Model 5

Lecture 30

Tutorial V

Course

On

Introductory Course in Real Analysis

So this is that again tutorial class, tutorial five, based on the lectures starting from 21st to 25th, based on this.

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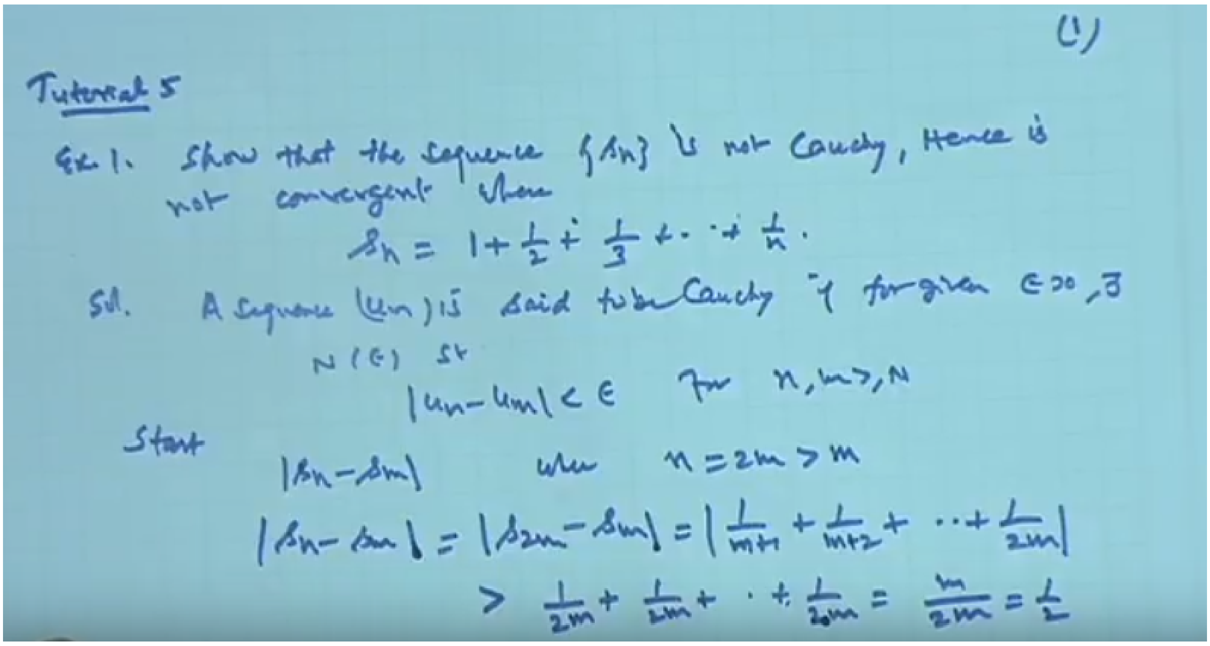

So let us see few problems, we will discuss. So that the sequence SN, sequence SN, is not Cauchy. Hence, is not convergent. The sequences in, where SN has, 1, plus 1 by 2, 1 by 3, plus 1 by. So we have to show this sequence is not a Cauchy sequence, hence it is not a convergent. So Cauchy sequence means, a sequence is said to be a Cauchy, a sequence UN, is said to be Cauchy, of real number, is said to be Cauchy, if for given epsilon, greater than 0, there exist a capital N, depending on epsilon, such that, mode of UN, minus um, less than epsilon, for all n m, greater than equal to capital N. So if it is not cauchy, it means we cannot get the difference of any two terms of the sequence, after a stage n, is less than epsilon. So let us start with, mode less of SN minus SM. And suppose n is greater than m, start with, where, say n is equal to say 2M greater than M. Start with this. Okay? So what will be the SN minus SM? So SN minus SM, this will be equal to, s, 2m, minus SM? And which is equal to 1 over M plus 1, M plus 2, up to say 1 by 2 m? So this will be, greater than. Now the smallest term is 1 by 2 m? So this is greater than 1 by 2 m, into 1 by, plus 1 by 2 m, up to n terms, n terms, so 1 by 2 m and so this is M over 2 m, which is ½. So what we get is s n minus s m, is greater than ½. Whatever the N and may be, large, so it cannot be.

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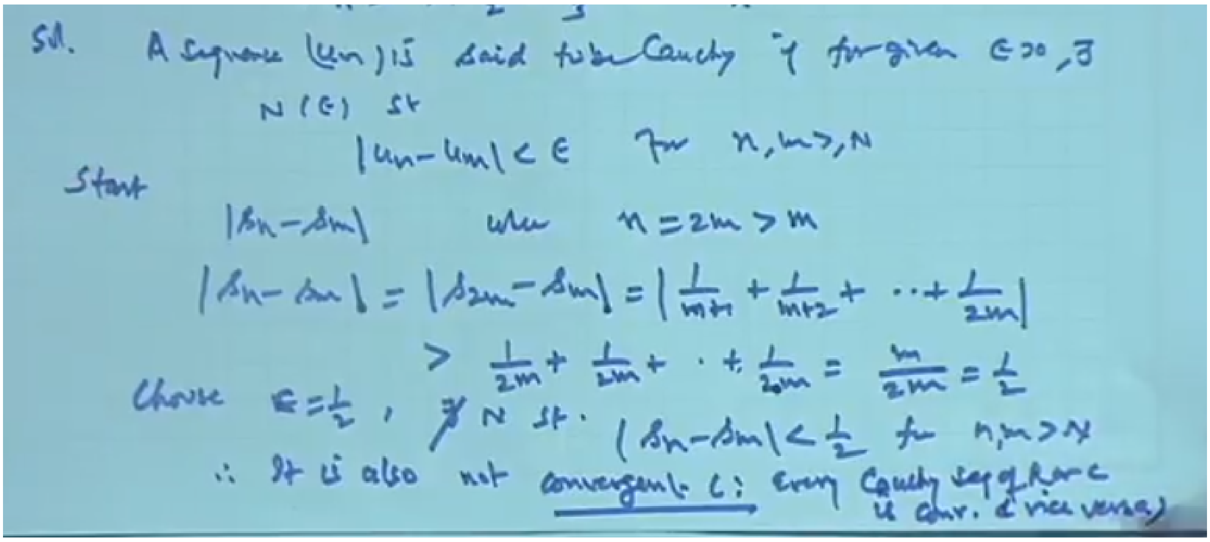

So if we choose, m epsilon to be 1/2, then we cannot find an H, there does not exist an N, such that, mode of SN, minus s m, less than half, for all n m, greater than equal to n. So this shows, it is not a Cauchy sequence. And we know if a Sequence, of real or complex number, if it is Cauchy, then it must be convergent. So since it is not cauchy, therefore it is also not convergent. Because of every Cauchy sequence of real complex number, is convergent, because every Cauchy sequence of real or complex number is convergent and vise versa. So this result we have already discussed so this shows.

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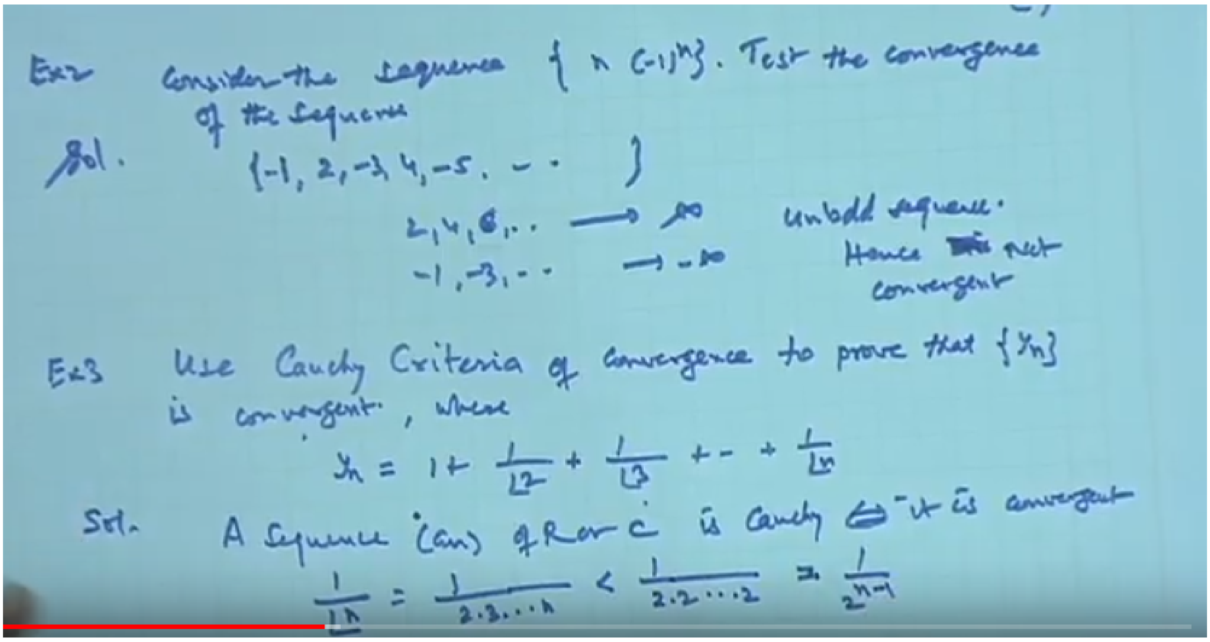

Now exercise, so we can now take a few problems. Suppose we have, contact if sequence, we can just stop, okay. Consider the sequence, sequence, n into minus 1 to the power n. Test the convergence of the sequence, convergence of the sequence. I think is a simple part. What is the sequence? If we look the sequence, the sequence comes out to be, minus 1, 2, minus 3, 4, minus 5, and so on. So we are getting a sequence, subsequence, 2, 4, 6, and so on, which diverges to infinity. Minus 1, minus 3, is also diverging to minus infinity. Therefore this is an unbounded sequence. Hence, hence diverges and not convergent not convergent. So that is really simple part for it, we will not go in detail. Exercise; Use the cauchy criteria, criteria, of convergence, Cauchy criteria of convergence, to prove, that sequence yn is convergent. Where YN is, 1 plus 1 by factorial 2, 1 by factorial 3, and so on, 1 by factorial n, this is. Okay? So what are the criteria of convergence? Cauchy convergence criteria means, every Cauchy sequence of real complex number, is convergent, or the. A sequence n, of real or complex number is Cauchy, if and only if, it is convergent. That result is so. So we will use this result to prove this. That it is a Cauchy sequence hence convergent. So, now let us see, what is our factrum. First see, 1 by factorial n. This is basically 2, 3 up to n. 1 is of course. Now each term is less than, is less than 2. So 1 by 2, 2, 2. So can we say, this is strictly less than 1 by, 2 to the power n minus 1. Is it ok? Because 3 is greater than 2, so 1 by 2, is greater than 1 by 3, and like this, using.

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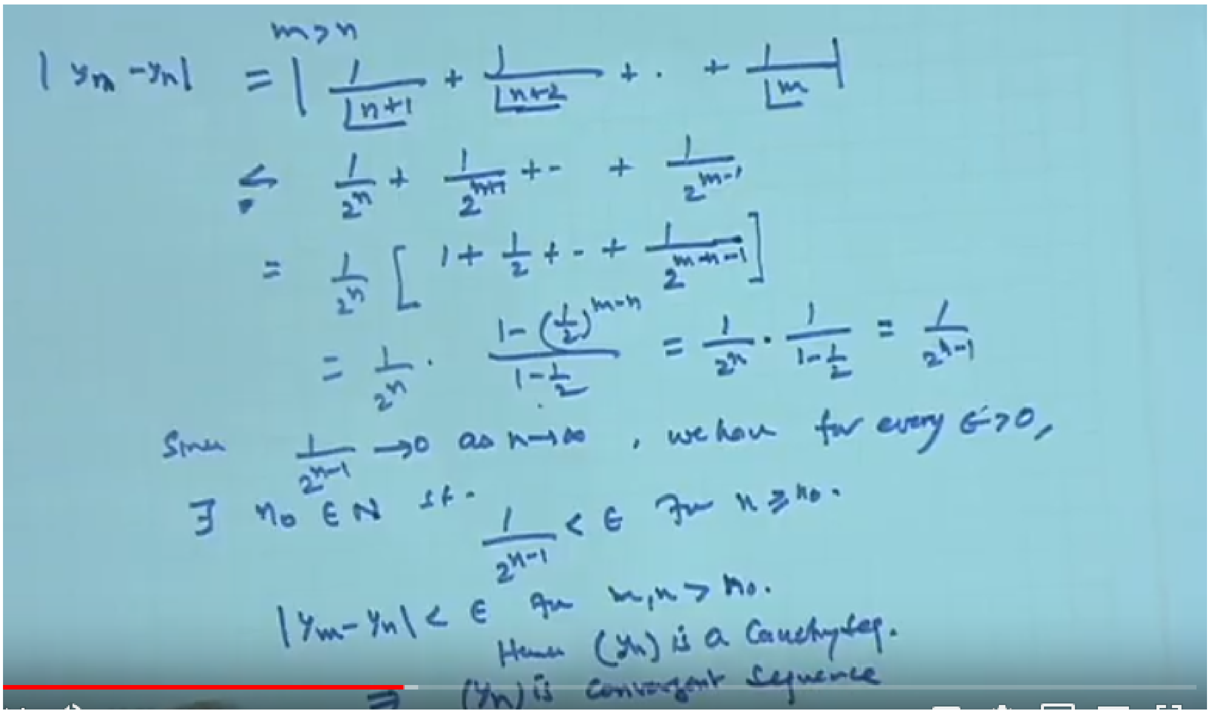

So use this thing, we now have, YM, minus YN. Where M, I am choosing, greater than n. So this is equal to, 1 by factorial n plus 1, factorial n plus 2, up to 1 by factorial M. This is the same as, less than, 1 by factorial, is less than 1 by 2 n, just now 1 by factorial n is less than 1 upon 2 to the power n minus 1? So it is less than 1 by 2 n. This is less than 1 by 2 n plus 1 and this is less than 2 to the power M minus 1. So from here we can say ,1 by 2 n if we take outside, then inside we get ,1 plus 1 by 2, plus 1 by 2 m minus n, minus 1, and that will be equal to limit 1 by 2 n, in geometric series, 1 minus 1/2 ,raised to the power M minus n term, over 1 minus ½, and that becomes ½. 1 by 2 n into, 1 over, 1 minus ½, as this is, 1 by 2n, because this is less than, 1 minus 1, is less than 1 and this is equal to, 1 minus ½. So totally is 1 by 2 n minus 1. So ym minus y n, is less than 2 n minus 1. Therefore, since, 1 by 2 n minus 1 goes to 0, as n tends to infinity, so we have, for every epsilon greater than 0, there exist an N naught, belonging to capital N, such that 1 by 2 n minus 1, is less than epsilon, for all N, than equal to n naught. Therefore YM minus YN is less than F epsilon, for all M N, greater than equal to M naught. This is true, for all n. Hence by YN is a Cauchy sequence. Therefore YN is also convergent, convergent. Okay? So that is what proving, is using this only, we are proving. Okay?

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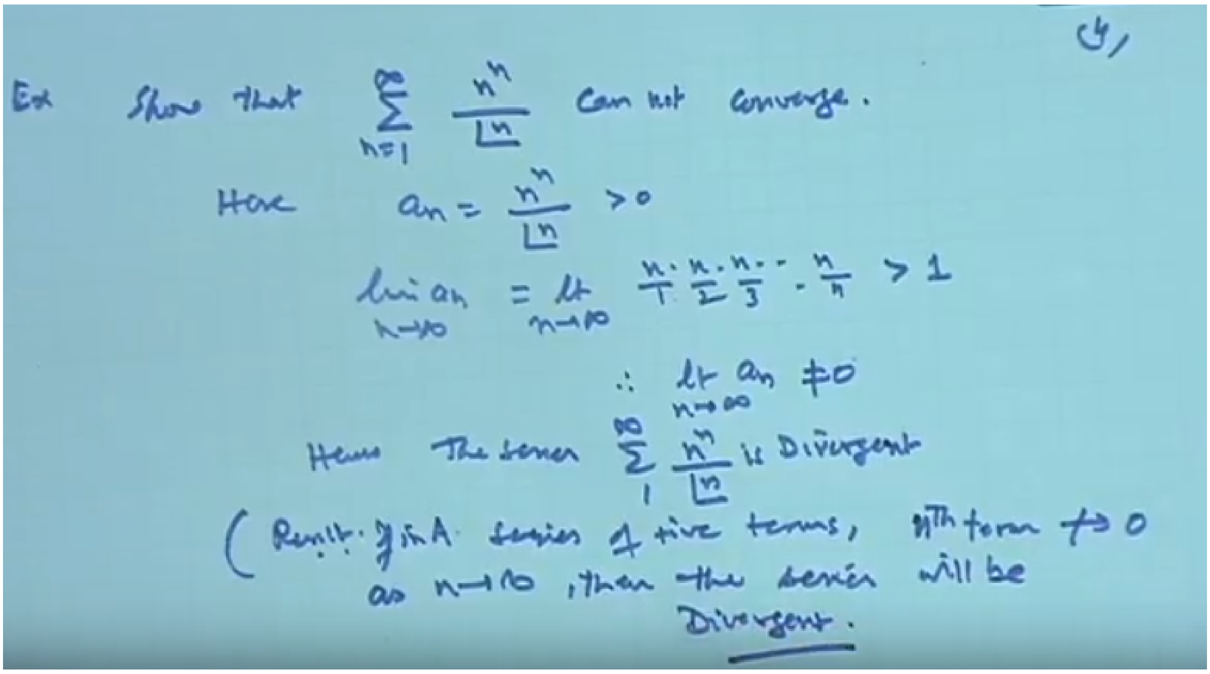

Next exercise, we see. So that, Sigma n is, 1 to infinity, n to the power n, divided by factorial n, cannot converge, cannot converge. Now these are the terms, of positive terms. Series of positive terms and if a series is convergent, then any term must go to zero. But if all the terms of the series are positive and limit of the N th term n, does not go to 0, then series has to be divergent. So here a n is, n to the power n, by factorial n, these are all terms are positive. And the limit of an, as n tends to infinity, this will be what? This is basically, limit of this, as n tends to infinity, n into n, into n, up to 1, divided by 1 2 3 and then. So this is, n by n is greater than 1, n by 2 is greater than 1, and by 3 is greater than 1. So basically this limit will always be greater than 1. Whatsoever the n, may be, Therefore, if the limit of this, does not tends to 0, as n tends to infinity, is not equal to 0, well the all the terms. Hence the series Sigma of this thing, n to the power factorial 1 to infinity, diverges, divergent, divergent. Yeah? Because of the, we are using this Result, a series of positive terms. If in a series of positive, if in a series of positive terms, the n th term, does not go to 0, as n tends to infinity, then the series will be divergent. So this is our. Okay? So using this, we can go for this.

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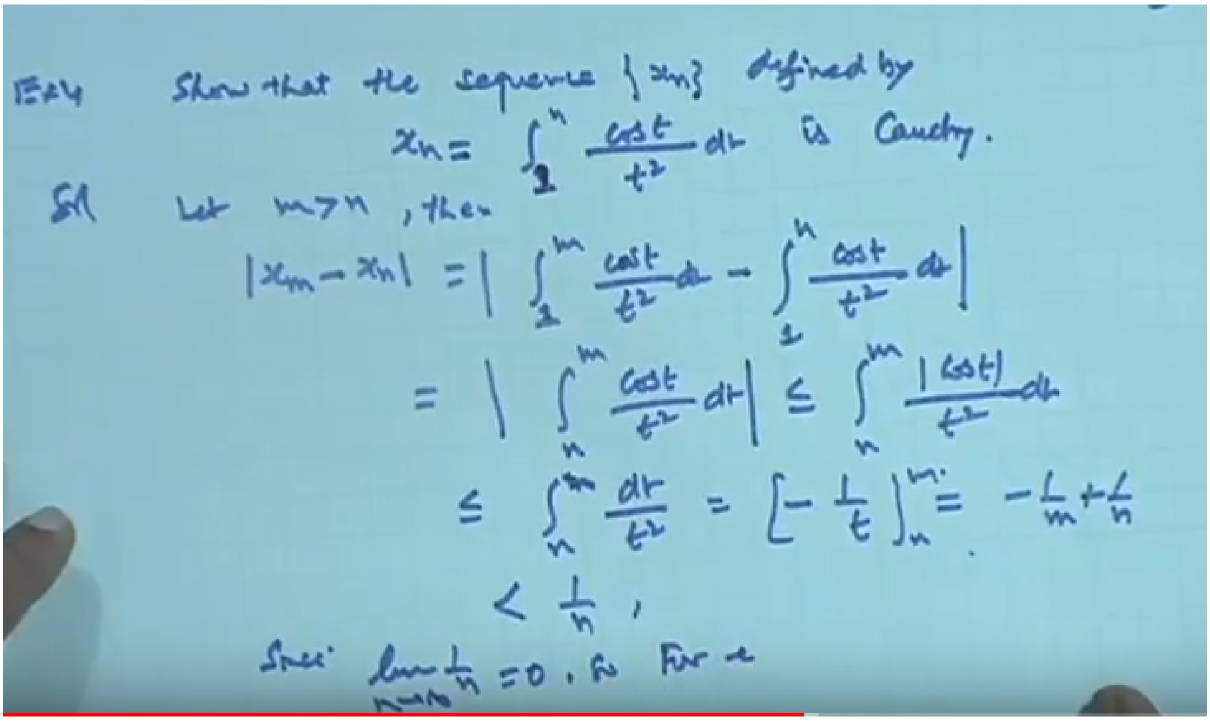

Next is, show that, the sequence xn, defined By, xn is equal to 0 to n, cos T, divided by T Square, DT is Cauchy. So let us suppose, M is greater than suppose N, is greater than n then mode of XM, minus X n. This is equal to mode, this is 1 2, sorry, this is 1 2 n. Because our T equal to 0, the function will not be defined, so 1 to M, so 1 to M. Cosine T over T Square, DT, minus 1 to n, cosine T, over T Square, DT. And that will be equal to modulus, n to M, n to M, cosine T, over T Square, DT. But this is less than equal to, integral, n to M, mode of cosine T, by T Square, DT. But cosine is bounded and bounded by 1? So it is less than equal to, integral, n to M, DT by, T Square, and this integral. The value will be minus 1 by T, and then, within the limit n to M, so this comes out to be minus 1 by M, plus 1 by n. Which is strictly less than 1 by n, and this is negative side. So, as n tends to infinity, but since, limit of this 1 by n, as n tends to infinity 0, then,

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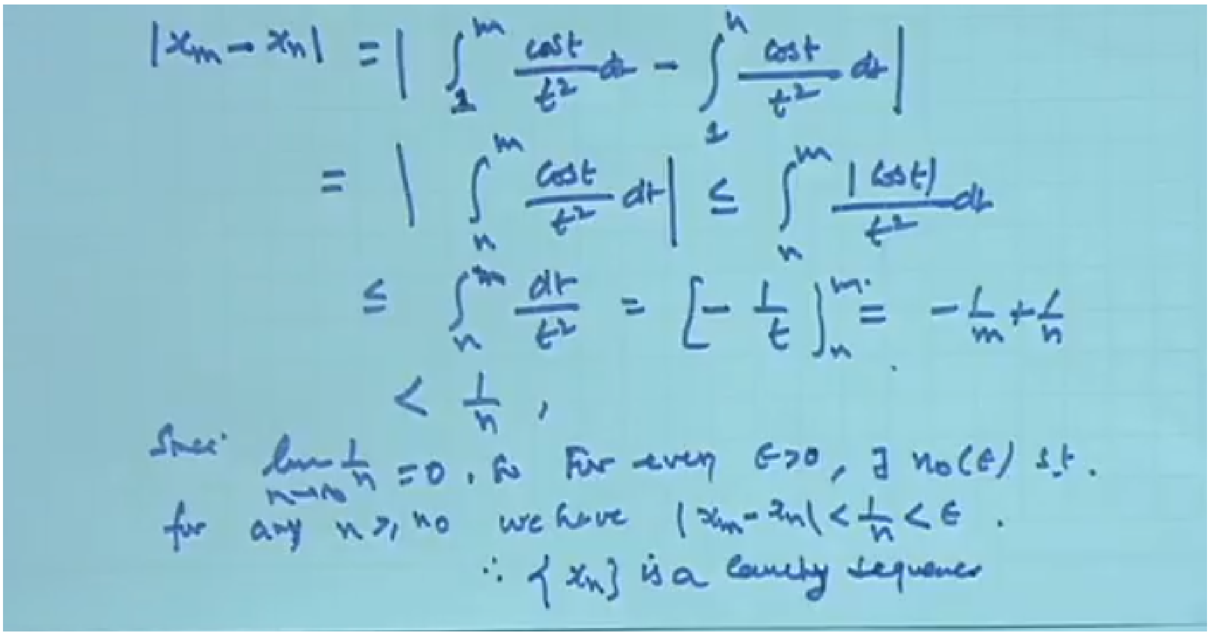

so for every epsilon greater than 0, there exist an N naught, N naught, depending on this epsilon, such that, so that, for, for any number, for any N, greater than equal to, N naught, we have, we have, mode of X mn minus X n, which is less than 1 by n, less than epsilon, and this is true, for all it.
Therefore the sequence xn, is a Cauchy Sequence, Cauchy sequence. Okay? So this is so. Because, the right hand side is less than epsilon, therefore this will go to. Okay.

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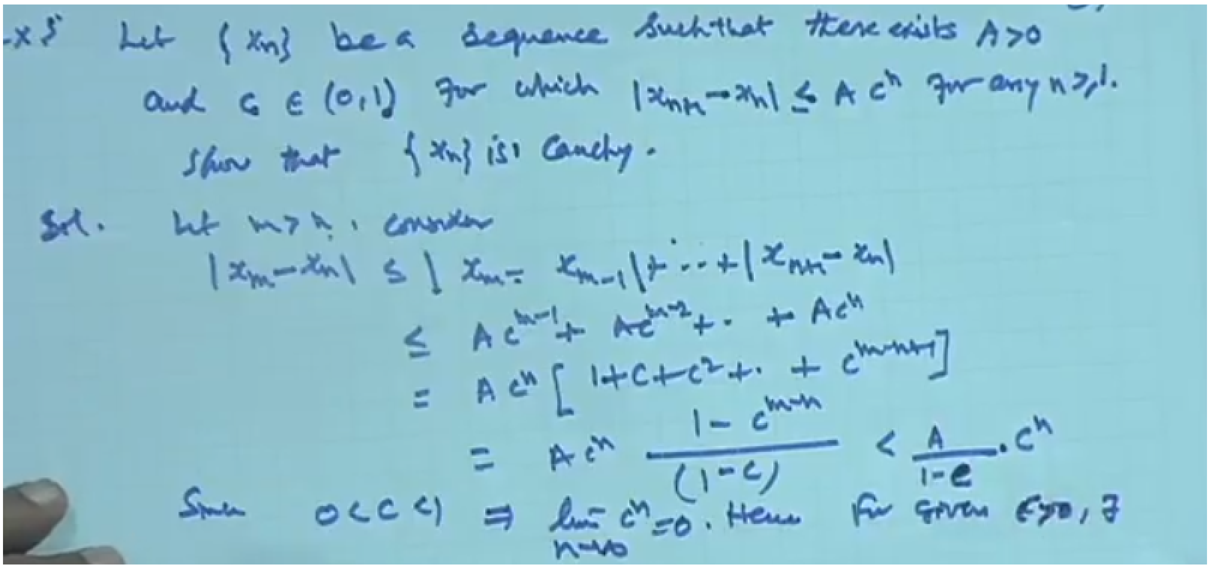

The next exercise; Let xn be a sequence, sequence, such that, such that, there exist, there exist, a, greater than zero, and C, and C belongs to 0 1, for which, for which mode of xn plus 1, minus xn, is less than or equal to, a into C to the power n, for any n, greater than equal to 1. Then show that, show that, xn is Cauchy, xn is cauchy. Okay? So let us see the solution for it. Suppose M is greater than 1? M is greater than n? Consider mode of X M minus X n. This is less than equal to mode XM, minus XM minus 1, plus mode of plus term, xn plus 1, minus xn, this term we are lighting. Now each one, use this result. So this is less than equal to, AC, m minus 1, AC, M minus 2, and so on. And last term is, ACN. So take ACN common. So, when you take ACN common, you are getting, 1 plus, 1 plus, C plus C square, up to C, M minus n plus 1, now this is a geometric series. And the sum will be, 1 minus, C to the power M minus n, divided by, 1 minus c. So this will be strictly less than for th, a over 1 minus C, into C to the power n, Now, C, lying between 1, 0 and 1, because it is given. C belongs to 0 1, So C, lies between 01. Therefore, this implies that limit of CN, as n tends to infinity, is 0. Hence for given epsilon, greater than given epsilon, greater than 0,

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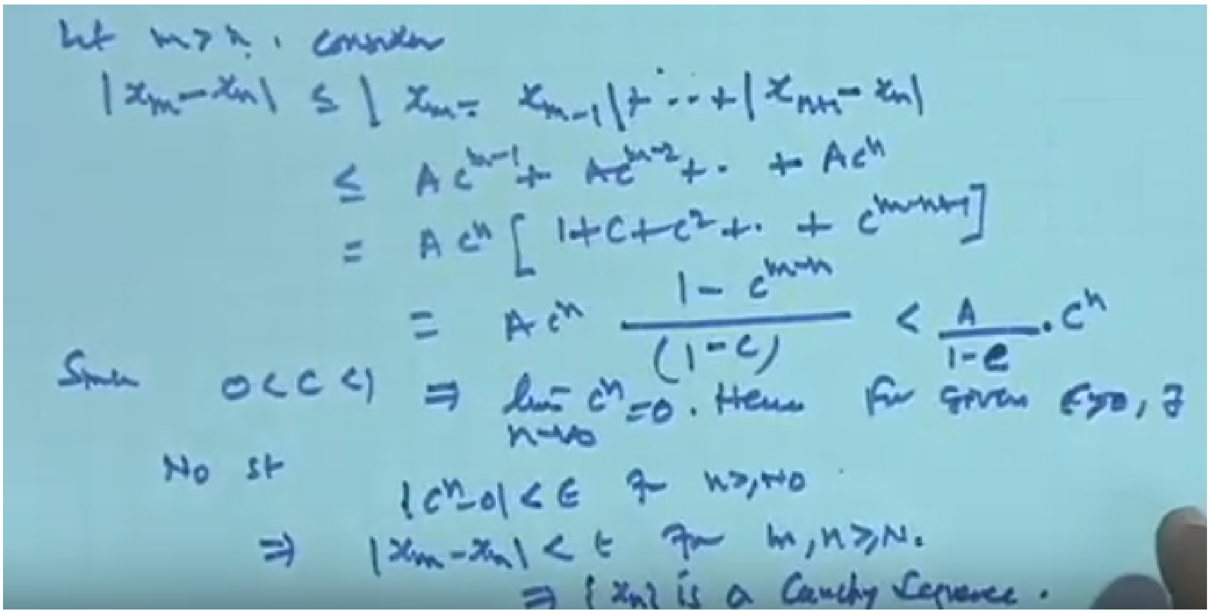

 there exists a capital N naught, such that, modulus of CN, minus 0, is less than epsilon, for all N greater than equal to n naught. Therefore this sequence XM minus xn, is less than epsilon, for all MN greater than equal to N. And this shows the sequence xn is a Cauchy sequence, Cauchy sequence, for that. Okay? So this will be our cauchy sequence.

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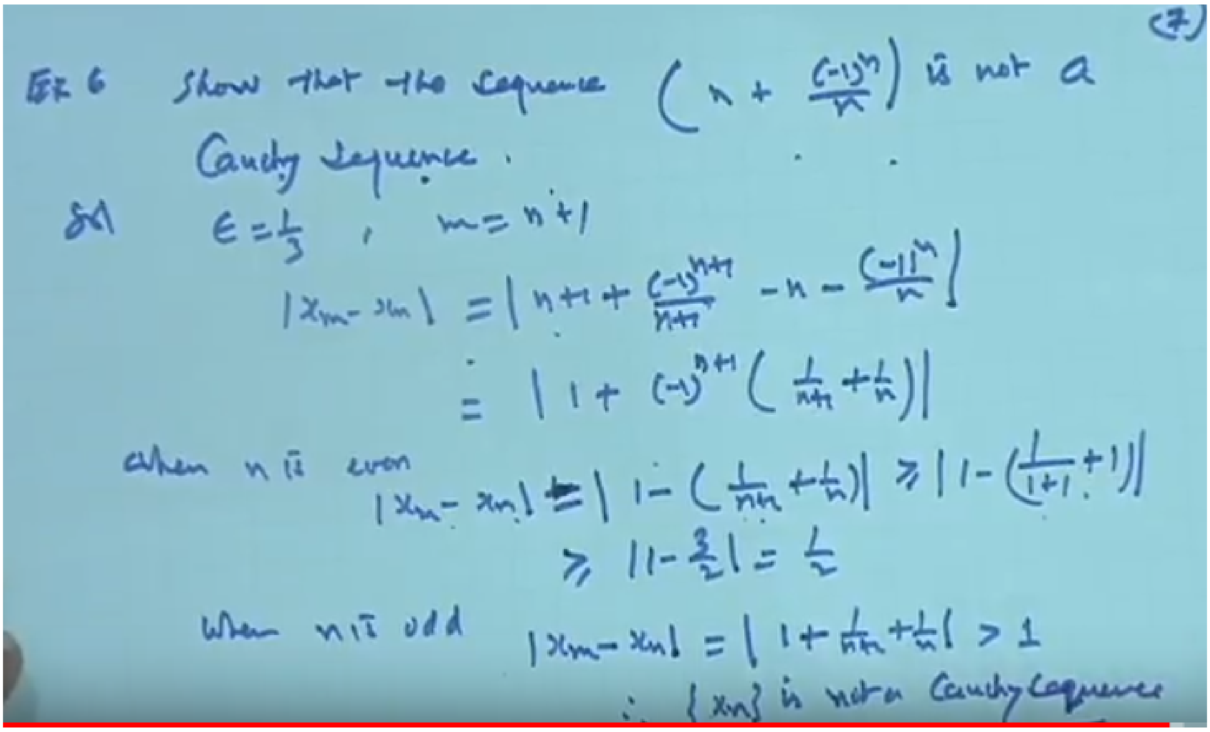

Exercise six; So that the sequence, the Sequence, n plus, minus 1, to the power n, divided by n, is not is not a Cauchy sequence, not a Cauchy sequence Solution. So let us start with say, epsilon is 1/3. If I prove that difference XM minus xn, is greater than 1/3, then it will not be cauchy sequence. So let M is suppose n, plus 1. Then what will be the XM minus xn? This is equal to n plus 1, plus minus 1 to the power n, plus 1 divided by n, plus 1 minus, n minus 1, to the power n by N. And that gives you 1 plus, minus 1 to the power n plus 1, 1 by n plus 1, plus 1 by n. Now take the two cases. Ehen n is even and when n is odd. So when n is even, you are getting, when n is even, what we are getting is, XM minus xn, is nothing but, greater than equal to, all is equal to, 1 minus, 1 over n plus 1, plus 1 by n, which is greater than equal to 1 minus, 1 over 1 plus 1, plus 1. Because this will be, n plus 1, is n is greater than 1, so 1 by s this, less than what with minus sign, again it is greater than. So this shows, this is greater than equal to 1 minus, 3 by 2, which is ½. So this part is. And when n is odd, then xn, XM, minus xn, will be equal to 1 plus, 1 by n plus 1, plus 1 by n Which is, obviously greater than, 1. So in any case this difference is not less than 1/3. Therefore the sequence xn, is not a Cauchy sequence, it is not a cauchy sequence. Okay?

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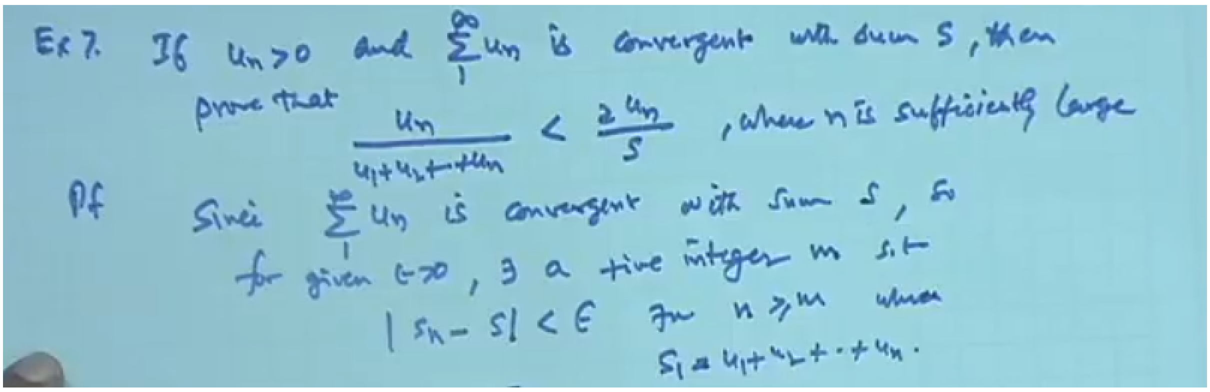

So, next question is. Say if n is, greater than 0 and Sigma u n, is convergent, with some s, with some s, then prove that, UN / u 1, plus into, plus un, is strictly less than, 2 times of un, by s. See the proof of it. Where N is sufficiently large, where, N is sufficiently large, sufficiently large quantity, N sufficiently large quantity.

So solution is. Now since Sigma u n, is convergent, is convergent, sigma un is convergent, with this sum, s. So for given epsilon, so for given Epsilon, greater than 0, there exists a positive integer m, integer m, such that the difference between SN and sum s, is less than epsilon, for all N, greater than equal to capital M, Where s n, means, the sum of the first n terms, of the sequence.

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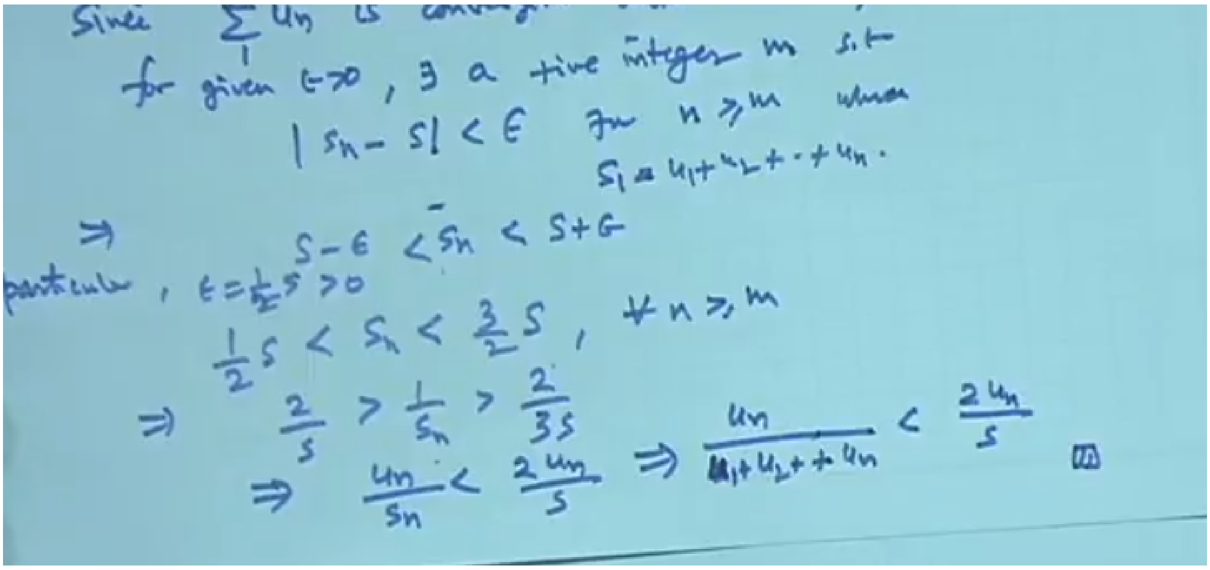

So this implies that, SN lies between, s plus epsilon, and s minus epsilon and in particular, if we take epsilon to be ½, then what we get? Half of s, Sorry, Epsilon is half of s, which is positive, which is positive. Then half of s, is less than s M, is less than, 3 by 2 s. And this is true for every N, greater than equal to M. So this implies that, 2YS, is greater than 1 by s n, is greater than 2 by 3 s, 3 s. NSD but s in, this one, un by is, so 2 by s is greater than this. And therefore what we get from here is, can we say u n, over SN, is less than 2 u n, by s? Multiply this. Okay? Multiply by UN, here, UN here. So u n, by SN, is less than 2 by UN and sn. And this shows, this implies, that U n, over s Me, SN means, S 1, u 1, sorry, u 1, plus u 2, plus u n. This is strictly less than, 2 times of U n, by s. And that is what we wanted to show. Is it okay? So this comes.

Thank you.