Model 4

Lecture – 27

Cauchy Convergence Criteria for Sequences

Course

On

Introductory Course in Real Analysis

So in the previous lecture we have discussed the Cauchy sequences of real numbers and its reason with the convergence sequence and bounded sequence. In this process we have proved basically the two lemmas.

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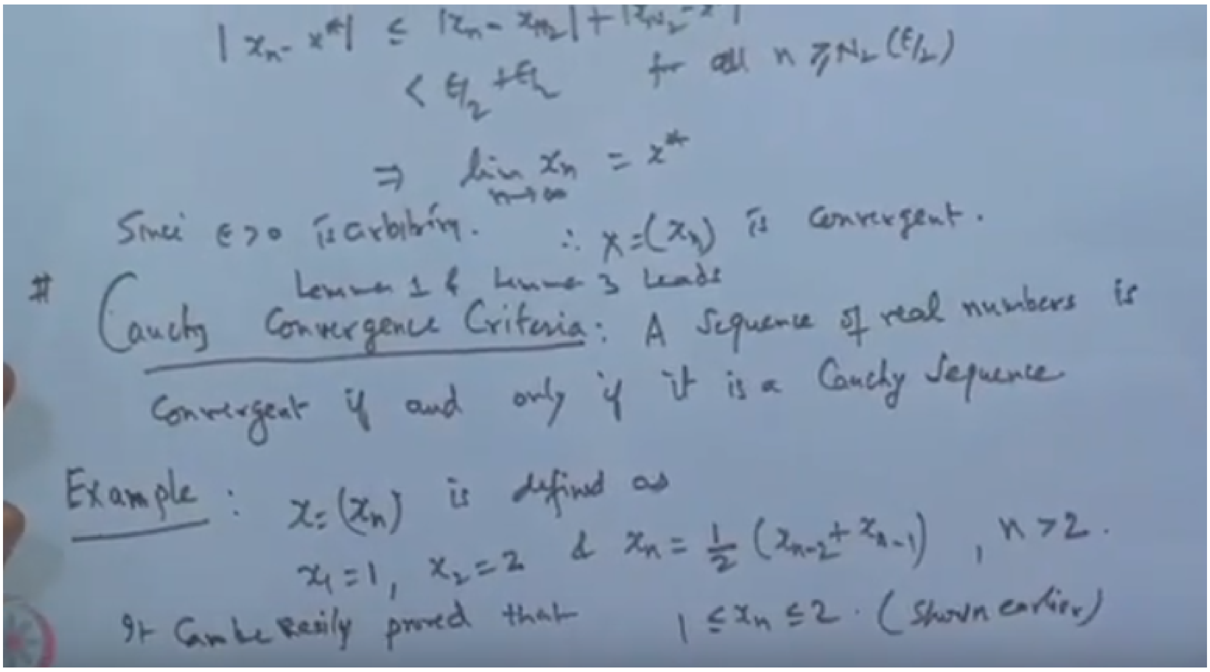

One is lemma one which says that every convergent sequence is Cauchy, sequence is Cauchy. well the secondly must says that lemma three lemma two limit says that every Cauchy sequence of real number, Cauchy sequence of real numbers is every Cauchy sequence of the real number is, convergent now combining this two lemma we have a Cauchy convergence criteria, Cauchy convergent condemned. So today we will discuss the Cauchy few examples of the Cauchy sequences, concept of the conductive map sequence of the real number and Cauchy convergence criteria. Now this lemma 1 and lemma 2 and 3 if we combined this sorry 1 & 3 if we combine we get the result, which is known as the Cauchy convergence criteria, Cauchy convergence criteria and there is very important result. What this criteria says a sequence of real numbers, a sequence of real numbers, is convergent is convergent, if and only if, if and only if it is Cauchy sequence, it is a Cauchy sequence, ok and thus proves the result. Combined the lemma say this lemma 3 every consequent real number is convergent and then lemma 1 that exit convergent sequence of real number then lemma 1 and lemma 3 will give so lemma 1 first lemma 1 and lemma 3 leads the following Cauchy convergence criteria ok now let us see few examples we are we can use this Cauchy convergence criteria to test whether the given sequence is a convergent one or not we are it's difficult to identify the limits for example if we take define the sequence xn, is suppose a sequence xn, is defined as x1, first term is 1, second term is 2 and there is a relation between the term after 2 on board after second term on what the reason is like this that xn can be obtained as a average of xn minus 1 and xn minus 2by2, average of so after second term number, third term will be obtained as the sum of the third term X 3 can be obtained as the sum of what X 1 plus, X 2 by 2, for ton like this way so this sequence this is 2 for this

is defined for N greater than 2. okay now this sequence if we find out the X ends, by simply choosing X 1 is this, X 2 this, then X 3, X 4 and so on then what we see that this sequence is basically all the terms of the sequence are bounded and bounded by 2 because that I think this result we have already seen in the first earlier 1. Because this is monotonic decreasing, when n is odd while decreasing when n is increasing, even like that way, so but if the all the terms cannot exceed by 2 and below it is always be greater than 1. so it can be easily it can be easily proved, that all the terms of the sequence xn are lying between 1 & 2, in fact it was shown earlier so. Earlier okay by choosing decreasing and increasing the odd terms and even terms separately and they are dominated by 2. Okay, but it's not a monotone sequence but the sequence is not monotone.

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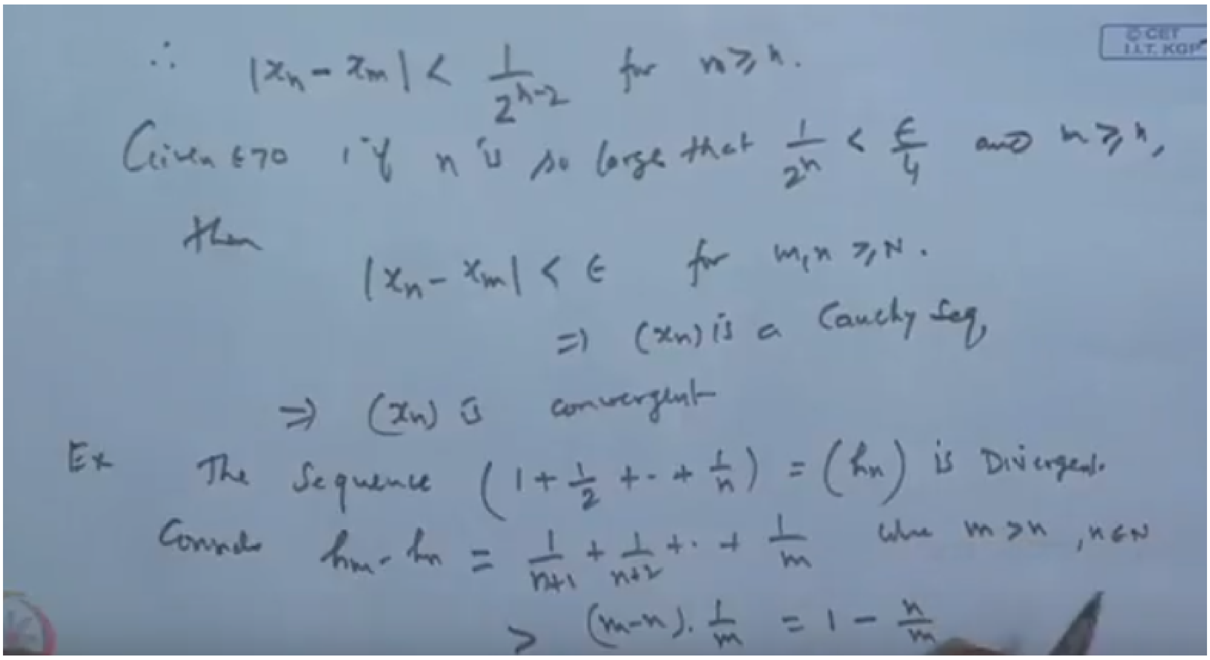

but sequence xn, is not a monotone sequence, it's even and odd terms are monotone it's even and odd terms but as a sequence it's not monotone, some terms are coming down than up and like this so as you hold that monotone sequence not there. so you cannot say the limit exists, because when you say it is monotone then only the limit will be you can say dominated by two bounded but here it's not a monotone sequence you cannot choose anything thing, about this however it is a positional however, it is a Cauchy sequence. The reason is because if we start with this xn minus XM, say we are M is greater than n, let let M is greater than n, consider mod xn minus now let us see this is less than equal to 1 X n, XM, xn minus xn plus 1 because n is less than n then xn plus 1 minus xn plus 2 like this and then our last term will be this, ok. Now this will be equal to 1 by 2 to the Power, n minus 1, plus 1 by 2 to the power N, up to 1 by 2 to the power M minus 2, why the reason is because if we look this term x1 by induction, induction one can show that mod of xn minus, xn plus 1 this is exactly same as 1 over 2n minus 1 to n minus 1 okay this can be said and this is true for all natural number n belongs to natural number n. it can be tested for n equal to 1 for n is equal to say 1 what happens mod of X 1 minus X 2 X 1 is given to be 1 X 2 is giving to be 2 so it is basically 1 which can be written as 1 to the power 0, for n is equal to 2 if we take then you take X 2 minus X 3 X 2 is 2 and what is X 3 X 3 from here is the average of when n is 3 is this X 1, plus X 2 by 2, X 1 plus X 2 either the 1 plus 2 by 3 that is 3 by 2 so this will be equal to 4, 1 by 2 which is 1 over 2 to the power or 2 minus 1. So suppose this is true for n, so suppose it is true for it is true for m. then M then what we get is we are taking mod of mod of X N and M, minus X M plus 1, this is given to be 1 by 2 to the power M minus 1 so consider now mod of X M plus 1 minus M plus 2 so that will be equal to what? now XM XM, plus 1 this is equal to XM plus 1 minus xn plus 2, if I write then this is equal to what X M plus 2 so X n XM plus X M plus 2 so M plus 1 by 2 and that is nothing but equal to X M plus 1, minus XM by 2 and that will be equal to according to this it will be half of this of 2 to the power M minus 1 that is half of 2 to the problem like this so it can be proved this way. So by induction we can show this result. now use this result here I use this result here and then once I use there then you can take common so 1 upon 2 to the power n minus 1 if I take common then you are getting 1 plus half up to 1 over 2 to the power M minus n minus 1 now this is the part of the geometric series. in fact geometric series is 1 plus half plus half square and so on in infinite terms are there well the sum is if you make the sum a over 1 minus all that is 2 so this is strictly less than 2 over 2 n minus 1 that is equal to equal to 2 over 2 to the power n minus 2**.**

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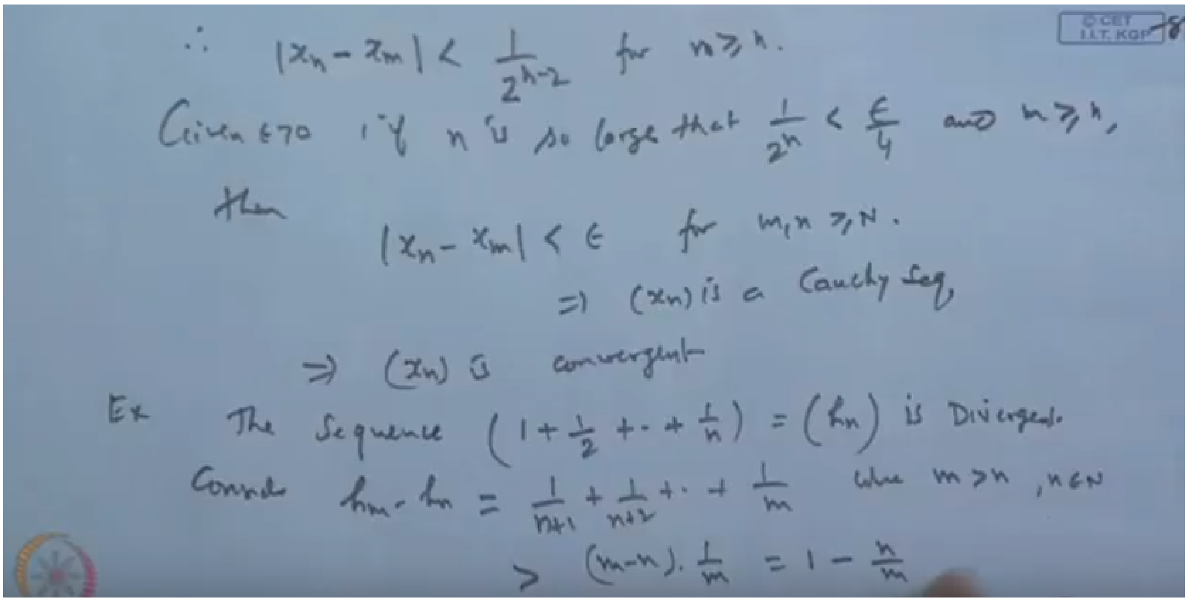

So what we get from here is therefore mod of xn minus XM is less than 2 to the power 1 upon 2 to the power n minus 2 and this is true for all n for all n for large n when M is greater than or equal to M, ok this one now. if I choose any epsilon given any epsilon greater than 0, again even yourself if I choose n if n is so large so large that 1 by 2 to the power n is less than epsilon by 4 and if we take m to be greater than or equal to n then what happens the mod of xn minus XM is less than this entire thing we want less than epsilon all by 4, so basically it is less than Epsilon and this is less than epsilon for all N greater than equal to capital n some integers n therefore this is a Cauchy sequence so this implies sequence xn is a Cauchy sequence. Ok once it is Cauchy therefore the sequence xn is convergent hence without calculating the limit one can go the sequence is convergent for it similarly if it is not Cauchy then one can so it is a diverging Sequence, for example, if we take the sequence, say the sequence 1 plus 1 by 2 plus 1 by n this sequence that is a sequence hn where H n is the sum of this now we claim that the sequence is diverging one. so obviously it if it is not Cauchy then it will be diverging because there every Cauchy sequence has to convergent, so consider this thing now consider HM – Hn ok when you take HM - Hn then what you get you are getting from here is 1 over n plus 1, 1 over n plus 2 and then 1 over we are M + n m is greater than n and this result is true for all n belongs to capital okay. Now this one is the lowest term is 1 by m, is you know, so it is greater than and total number term is M minus N, is greater than this number which is greater than say equal to 1 minus this is equal to 1 minus n by M. 1 by n now MN n are our arbitrary number because one can choose MN any integers.

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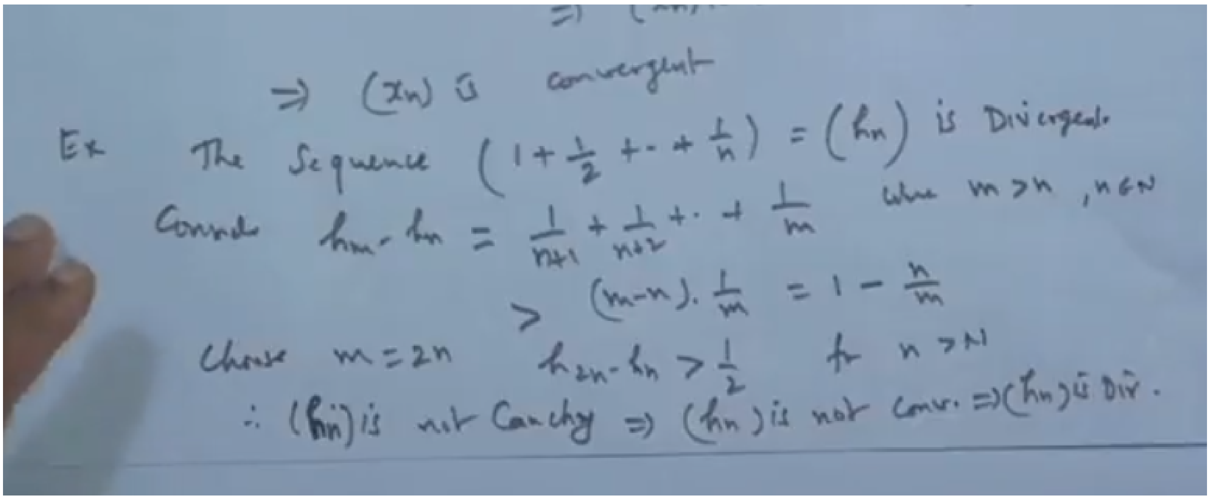

so I chose MN choose M to be equal to 2m, then what happens this is this H 2 n minus H n this difference is greater than half, and this is 2 for all N greater than capital N after a certain stage so this sequence cannot be Cauchy sequence so the sequence is not coaching HN is not Cauchy. So once it is not catchy, it means the sequence a chain is not convergent so it is diverging that's what. In and this is known as the harmonic series which we will come later on discussion forward for it okay.

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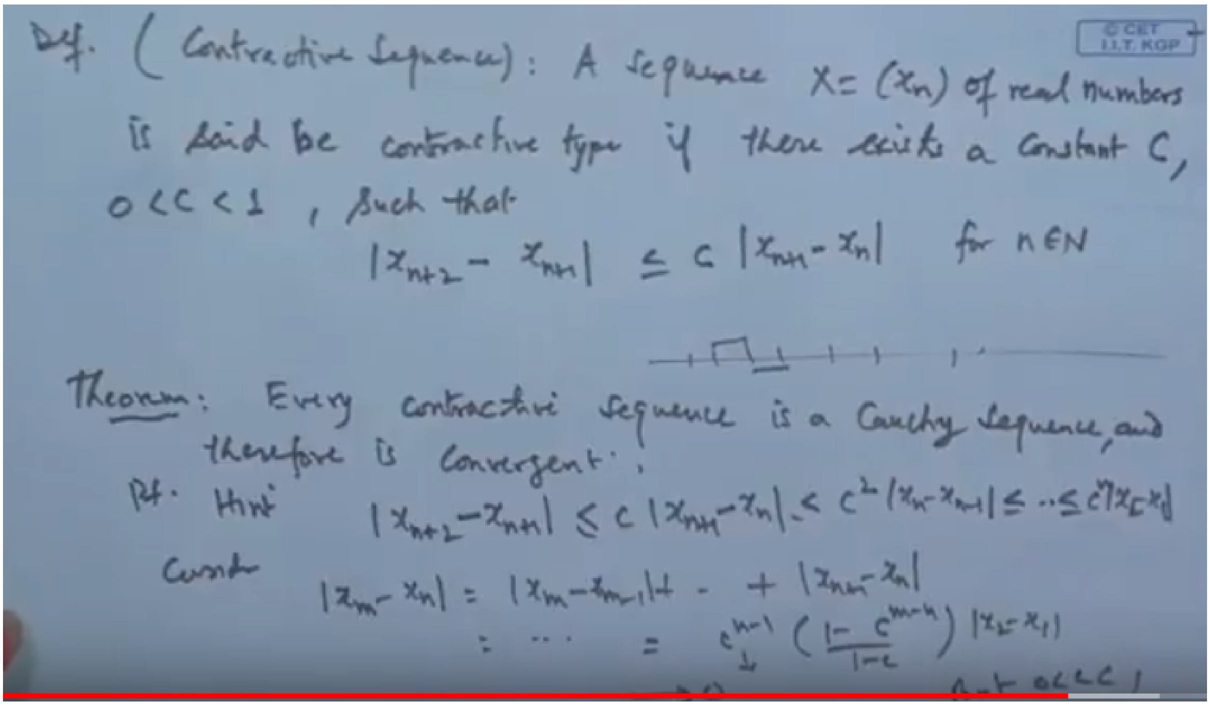

Now there is a another type of the sequence, which we call it as a contact of Stipe sequence ,so contractive sequence a sequence xn a sequence X is xn of real numbers, the second xn of real number, is said to be contacted .is said to be contacted Type O contractive sequence if there exists, there exists a constant skepta we are seen lying between 0 & 1 such the istic line less than greater than 0 is strictly less than 1 such that such that mod of xn plus 2 minus xn plus 1 this mod of xn plus 2 is less than equal to c times of mod xn plus 1 minus xn and this is true for all natural number n ok. This it means that sequence in like this that if we take the difference between two consecutive terms of the sequence said this difference then this difference cannot exceed by the difference of the previous one that is AC constant times is always be less than equal to some number which is less than this because she is less lying between 0 & 1 if I take C equal to 1 then only it is equal to so it is less than equal to so it keep deep differences keep on reducing keep on reducing for that now what this result says which is really interesting the result is every contacted sequence, contractive sequence is a Cauchy sequence, every conductive sequence is a Cauchy sequence and therefore, and therefore is convergent, okay. now again we will use the similar type of tricks as we did earlier that first you make this difference xn minus 2 and so on like this prove it in fact to prove this thing I just give the hint, hint say that first you saw this part xn plus 2 minus xn plus 1 this will be less than or equal to C times mod xn plus 1 minus xn again apply this so it is less than C square mod xn minus xn and continue this so finally what we are getting is finally you are getting less than equal to C to the power n x2 minus x1 so this is coming like now consider mod of xn XM minus XM and when you break up into the parts mod XM minus XM minus 1 up to mod xn plus 1 minus xn then substituting these values and finally you will get this thing we'll come out to be C to the power n minus 1 1 minus C M minus n divided by 1 minus C and then X 2 minus X 1 but C is lying between 0 & 1 so this part will tends to 0 as n tends to infinity so it is a Cauchy sequence so Cauchy’s, okay.

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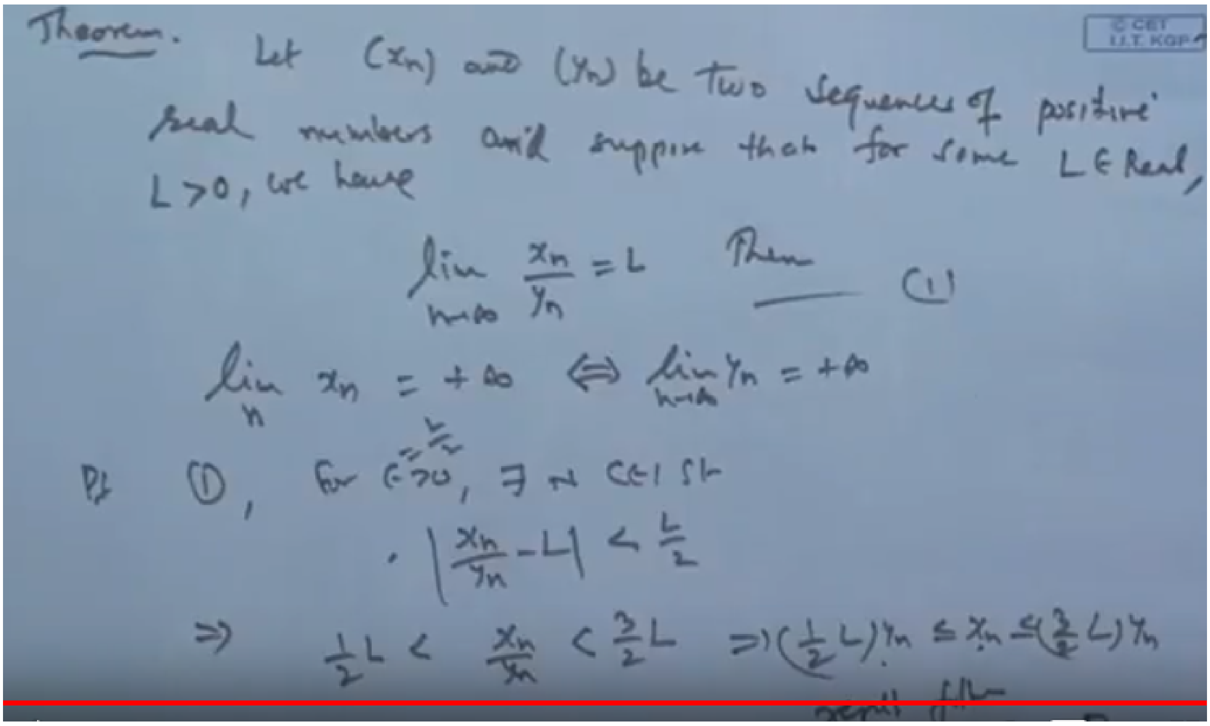

Now there are only one result which I will show before the divergence you can this result is very useful for the diverging let xn, yn be two sequences of sequences of positive real numbers, positive real numbers and suppose that for some L which is real belongs to L and positive we have limit of this xn, over yn as n tends to infinity here, then limit of this xn, over N is infinity, if and only if, if and only if limit of yn is infinity. Okay the proof is very simple this is given. So from 1 we say for given epsilon greater than 0, there exist n and such that depends on epsilon such that mod xn yn minus L, is less than epsilon but epsilon I choose to be l by 2. So is less than L by 2 and if I open this way we get from here is half L, is less than xn by yn, is less than 3 by 2L, when positing it now if we multiply by Yn then what happen this implies that 1/2 L into sequence by n is less than equal to xn less than equal to T by 2 L into yn. Now these are finite so if X n, yn goes to infinity, xn will go to infinity if Xn goes to infinity by so the result follows That.

So thank you very much,

Thanks.