly greater than 1 and diverges if P is strictly less than or equal to 1, so basically the Y N should be chosen in such a way so that it will fall in one of the in this category, so here what the trick is let us take the look, here term and anything common here outside any n square if you take outside common from the denominator take the largest power of n say outside, then what happens? If you choose by n to be 1 by n square then when you are take choosing n square outside that this becomes 1 over 1 plus 1 by n so limit of this xn over yn will exist. So clearly limit of xn over yn when n tends to infinity is nothing but what? This is equal to n square over N square plus n, limit as n tends to infinity divided by n square so 1 plus 1 by N and limit as n tends to infinity and this limit is 1 which is different from 0. So, what we see here that if you xn and by n both are thus strictly positive real sequence of real numbers, such that limit of xn over yn exists, which is different from 0, therefore this series is convergent if and only if the series convergent, but this series is of the form Sigma 1 upon n to the power P where P is greater than 1, so series converges so this implies that Sigma 1 over n square plus n is convergent, if and only if Sigma 1 by n square is convergent, which is true which is true, because of this therefore answer is the series will be convergent, so answer is the series 1 to infinity 1 over n square plus n is convergent, is convergent, Ok? Now similarly, if you go for the another example set, let's take this example Sigma test the series, Sigma 1 over under root n plus 1, 1 to infinity again this is a sequence of non-negative real number strictly positive real numbers, so to test this we will apply the limit comparison test,

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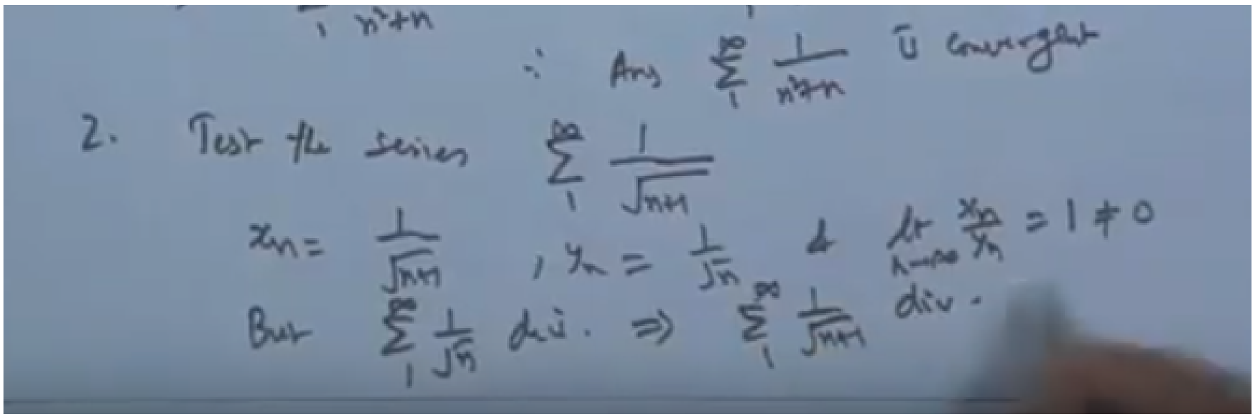

so let us take here to xn here is 1 over root n plus 1 then by n you take if the term which is highest power here is highest power here, so highest power is 1 there are no problem and if the a when you take outside it is becomes 1 by root n, so if I take this and the limit of xn over by n when we choose the limit, the limit will come out to be 1 which is different from 0, you just see take root N and divide, by it means the nature of these two series are identical but the series, but the series Sigma 1 by root N 1 to infinity this is of the form Sigma n to the power 1 upon n to the power P we are P is less than 1 so diverges, therefore, the series Sigma 1 by under root n plus 1 will also diverge and that's the answer,

Okay? So we can get it this way. So main idea is that, you have to pick up the suitably, the terms by n so that we can compare it with our given sequence, given series xn the terms of the series xn and hence, the one can identify the nature. Now this is the one which we were, very important these tests are very interesting important because it gives immediately the series nature of the series without going for this sum, because sometimes we are not interested, in getting exactly the sum of the series or the limit of the sequence of sum, because it will not help us, we are not interested in finding the sum of the series we are all interested only finding the series whether it is convergent or divergent, and for this, this tests are very much helpful.