Module: 4

Lecture: 28

Infinite series of real numbers

Course on introductory in real analysis

So today we will discuss infinite series of real numbers, we have already discussed the sequences and various concepts, regarding the sequence, so today we will talk about that infinite series. We define normally when we say if a set of a remain takes 1 X 2 xn are given a sequence, when they are added or subtracted and put it in the form of X 1 plus minus X 2 and like this, then we call it as series, but that is not a authentic way of defining the series, because this does not solve anything whether the limit convergent or divergent whatever so, we define the series as follows.

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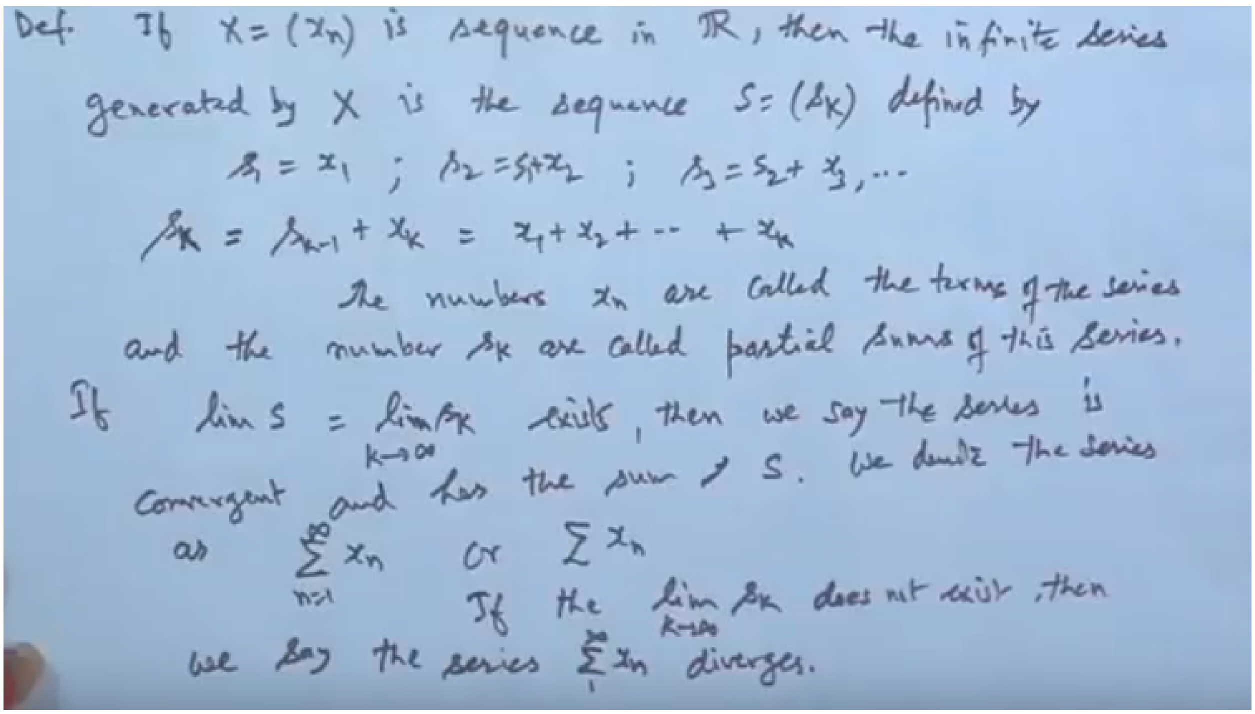

if X which is a sequence of real numbers xn, is a sequence in r in our set of real number, then the infinite series, infinite series generated by X, is this terms sequence, is the sequence, is the sequence s of partial sum XK defined by s1 is the first term, say X 1, s 2 is the sum of the first term that s 1 plus X 2, s 3 is s 2 plus X 3 and so on so SN will be or SK will be s K minus, 1 plus XK that is X 1 plus X 2 plus X 3 up to XK, the number X K the number, numbers XK or X n this is these are called the terms of the series, of the series and the number SK, SK are called partial ,sums are called partial sum, partial sums of this series, this s came if sum of the first K term of the series and so on.

Now if this limit if the limit of s that is the limit of s K when K tends to infinity, if this limit exists, then we say the series, then we say the series is convergent, is convergent and has the sum s, is the sum. We denote the series as Sigma X and n is 1 to infinity, if first term is X 1 or sometimes we also do not like xn without the suffix that so that summation is taken from 1 to infinity, so this is the way of defining it or some other if the first term is starting from with 0 then the summation will be taken from 0 to infinity and like this, well and if the series whose first term is say X hundred then the series will start from n equal to one day to infinity, so that way we can identify them, Ok? Now if this limit does not exist then in that case we say the series diverges, if the limit s K when K tends to infinity does not exist it means either it weighs in finite plus infinity or minus infinity or has a various build limits does not exist, then we say, then we say then we say, the series Sigma xn, 1 to infinity diverges, Okay? So that's then.

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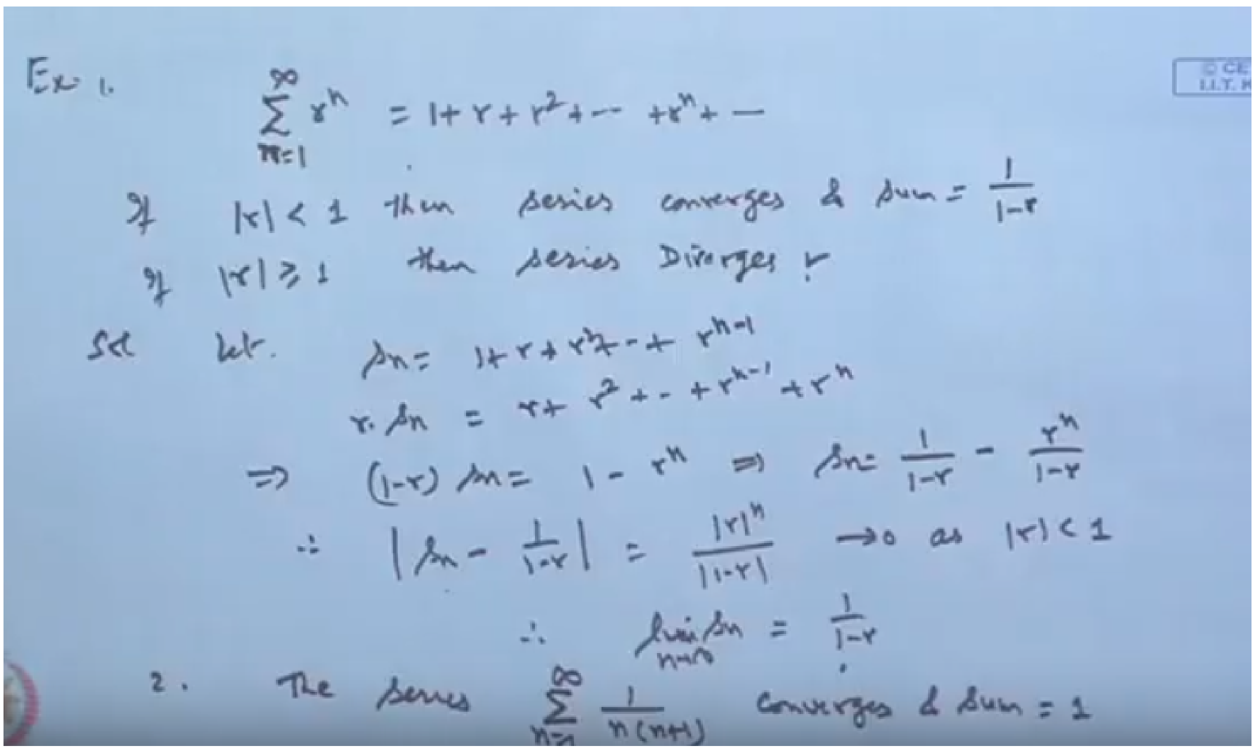

for example, we have a series, suppose we have the series Sigma R to the power n any n is 1 to infinity now we that is the series 1 plus R plus R square and so, on plus R to the power n like this, so this is the series now we claim if mod of R it is strictly less than 1, then series converges, converges and the sum of the series and sum is equal to 1 over 1 minus R, sum will be this, but if R is equal to 1 if R is mod R is greater than or equal to 1 then the series diverges.

Diverge, the proof is very simple solution let us find the first sum of the first n terms, so let s n is the sum of the first n terms 1 plus R, plus R square, plus R to the power this is the sum of the first term are n minus 1, because this will be first term second and up to n terms, Okay? Then if I multiply this by R then what happen R times SN this is equal to what? R plus R square, plus R n minus 1, plus R now subtract it, so this implies that 1 minus r SN you, when subtract it, it get cancelled in that and we get finally 1 minus R to the power n, so what we get it is SN becomes 1 by 1 minus R minus R to the power n over 1 minus R, ok? So consider, mod of SN minus 1 over 1 minus R that is equal to mod of R to the power N over mod 1 minus R, now R I am choosing fixed but it is less than 1, if it is less than 1 this term will go to 0,so this tends to 0 as mod R is strictly less than 1, therefore, the limit of this SN as n tends to infinity, is exist n equal to 1 minus R, so this series converges and converges to the sound 1 over 1 minus R, for R is equal to 1 obviously the terms are 1 plus 1 plus 1 and so on so pre SN will be n so when n is infinity limit SN will tends to infinity, so it will divert, for R greater than 1 we will see in due course that it are so diverges in fact it will limit will not exist and it will go to infinity, so that will come in obviously, we can find out the Sum so this is really interesting, one similarly another examples this part we will take up after going for the further test, then we say automatically by pH test it will come diverges and then let us take this sum this series Sigma 1 over N n plus 1 n is 1 to infinity we claim this converges and sum is sum is 1.

The solution is again simple, if I take the terms SN what happen this term SN 1 over 1 into 2, plus 1 over 2 into 3, plus 1 over up to say n is 1 into n plus one, so this will be if we write it in the form since,

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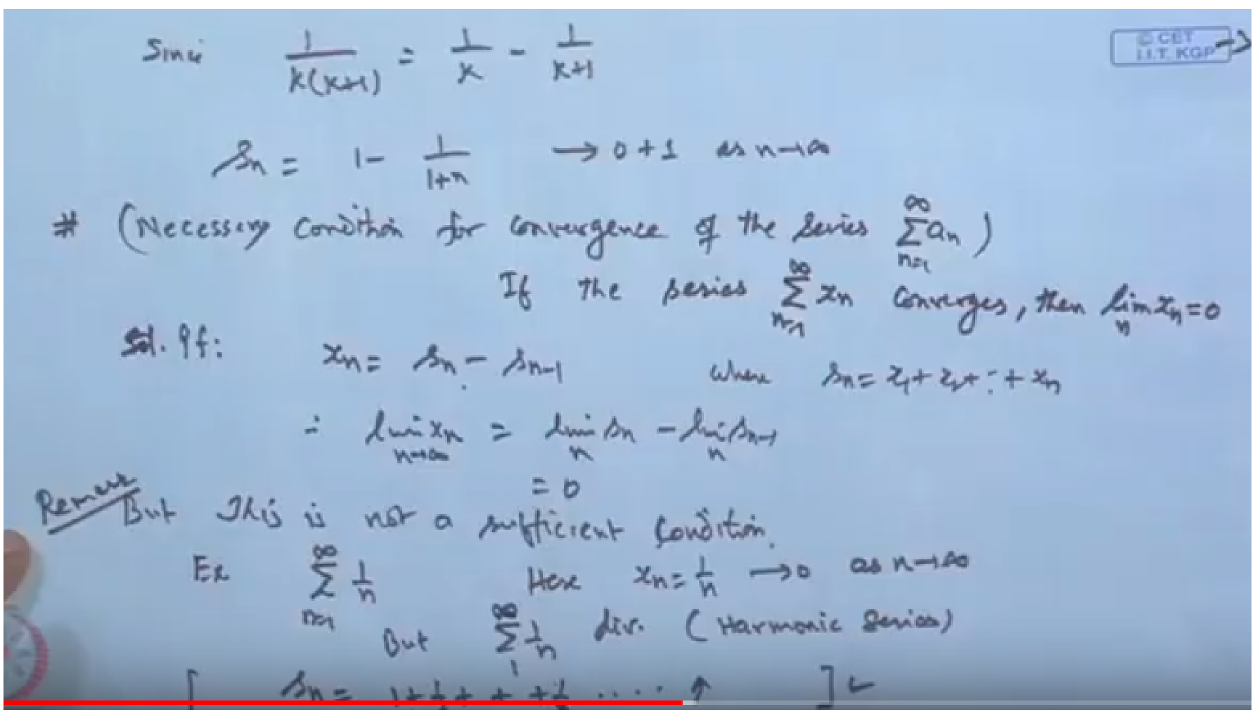

since one over K, k plus one, we can break up and put it in the form of 1 over K minus 1 over k plus 1, so using this each term we can put it, each term we can put it in this way and once you put it this thing then SN becomes this is 1 over 1 minus 2 that is 1 over 1 minus 1 by 2, this will be then 1 by 2 minus 1 by 3 like this, 1 by n minus 1 by n plus 1 so get cancel and finally you are getting 1 and 1 minus 1 over 1 plus so SN becomes 1 over 1 minus 1 over 1 plus n all n plus 1, now as n tends to infinity it goes to 1 minus 1 minus 0 that is goes to 1 as n tends to infinity, this term will go to n so this will go to 0 and this is 1, so the limit of this exist and equal to 1 so this way we can identify this, so these are fewer examples now in case of the convergent series there is a necessary condition, that if a series is convergent, then it's any term will always go to G, that is the necessary condition for so if a series whose any term does not go to 0 the series cannot be a convergent series, so that is the criteria which we have the necessary condition, the necessary condition for convergence of the series, Sigma a n is 1 to infinity, so this we will put it in the form, if the series Sigma X and we are doing very extensive letter put it xn Sigma xn n is 1 to infinity converges, converges, then limit of the xn must go to 0, limit of xn as n tends to infinity will be 0, that is any term will always go to 0, okay? So of proof let us see the solution or proof, so what is our SN, SN basically in xn this is nothing but what? SN minus SN minus 1 because SN is the sum of the first term we have SN is stands for X 1 plus X 2 plus X n so SN minus 1 means up to xn minus 1 so when you subtract you are getting SN, now this series is convergent so limit of the SN will exist, so limit of SN and limit of SN minus 1 will be same so therefore, limit of xn when n tends to infinity this will be the limit of SN minus limit of SN minus 1, but both the limit will be the same so it will come off to 0, so this is the necessary condition for a series to be convergent, but this is not a sufficient condition, but this is not remark. But this is not a sufficient condition, sufficient condition, that is if at any term of a series converges to zero then it may or may not be a convergent series, for example, suppose I take this series Sigma 1 by n, n is 1 to infinity here xn is what? 1 by n which tends to 0 as n tends to infinity and this we have seen that this series and but the series SN, the series Sigma 1 by N 1 to infinity diverges, diverges as we have seen earlier because it is a harmonic series, harmonic series, in fact the proofs we have gone already see because the if you remember we got the SN to be 1 plus 1 by 2 plus 1 by 3 up to 1 by n is it not and then it keeps on increasing when you take the n then this keeps on increasing in fact limit of SN will go to this keeps on increasing, okay? Diverges so this we have already increasing functions like this and keeps on the unbounded one.

So this we are not already shown in the first lectures we are going for this series sequences in case of sequence this example we have taken and so on the series diverges so here this divergent.

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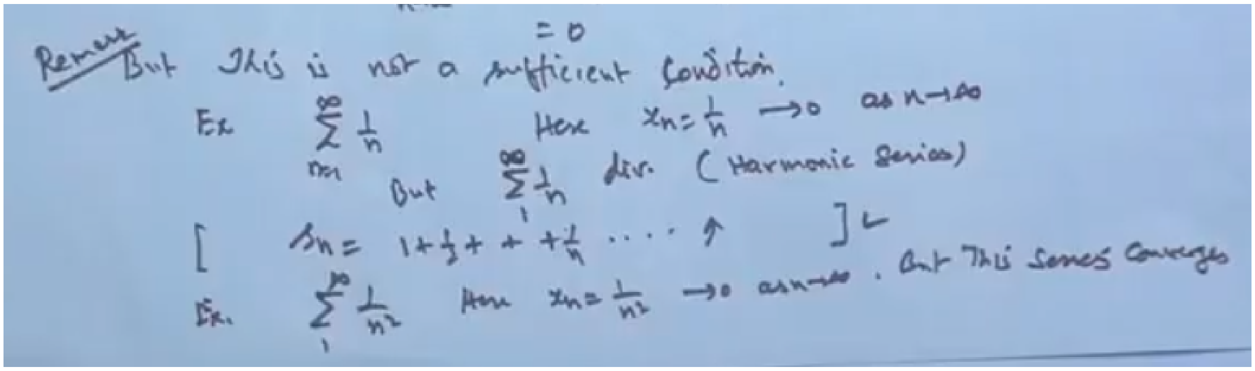

If we take another example, say Sigma of 1 over n square 1 to infinity, now here the xn which is 1 by n square goes to 0 as n tends to infinity, but this series converges but this series converges, that will also be shown in the next few, after few, article to be covered and test for the series is there then this is one upon Sigma n to the power P type where P is greater than one series will converge, so that we will prove it hence from there you can say this is convergent, but what the day you see that these two series in both the series the any term goes to zero, here also it goes to zero but once it is diverges adult converges so the taking the sequence xn the net term change checking whether it is tending to zero or not, this will not give a conclude is a just tending to 0 will not implies the series is convergent, because may be divergent also however, if we take thus any sequence xn, any series xn, whose electrum does not go to 0 then series has to be a diverging series because if it is convergent then correspondingly the inner term must go to 0, Okay? So that's why here, this will, third point remark.

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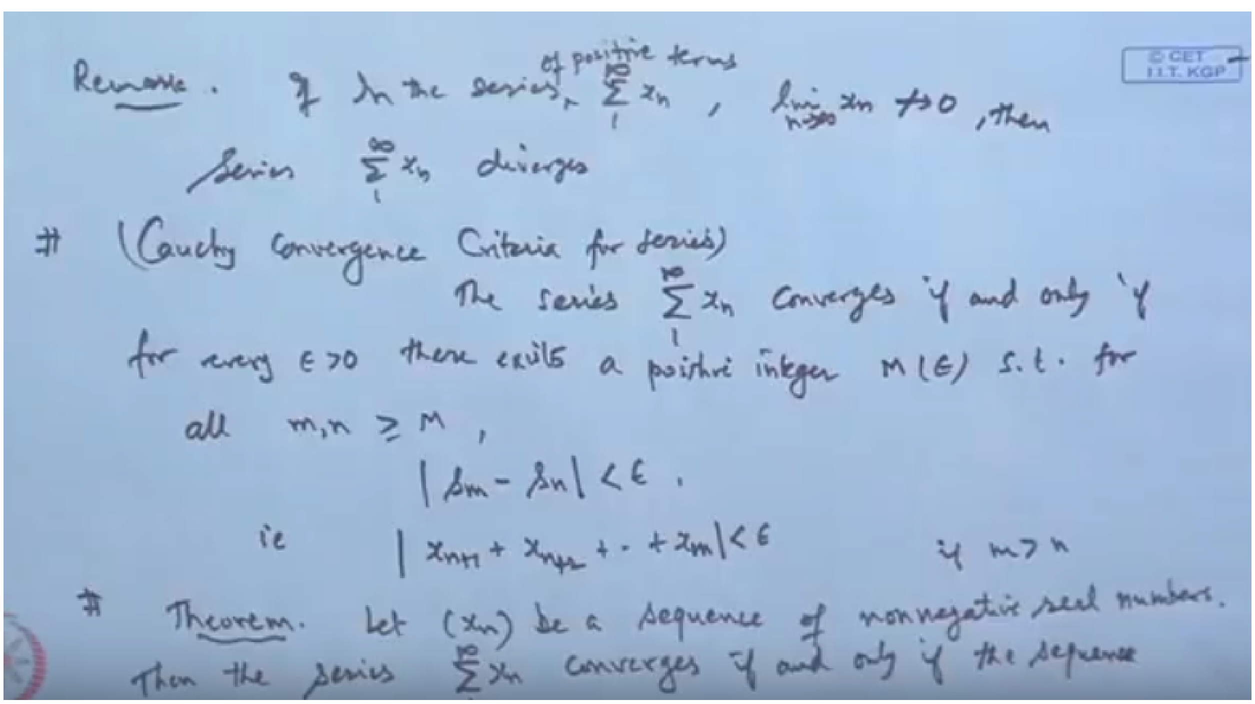

if in the series, Sigma xn 1 to infinity the any term of positive term Sigma xn s series if the xn is a sequence of terms, if the limit of xn when n tends to infinity, does not tends to 0, that's not tends to 0, then the series diverges then the series Sigma xn 1 to infinity diverges, Diverging. And the reason is very simple the reason is why because if it is convergent then according to the necessary result earlier the necessary condition for convergence of the series any term must go to 0, so this is not happening therefore, that much so if the series a positive from let us take the positive we got the series or positive terms, also positive terms, Okay? We have this course, so this was part of it, Okay? Now there's another in result just like a Cauchy convergence criteria, we have a Cauchy convergence criteria for the sequence any sequence of real number is convergent if and only if it is Cauchy that is it satisfies that converts Cauchy convergence criteria that after a certain restate the difference between any two arbitrary terms of the sequence is less than epsilon, given epsilon of then we say the sequence the sequence is convergent. So correspondingly we also have a result for a series that is known as the Cauchy convergence criteria for series, so next is Cauchy convergence criteria for series, the result is the series, Sigma xn 1 to infinity, converges if and only if, for every epsilon, greater than zero for given epsilon greater than 0, there exists a positive integer and there exists a sequence as there exist a positive integer, capital M which depends on epsilon, such that, such that for all M N greater than or equal to capital M, the following condition holds the mod of SM minus SN, the partial sum of the series a m term of premium and a n term is less than epsilon, is less than epsilon, for all that is equivalent to say is that mod of X n plus 1 if, if I choose M is greater than n if M is greater than n then xn plus 1, xn plus 2 up to X M is less than epsilon, that is upto.

So this is known as the Cauchy convergence criteria, and this part is if and only if, that is if the series converges then sequence of its partial sum will satisfy this condition, and obviously we have defined the convergence of the sequences when the limit of the SM goes to a limit exists, so when the limit of the sequence of partial sum exists then only we say the series converges, so suppose the series converges, so limit of SN exists it means the limit of SN exists means it must satisfy the Cauchy convergence criteria, so this is nothing but the Cauchy convergence criteria and conversely if this is true then the limit of the sequence SN must exist, limit of the sequence exists means the series must be a converging one, so that's so the connectivity, Okay? Now based on this we have a very interesting results, the result is in the form of theorem, let xn be, let xn be a sequence of, non-negative real number, real numbers, then the series, then the series, Sigma xn 1 to infinity converges if and only

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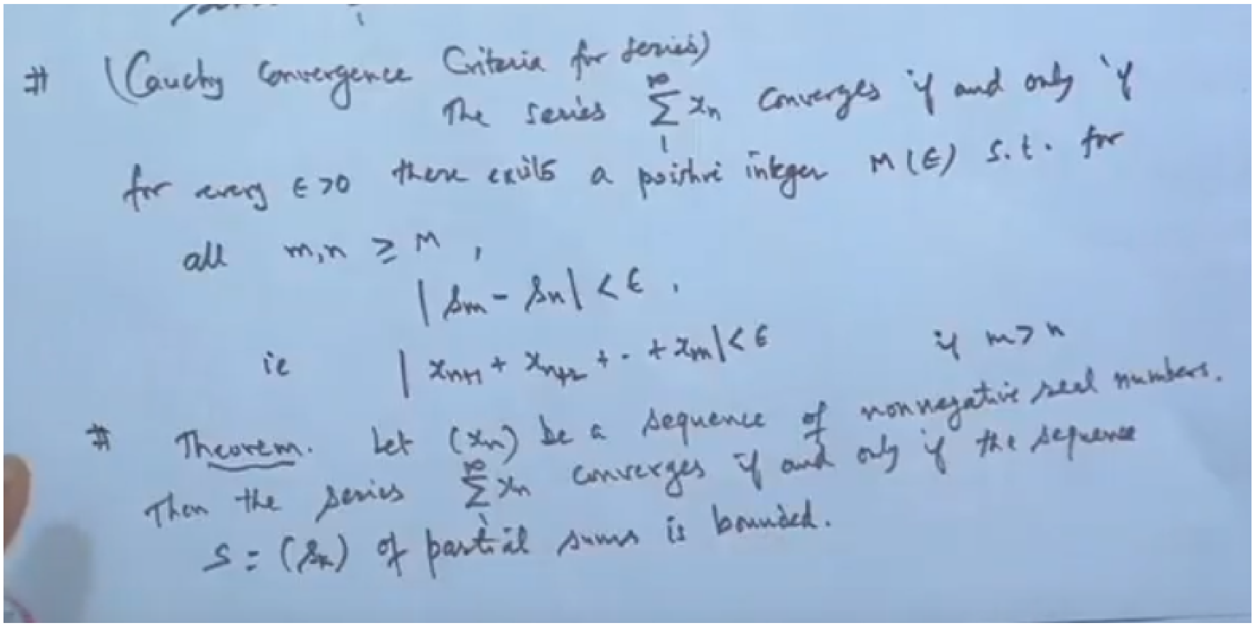

if the sequence of partial sum S or partial sums or partial sums, sequence s of sk of partial sum, each sums is bounded, is bounded, Okay? And in fact when it is bounded the limit superior of this limit of SN will come out to the sum of this, so let's see the proof of it, Okay? What is given is the sequence xn is a sequence of non-negative real numbers positive numbers, Okay? Non-negative s may be 0 the series is convergent, if and only the sequences of the process on each done. So suppose the series converges, it means limit of the SN will be exists and what is our SN?

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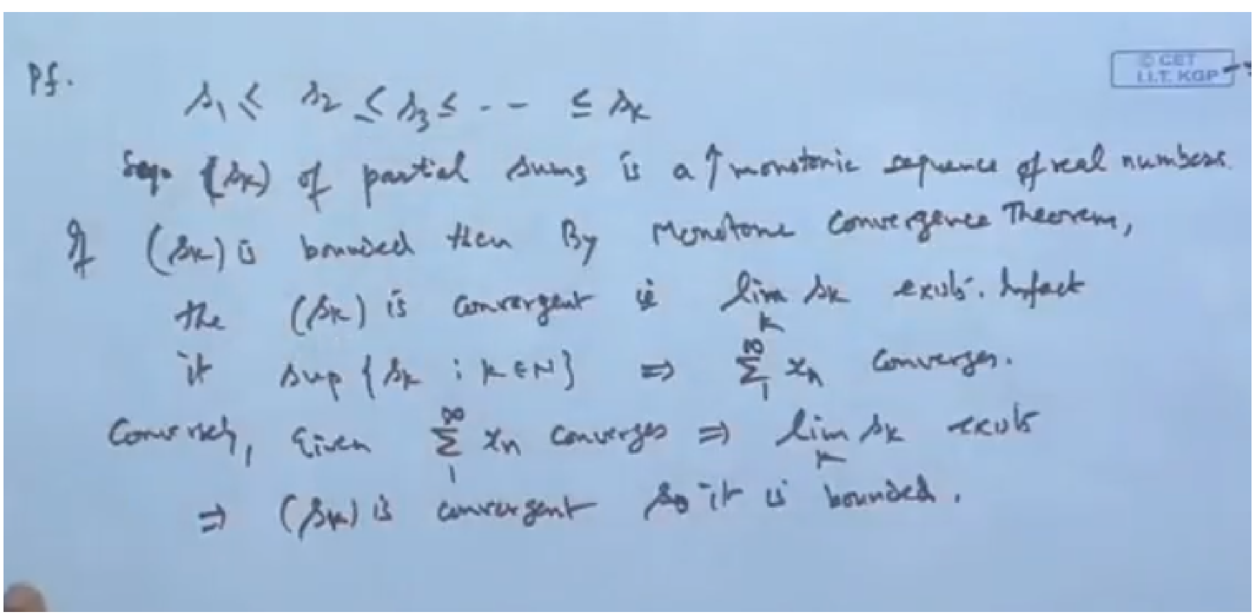

SN is basically the partial sum s 1 is the first, term s 2 is the sum of the two terms, 1 and 2 and since X 1, X 2 xn are non-negative real numbers. So may be few may be 0 or may be strictly greater than 0 also positive number, so s 1 definitely less than equal to s 2 which is less than equal to s 3, and so on like this, so what we are getting is the sequence of the partial sums, so the sequence of partial sum, of partial sums sequence of partial sums, is a monotonic sequence, monotonic sequence,, of increasing numbers increasing monotonic sequence, monotone sequence increasing monotonic sequence of real numbers.

Now what is the monotonic convergence Theorem? Monotone convergence theorem Says any monotonic sequence which is either increasing or decreasing and if it is bounded a monotonic sequence which increasing and bounded above then it must be convergent or monotone sequence beside decreasing and bounded below must be convergent, here this is a monotonic sequence increasing sequence of real numbers, and what is given is as series is convergent, if this sequence is given to be bounded suppose I take this part if sequence is bounded then it means this monotonic sequence is bounded, so once it is bounded it has to be convergent, Okay? So if the sequence SK is bounded then by monotone convergence theorem, monotone convergence theorem, the sequence SK is convergent, that is limit of XK exist, exist in fact it Is, in fact it is the upper bound for this it is the supreme value of all s k then the K belongs to integer, Supreme its upper bound for this we have not discussed upper bound we will take up after that, Okay? In fact this limit, so if the SK is bound partial sum is bounded then we can immediately say by monotone convergence theorem, this limit will exist hence the series is convergent, so this implies the series Sigma X N 1 to infinity converges. Conversely, if given the series Sigma xn 1 to infinity converges, it means what? That means the limit of the sequence s K over K exists, finite now every convergent sequence is a bounded sequence so this means that sequence SK is convergent, so it is bounded, so this proves the both way, that if this series is convergent then the sequence of the partial sum will be bounded sequence and vice versa if sequence of the power system is bounded then the corresponding series will be convergent, so that's the convergence criteria for this.