Model 5

Lecture 29

Convergence Criteria for Series of Positive Real Numbers

Course

on

Introductory Course in Real Analysis

So this is in continuation of my previous talk, where we have discussed infinite series of real numbers, and its convergence or divergence. The discussion was supported by giving few examples. We will continue the discussion further, by giving few more convulsions criterion, or for infinite series of positive real numbers. First we prove an important theorem, on convergence criteria, of the series Sigma, except 1 to infinity. And then, will drive as a corollary, many other results, which are helpful in identifying the nature of the series.

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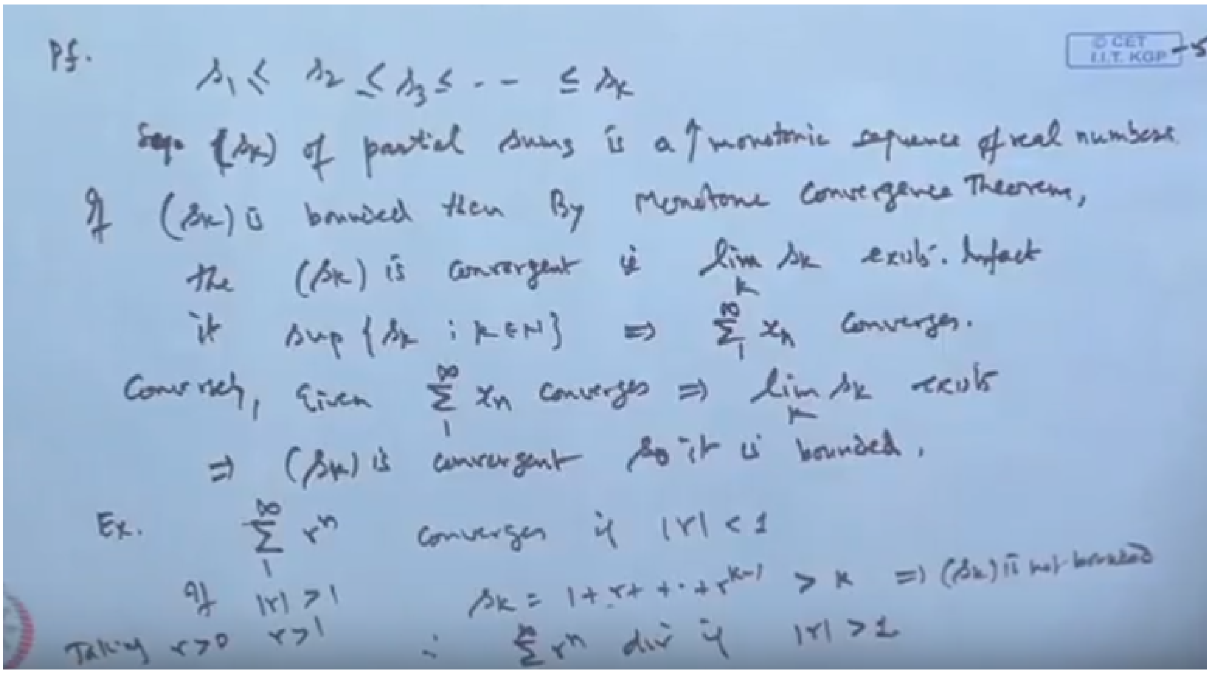

We have seen this example Sigma, R to the power N, 1 to infinity. And we have seen that this series converges, if mode R is less than 1. Okay. But if mode R is greater than or equal to 1, then what happens? What is our SK? SK becomes 1 plus R, plus R, R to the power K minus 1. Now this each one is greater than 1. So it is greater than K. is it not? because, it is greater than 1? So limit of SK does not exist, unbounded. Therefore SK sequence is not bounded, Okay? So this series cannot be convergent, because, if it is convergent, the sequence of the partial sum, must be bounded, as well this result. So this show, the series R to the power N 1 to infinity, diverges, if mode R is greater than 1. Here we are taking 1 one more thing is, here we are taking all the terms of the sequences, to be non-negative.

We cannot apply this result for the series, whose terms are, some are positive, some are negative, like this. We cannot apply this result. Or we cannot do for the non negative terms, if all the terms are negative, what we can do, we can take the minus sign outside and make it all the term to be positive. But if the series is alternately positive negative, then of course this result is not helpful. If result is only valid, when the terms of the series are all non negative. So here when you are taking mode R is greater than 1, basically we are choosing all R to be greater than 1, positive, this is positive. So R is greater than 1, taking R to be greater than 0 or positive, then this diverge. And r is equal to 1 obviously, again we take 1 + 1 + n, so, SN diverges. Therefore it Unbounded, therefore it is not convergent. Okay? similarly, for this. Now we have another results, which is also very this thing. And that result is, yes, the next result, which will help you driving the few more results. This result is in the form of theorem.

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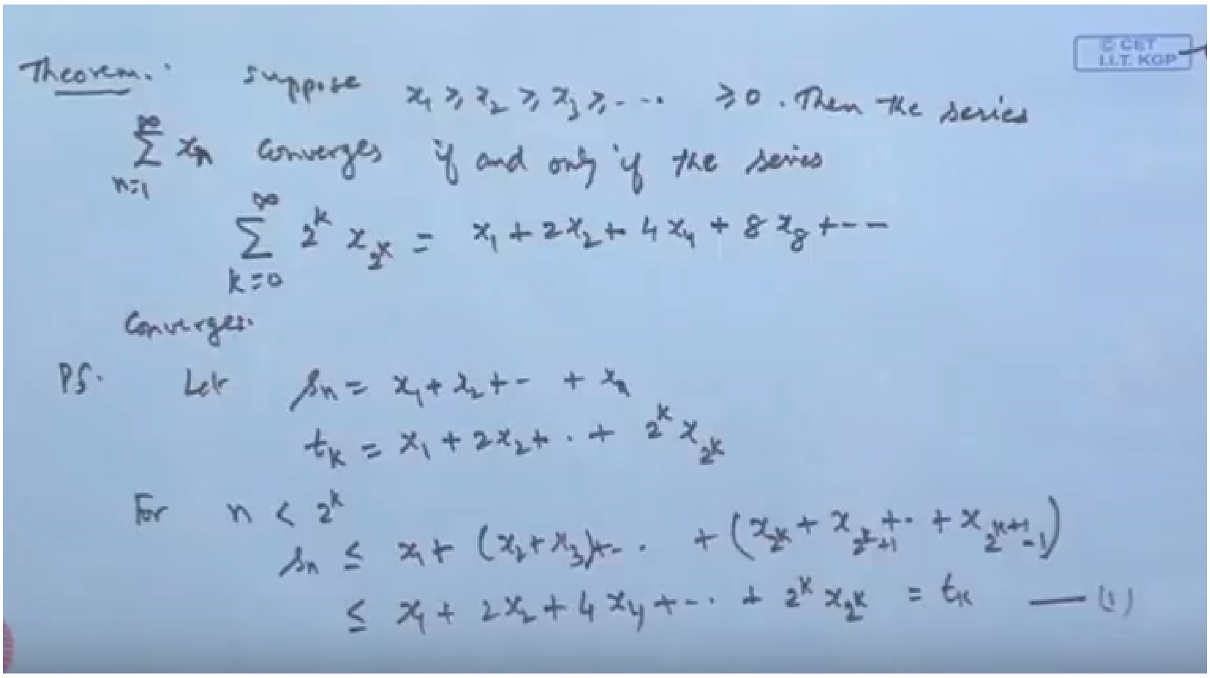

What this result says is, suppose, suppose X 1 is greater than equal to X 2, is greater than equal to X 3, is greater than equal to, all terms are greater than equal to 0. Means non negative terms. Then the series, then the series, Sigma XK or X n, n is 1 to infinity, xn 1 to infinity, converges if and only if, if and only if, the series, if and only if the series Sigma K is equal to, Sigma k equal to 0, to infinity, 2 to the power K, X to the power K, X 2 to the power K, X suffix, 2 to the power K, that is a series is X 1, plus 2 times of X 2, plus 4 times of X 4, X 4, plus 8 times of X 8 and so on. This series converges. Okay. So this is very result, which will help in driving the few more results, with the help of this. So this is the, so proof le us. What it says is, suppose a series Sigma xn is given, whose all terms are non-negative, then the nature of this series, that is, a series will be convergent, if the corresponding this series, will converge and vise versa. Okay?

So in order to test this series to be convergent you come, convert the series into this form, which will be more comfortable, in a suitable form, which can easily be proved to be a convergence or divergence. So corresponding nature of this, if it is convergent, this will converge. Okay? That is what it says. Okay. So let us consider the partial sum. Let s n stands for the partial sum of the first series. Say X 1 plus X 2, plus xn. And let TK ,it stands for the partial sum of the second series, X 1 plus 2 times of X 2, plus 2 to the power K, X suffix 2 to the power K. Suppose we are taking this term up to, say 2 to the power K, 2 K, where the few , there are few gaps. Is it not? Because, it is not in continuation. X1, X2 then after that, x3 is missing, x4, like this. So we are choosing this term. Now let us take the different cases. For n, this is strictly, less than say 2 to the power K. Then what happened, the SN, this will be less than or equal to, x1 plus x2, plus x3, if we combine this, and then let us write this form. X, 2 K, 2 to the power K, plus X 2 to the power k plus 1, k plus 1, up to say 2 to the power X, 2 to the power K, plus 1 minus 1. Say I am taking xn, to be less than this number, obviously when I am taking n is less than 2 to the power K, so obviously these terms, which we are taking, is much higher than this. Okay? And total terms, in this case will be what? Now this will be X, now X 1 is greater than X 2, X 2 is greater than X 3. So basically, this will be X 2 is greater than X 3. So we can say this is X 3 is lower than X 2, so it is less than 2 of X 2. So this will be less than X 1, plus 2 times

of X 2. Similarly when you go for x4, x5, x6, x7 and x8, then we will get next term will be, 4 less than X 4. Okay? That sum. And continue this, what will happen to this?
There are only 2 to the power K terms, So 2 to the power K, into X, 2 to the power 2 K. This would be. Because this is the smallest term and this is, sorry, this is the largest term and rest all decreasing, Is it not? So it would be less than this. But this term is nothing but what? TK? So what we get is, that SN is less than equal to Tk. Okay? This is one thing.

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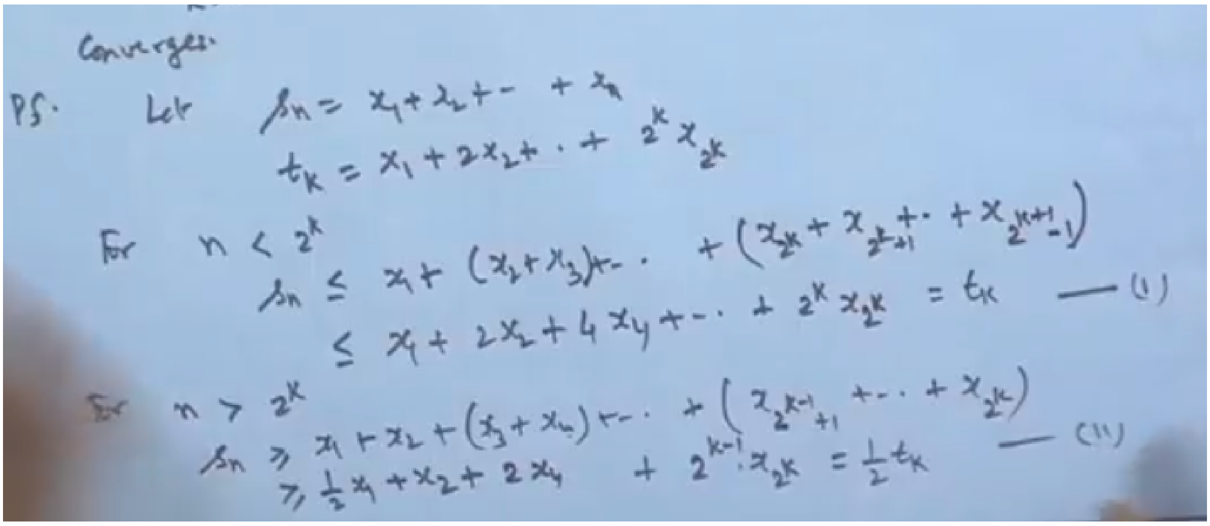

Now for n, let us take n, to be strictly greater than 2 to the power K. And then again you rewrite, suitably, SN can be written H, greater than equal to x1, plus x2 and then combine this term. X3 plus x4, like this and last term, start with the X, 2 to the power K, minus 1, plus 1, this term and up to go x to the, X, 2 to the power K, up to this. So what happen is, this is greater than, equal to. Now let us write, x1, is obviously greater than half of x1? Is it not? because, it is positive term, Then plus X2, remain as it is, then x3, now X 3, X 4, X 3 is less than X 4, or X 4 is greater than. So if I replace X 3 by X 4, then it is greater, so it is greater then equal to, 2 times of X 4, 2 times of X 4, this will be, yes. Is it not? 2 times of X 4 and like this, up to 2 to the power K minus 1, into X, 2 to the power K, like this. Okay? So X 3 is greater, X 3 is greater, like this. Now, you check it. This will be what? If I take the TK, TK is coming to be X 1 plus 2, X 2 plus 4, X 4 like this? So it is half of this. So basically it is the half of TK. So what we get from one and two, we get from this. So from one and two we get,

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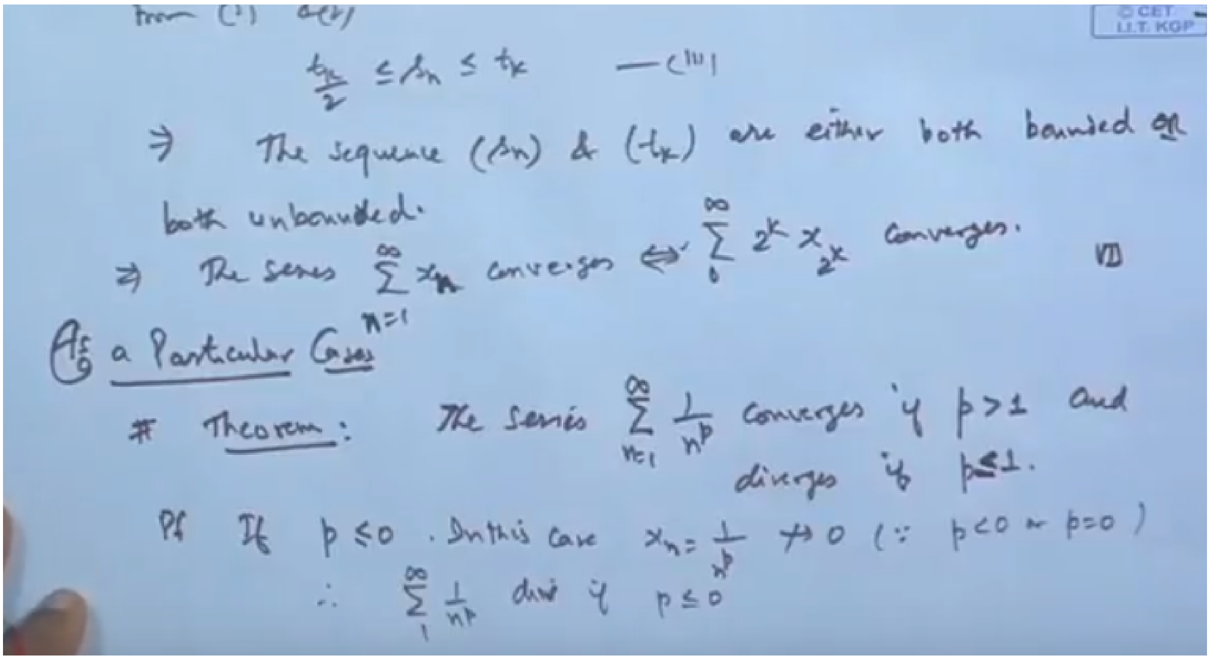

from one and two, what we get is, that SN is less than equal to Tk, and SN is greater than equal to Tk by two.

But what is TK and SN? Let us say, let TK and SN, the T, SN is the partial sum, and, it, partial sum of the series Sigma of xn, and TK is the partial sum of the second series, Sigma 2 to the power K, X, 2 to the power K. Now this partial sum, satisfy this condition, inequality. So if TK is bounded, SN has to be bounded. Now if SN is unbounded, TK has to be unbounded. So this implies the third criteria implies, that the sequence, SN and the sequence TK, TK, are either both or either both bounded or unbounded, or both unbounded or both unbounded. That is it. So once they are bound, they and only the result follow. It means the nature of this series converges, if only this convergent. So if they are bounded, then will be the. So this shows, so this implies, the series Sigma xn, converges if and only if, Sigma 2 to the power K, 0 to infinity, here 1 K is equal to 1 to infinity, I okay, in, should I write. Sigma to the power K, X, X, 2 to the power K, converges. Okay? So this is what we get it. And this should be right, n, because otherwise that will be different. Okay? So n is 1 to converge if only this one and this completes the proof. Okay? So this completes. As a corollary to this, or as a particular case, as a particular cases. Okay?

We have this result, theorem, in the form the series Sigma 1 by n to the power P, n is 1 to infinity, converges, if P is strictly greater than 1 and diverges, if P is less than or equal to 1. Okay? So this we get. Let us see the proof of this; it will follow from the previous results. Now obviously if P is less than or equal to 0, suppose P is 0, then what happen? This each term becomes 1. So each term becomes 1 means that any term will not go to 0. So when P is 0, so in this case, in this case, the NH, the xn which is 1 by NP, n to the power P, will not go to 0. Because when P is less than equal to 0, this will come in the up and it will not go to 0, it will diverge because, P is negative or 0. So by necessary condition does not satisfy. Therefore the series, Sigma 1 by n to the power P, 1 to infinity, diverges, if P is less than equal to 0. Okay? Now take that case, when P is, is strictly greater than 0.

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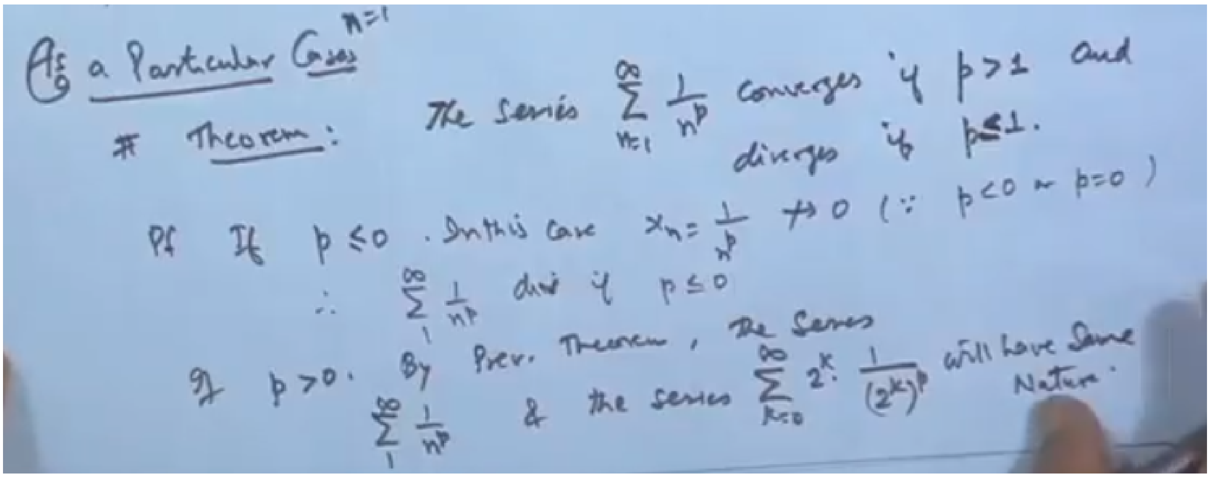

So if, P is, is strictly greater than 0? Then what we claim is, when P lying between 0 and 1, even 1, it will diverge and when P is greater than 1, it will converge. Okay?

So now, apply this result. By the previous result, by previous theorem, the two series will happen in, the two series, the series, Sigma 1 by n to the power P, 1 to infinity. Okay? And the series Sigma, k equal to 0 to infinity, 2 to the power K, 1 by, replace n by, 2 to the power K, 2 to the power K, power P. Because n to the power P, I am replacing n, xn, xn by this, xn is 1 upon n to the power P. So replace n by, 2 to the power K, So, n to the power P. These two series, so the series will have same nature, that, if they are, if this is convergent, it has to be converged and vise versa. This is convergent this and similarly divergent point. Okay? We will have the same nature.

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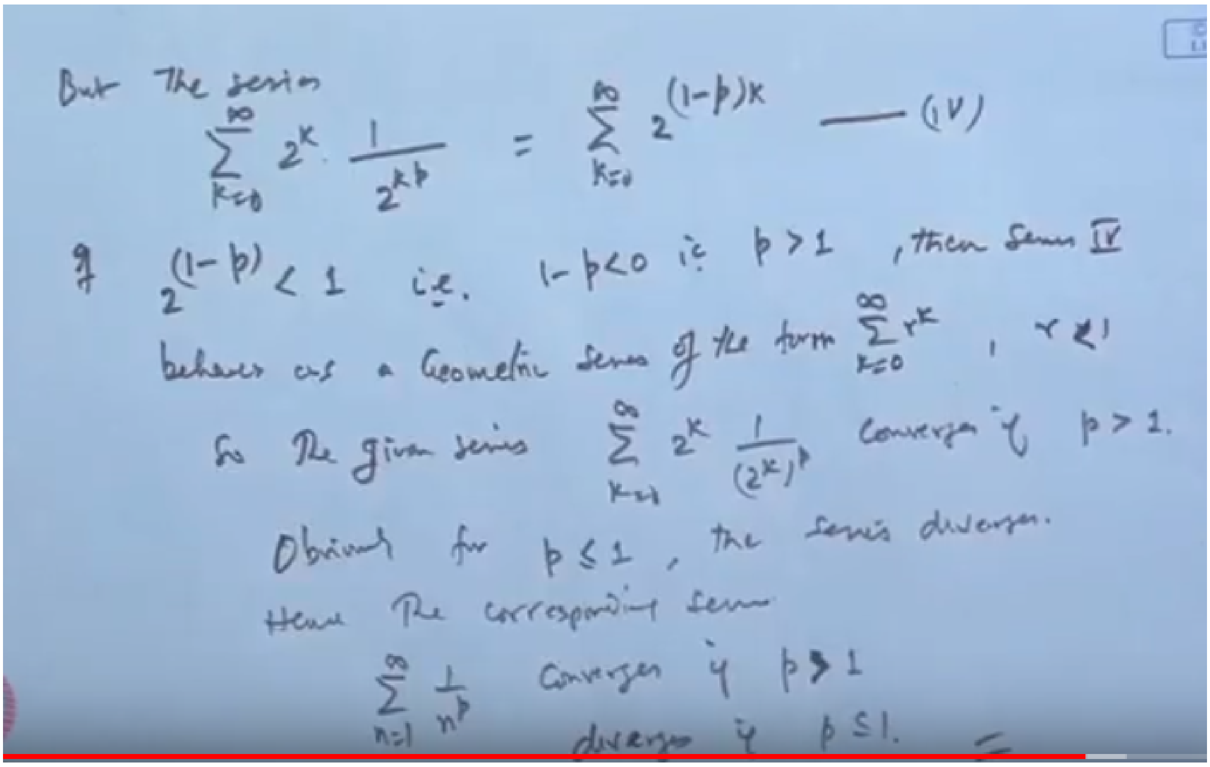

But what is this? The series, but the series, Sigma K, 0 to infinity, 2 to the power K, 1 by, 2 to the power K P, K into P, this is nothing. But what? This is equivalent to Sigma, k equal to 0 to infinity, 2 to the power, 1 minus, 1 minus P, into K. Okay? Now this is basically a geometric series, this is a geometric series. So if we take, if we take, if 1 minus P, If 1 minus P, 2 to the power, 1 minus P, if this part is less than 1, is strictly less than 1, then it will behave as a geometric series, Sigma 1, R to the power K, where R is, mode R is less than 1. So this will be, this series, then it means that is, that is what? When, it is less than 1, When 1 minus P, is negative, negative. That is P is strictly greater than 1. So in this case, the series say fourth, if this. Then the series fourth behaves as, a geometric series, geometric series, of the type, of the form, Sigma, R to the power, say K. Where K, 0 to infinity and R is greater than 1, R is sorry, less than 1, because R is less than, R is this term.

So in that case, it converges. So, thus given series, given series, Sigma k equal to 0 to infinity, 2 to the power K, 1 by 2 to the power K, power P, converges, if P is strictly less greater than one. And if P is less than, if this thing is greater than one, obviously then greater than one, so it is, the limit of this innate term does not go to zero? Therefore it diverges, when equal to one, also it will diverge. Okay? So, this so. Otherwise and obviously, and obviously, for P, less than or equal to one, the series diverges. Hence the corresponding series, series, Sigma 1 by n, to the power P, n is, 1 to infinity ,converges if P is, P is strictly greater than 1, strictly greater than 1 and diverges, if P is less than equal to one and that proves the result. Now when P is equal to 1, it is a harmonic series, and it divergent that what we told. When P is equal to 2, it is convergent and the example which we have choosen that necessary condition, XL tends to 0 is not a sufficient condition, is justified from here. Okay? So that is what. Now, based on this we can also say another example, let us say, suppose or result, because this will also be useful.

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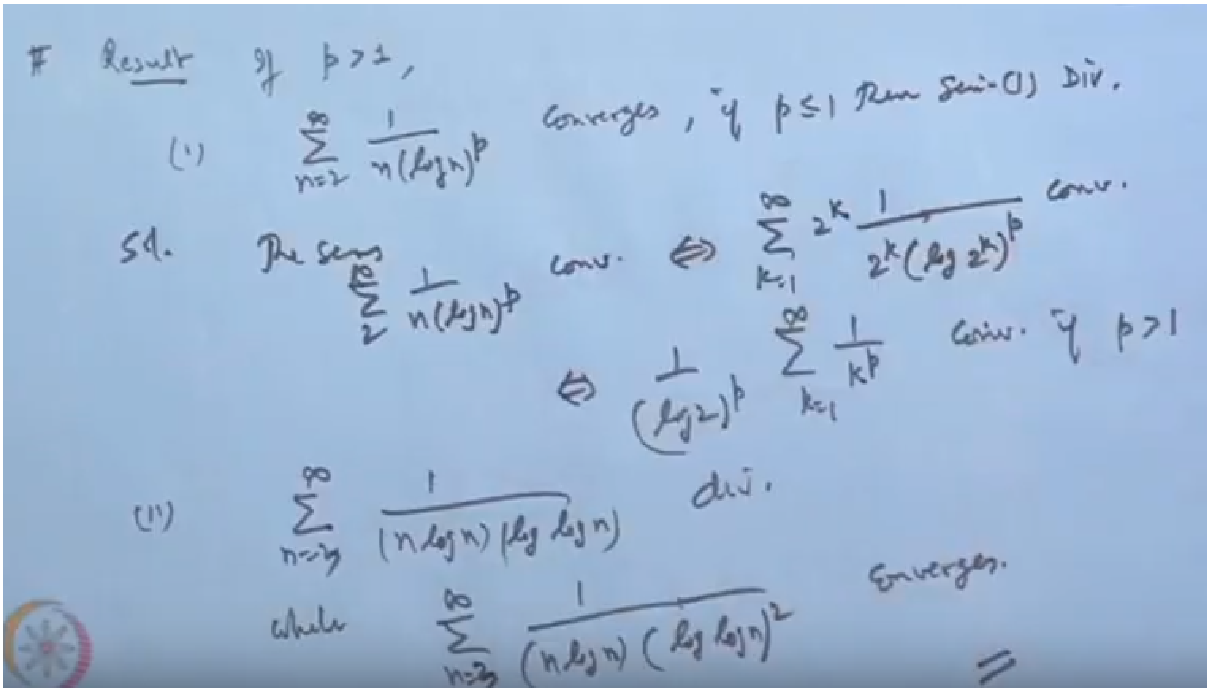

If P is greater than 1, if P is greater than 1, then the following series converge, then no 1; Sigma n equal to 2 to infinity, 1 by N, log n to the power P, converges, log n to the power fee converges. Okay?

when P is greater than 1 converges. And if, and otherwise diverge, and if P is less than equal to 1, then the series 1, diverges, diverges. Again it follows from the same thing. If you go through that one, then that follows the same thing. How? How the solution will come?
To test the Sigma xn convergent, what we do is, we find out that, 2 to the power K. So the series Sigma, 1 by n, log n to the power P, 2 to infinity converges, if and only if, if and only if, Sigma k equal to 1 to infinity, 2 to the power K, log, 1 by, 2 to the power K, log, 2 to the power K, power P. Obviously when K is a 0, we cannot choose, because log 1 is 0. That is why we are avoiding that one, this is. So this converges means, if this converges by. But this converges equivalent to, what? Is it not the same as 1 by log 2, power P, Sigma of 1 by, K to the power P, when K is 1 to infinity. And this series converges; this is convergent, if P is greater than 1, because, this is the NF power test. So this series converge. Similarly, if I take the second power, suppose I take this series, Sigma, n is equal to 2 to infinity, 1 over n, log N, and log N, into log, log, n, into log, log, n. Log, log, n. This diverges, while the series Sigma, n is 2 to infinity, 3 to infinity, because this is and it is, I think 3 because, otherwise this 2 will not. n is 3 to infinity and n is 3 to infinity, 1 by n, log n, then log, log, n, 2 power 2, converges. And the reason is the same as above. Okay? so, this.

Thank you very much.
Thanks.