**Model 4**

**Lecture – 26**

# Cauchy Sequence

**Course on Introductory Course in Real Analysis**

So we were discussing about the criteria, for the convergence of sequences and in this process we have seen two criteria one is the monotonic convergence criteria for monotone convergence sequence, but that monotone convergence theorem which is very wonderful result, which gives the convergence criteria for a sequence particular types you can which are monotone, either monotone increasing or monotone decreasing sequence, and another criteria which is the general in nature that if a given sequence is given one can find the limit of the sequence, and if the limit exists we say the sequence converges, if the limit does not exist or goes to infinity or minus infinity say the sequence diverges. But you see in both the cases there is it in the first case particularly there is a drawback in the monotone convergence theorem the drawback, main drawback is that this result is only applicable to those sequences, which are monotonic in nature, but the sequence may not always be monotone sequence, so in such a case the monotone convergence theorem is not applicable to charge the convergence of the sequence, and second criteria which we told though it is valid for any type of the sequence but in identifying the limit itself is problem, is a difficult one, so what we want is will there be any criteria which can directly say the sequence given sequence is a convergent one or divergent one without going for the limit, and this is given by Cauchy and is known as the Cauchy convergence criteria and it's the one of the important result which relates the convergence of a sequence of real or complex number with its cushiness, so in fact, if a sequence is Cauchy then automatically it will be a convergent and vice versa, now this is valid in case of the real complex numbers but when we go for an arbitrary metric space which we have not taken up then the convergence criteria obviously does not help there, so there will be defined in a different way, the convergence of a sequence, okay? So we will go on since we have restricted our studying only for the real number or complex number, so therefore this convergence criterion is very much helpful in getting the result following nature of the sequence which is convergent or divergent.

So before going with the Cauchy convergence carrier what is the Cauchy sequence?

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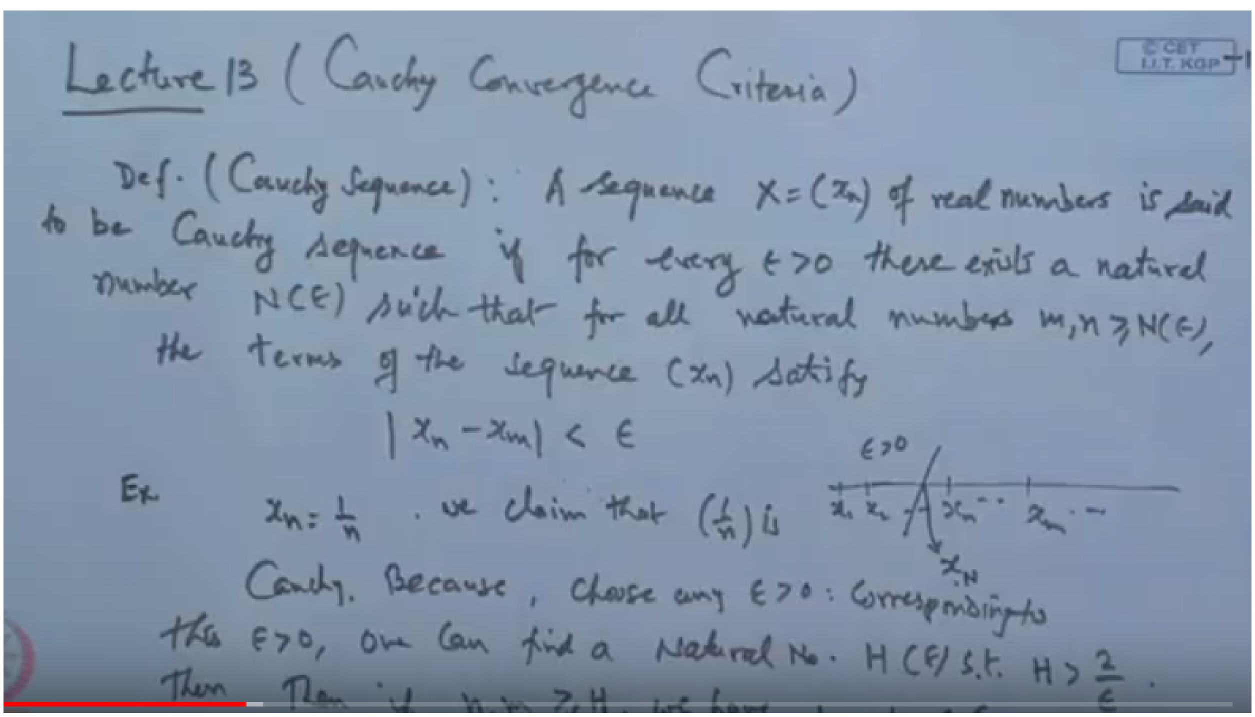

So we say different first definition of a Cauchy sequence of real numbers, a sequence xn, X which is xn of real numbers is said to be, is said to be, is said to be, Cauchy sequence, Cauchy sequence if for every epsilon greater than zero, for a given epsilon greater than there for every epsilon greater than 0, there exists a natural number of positive integer you can say, natural number, say N which depends on epsilon, such that such that, for all MN natural number, numbers natural numbers m n greater than equal to capital N which depends on epsilon of course, the sequence of the terms satisfies the following condition. These terms of the sequence xn satisfy the following criteria that mod of xn minus XM is less than Epsilon it means a sequence is Cauchy if X 1, X 2 xn this is a sequence of Cauchy numbers a sequence of real and complex numbers but we say this sequence is a Cauchy sequence if for every epsilon greater than 0, so first we choose the epsilon then corresponding to this epsilon one can identify a point xn such that when we take all the terms after this xn onward the difference between any two arbitrate terms after xn will remain less than Epsilon distance between xn and xn we'll remain less than Epsilon, it is in the epsilon neighborhood of one of the say m is fixed then one of the point in this, so they are very close to each other the difference distance between any two arbitrary point after the certain stage capital in on but the difference is very, very small or is less than the given number of epsilon. if this happened then we say the sequence is a Cauchy sequence for example if we look the sequence say xn is say 1 by n we claim that the sequence 1 by N is a Cauchy sequence, so it means for any epsilon u greater than 0 so let because if we choose any epsilon because let us pick up choose any epsilon greater than 0 first then corresponding to this epsilon 1 can find a natural number then corresponding to this epsilon to this epsilon greater than 0 one can find a natural number say capital H which depends on epsilon such that such that, let us say H is such which is greater than 2 by Epsilon so H will depend on epsilon we can choose a natural number which is greater than 2 by F epsilon, then

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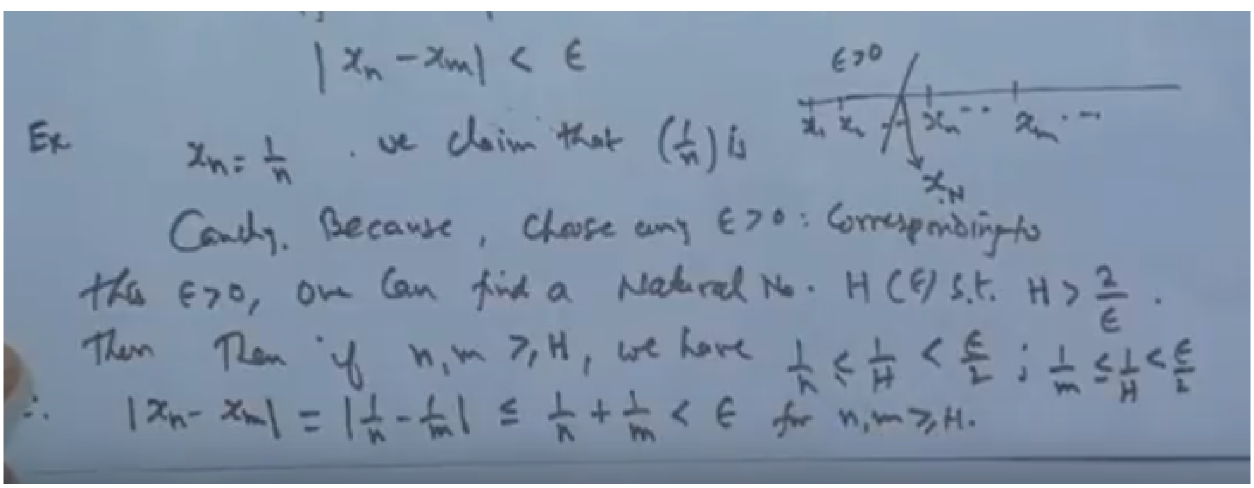


According to the Cauchy definition this difference must be less than Epsilon, Okay. So if I choose MN greater than n so if we then if we take n and M natural numbers which is greater than equal to H we have one by n is less than 1 by H, Okay.

All less than equal to 1 by H and but H is greater than this so it is less than epsilon by 2 similarly 1 by M is also similarly 1 by M which is also less than equal to 1 by H so it is mystical less than epsilon by 2 so these two are less than ever therefore when we take the difference between mod of a therefore mod of xn minus XM when n m are greater than H, this is nothing but the 1 by n minus 1 by M and by Tanglin equality it is nothing less than equal to 1 by n plus 1 by M but 1 by n is less than epsilon by 2 and 1 by M is also less than epsilon by 2 so this is less than epsilon for all n, m greater than equal to H. It means the difference orbit difference between two orbit terms of the sequence after the certain stage H onward is less than epsilon so sequence 1 by N is a Cauchy sequence, Ok? That is what now.

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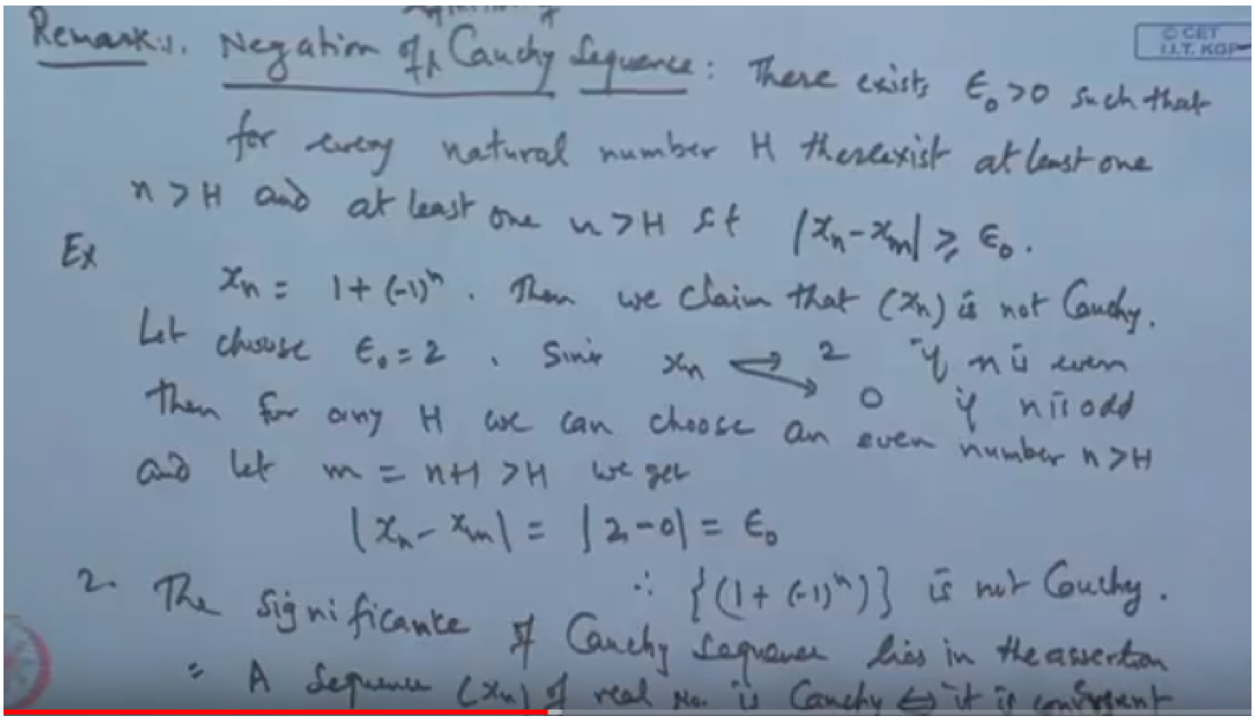


What is the negation of the Cauchy sequence?  the negation of because that will be required, negation of Cauchy sequence if a Cauchy sequence does not satisfy, if you see every sequence xn does not satisfy the Cauchy convert cushiness so what is the negation of Cauchy? negation of the definition of Cauchy sequence, negation of definition of Cauchy secrets. so different on Cauchy sequence, the negation is like this the Cauchy sequence says, that for every epsilon greater than zero, so negation will be there exists an epsilon naught and this shows there exists a natural number, so we can say that for every natural number H, then for all natural numbers so there will be a natural number m and n which are greater than H and satisfy this condition, Okay?

So we can say the negation like this there exists, there  exists epsilon naught greater than zero a small number a epsilon greater than zero such that, such that for every natural number H, for every natural number H there exist, there exist for every natural number there exist, exist say at least one and which is greater  than H and 1 M and 1 M at least one M which is also greater than H such that the difference between the terms of the sequence xn and XM h greater than equal to epsilon not, ok. Then this will be a negation of the definition of Cauchy scale, it means if you want to show the sequence is not Cauchy then this criteria must be Satisfied. For example, if we take this sequence say X n is the sequence like 1 plus minus 1 to the power n this sequence so then we claim that the sequence xn is not Cauchy. This is a work it means this condition is not satisfied so let us pick up the epsilon naught, Okay? So let us choose epsilon naught equal to 2, then exists an epsilon naught we take epsilon then since the sequence xn is such which goes to 2 if n is even and goes to 0 if n is odd, so if we take that n is even and n plus 1 becomes odd so difference between these two will be 2 which is epsilon naught so in fact then for every for any than for any H natural number H, we can choose we can choose an even number and get out then H and then M let M is n plus 1 which is also greater than H we get mod of xn minus XM is nothing but what, when n is even the xn comes out to 2 and when M which is n plus 1 or so it is 0 this is exactly same as epsilon naught, so this satisfies the convergence negation of the definition of Cauchy sequence. Therefore, sequence xn 1 plus minus 1 to the power n this sequence is not Cauchy, Ok? So this the significance of the Cauchy sequence is, the significance of Cauchy sequence lies in the assertion, lies in the assertion, the assertion is that a sequence is Cauchy sequence, a sequence of real number xn of real numbers is Cauchy if and only if it is convergent, convergent if and only if it is convergent so that's the main idea that if a sequence is Cauchy then it has to be convergent and if a sequence of real number is convergent then it will be Cauchy, so this is the main result which we will show it here and that's why they exist in the definition of the cushiness or Cauchy sequence plays the vital role in the set of set of real numbers system of the real numbers, Ok?

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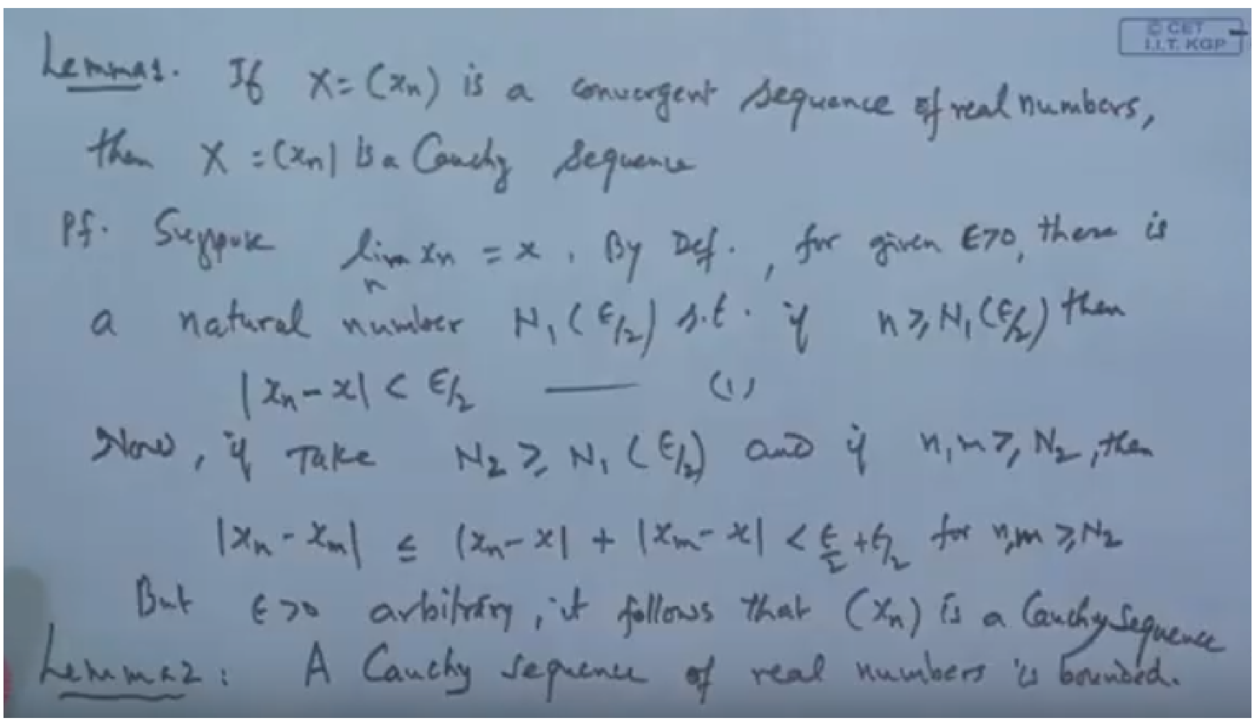


Let's see the first lemma which say, which is used say lemma 1 if X which is xn is a convergent sequence, is a convergent sequence of real numbers then the sequence X is Cauchy, is a Cauchy sequence. So this is the first result which one two so that every convergent sequence is a Cauchy sequence the proof of this is very simple suppose, suppose limit of the xn as n dash X because it is a convergent sequence or limit will exist and suppose this limit is X, then by definition of the limit for given epsilon greater than 0 there exists, there is a natural number there is a natural number say n1 which depends on say epsilon 2 such that if we take n greater then equal to N 1 depends on epsilon 2 then mod of xn minus X is less than epsilon

by 2 let it be 1, okay. now if we take now so now if we take another number n 2, we take n 2 which is greater than or equal to say n 1 depending on say epsilon 2 and if we choose N and M greater than or equal to n 2 then obviously for this particular into this result is also valid because this is true for all N greater than 1 so the we take and n which is greater than n 2 that will also be satisfied because it is greater than N 1 so it will be satisfied, then we have mod of xn minus XM apply the time this is less than equal to xn minus X plus XM minus X, now this is less than epsilon by 2 this is less than epsilon by 2 for all n m, for all n m greater than n 2 then in that case the difference between n xn minus XM is less than Epsilon so this shows that but epsilon is arbitrary, any choose, any number you can choose therefore it follows that the sequence xn is a Cauchy sequence, is a Cauchy sequence so that's what we wanted to okay? Okay.
Now as we have seen that a convergent sequence is a bounded sequence and we have seen that every convergent sequence is a Cauchy sequence, so basically that we are going to show this also that every Cauchy sequence is also bounded sequence, okay. So that will the next result is lemma two says that a Cauchy sequence of real numbers, of real numbers is bounded. So the proof is just based on the similar lines as we have done it in case of year so what we do is we take the sequence xn which is a Cauchy sequence since Cauchy criteria is satisfied so applying the Cauchy criteria you can get the sequence xn is dominated by a certain number of resultant states and for the remaining one finite sum take the maximum value so that's what to do. So let X be a proof.

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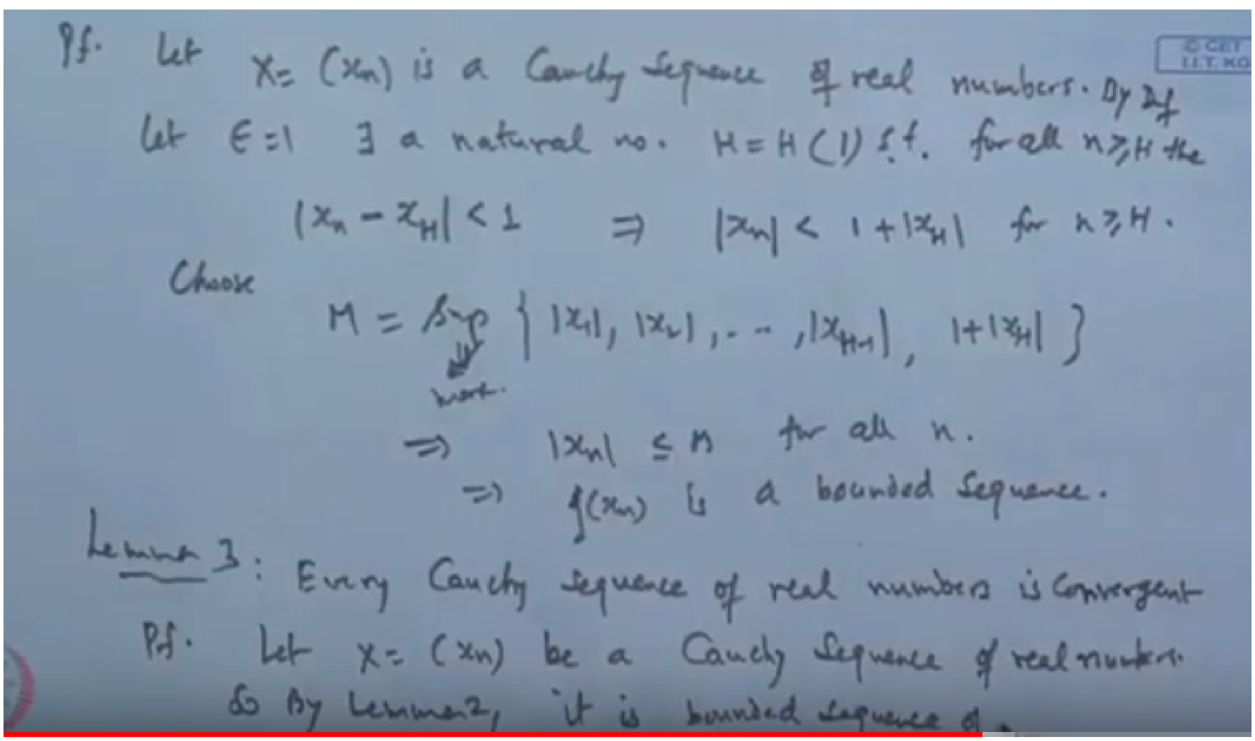


Let us see the proof, is so let our sequence X which is xn is a Cauchy sequence of real numbers. So by definition, so by definition for Epsilon any epsilon greater than 0 so if we take epsilon let epsilon is equal to 1 so corresponding to this epsilon there exists a natural number this is signed for there exists natural number H which depends on epsilon, epsilon means 1 such that such that for all N greater than equal to H the difference between xn minus X H is less than 1. Because, m we can choose so M is H so anyway this implies that the sequence xn is less than 1 plus mod X H for all N greater than or equal to H, by modulus well because if this is greater than we can just take it or less than greater than equal to mod xn and then so basically this is mod xn mod xn is less than this, Ok?

Let it be and then further you choose capital m as the maximum value or supreme value of mod X 1, mod X 2, mod X, X H minus 1 and then 1 plus mod X H we can also use the maximum in fact supreme are not defined because these are finite number so we can change this by a maximum, also make suit instead of supreme that is, Okay? We can also take this makeover so when you take the maximum of these two suppose m then what happens to the all the terms of the sequence xn will be dominated by m and this is true for all n so this shows the sequence X n this sequence is a bounded sequence, bounded sequence of real numbers so every Cauchy sequence of real number is bounded let's see the converse part and converse well will prove the Cauchy convergence criteria that is complete so what the lemma 3

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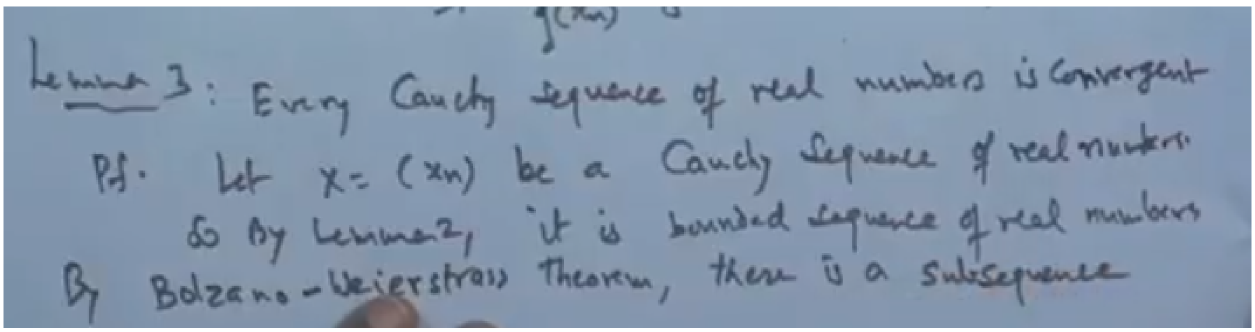


says, a lemma 3 or you can say every Cauchy sequence, every Cauchy sequence of real number, if real number every Cauchy sequence of real number is convergent so combining 1st and 2nd Cauchy sequence of real number is bounded and sorry, Cauchy sequence of the real number is already we have seen a convergence occurs real number is Cauchy and if we prove the Cauchy sequence every Cauchy sequence is convergent then we can go for the Cauchy convergence criteria, Ok. So let us see the proof for suppose we have a sequence X which is xn is a Cauchy sequence be a let be a Cauchy sequence of real numbers, of real numbers, Ok?

Now this Cauchy sequence by previous result a Cauchy sequence of real number is bounded by lemma two if you see every Cauchy sequence of real number is bounded so by lemma two it is a boundary sequence, so by lemma two it is a bounded sequence of real numbers and by Bolzano theorem says if a sequence is bounded then it must have a convergence of sequence so by Bolzano west's theorem always trust or owned by Bolzano weierstrass theorem, says that, there is a there is a sub sequence,

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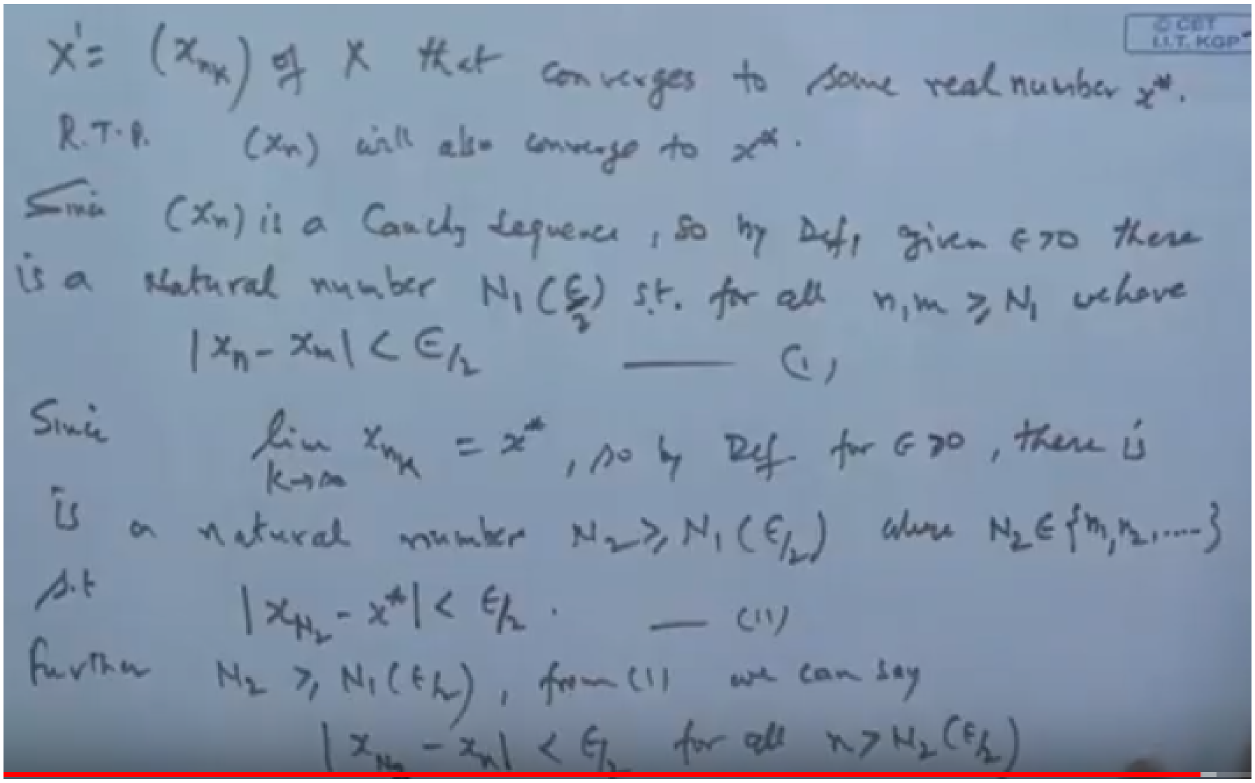


X dash, say equal to X n K, X, suffix n K, of X, X, that by Bolzano of sequence X that converges, to the some real number X system, real number X system, some real next, now if I prove that our sequence xn also converges to the same number X star then the result is complete, so what is equal to prove is, that the sequence xn will converge, will also converge to X star, this is our problem to solve. Okay? So let us see how do. Now what is given the sequence xn, is a Cauchy sequence. So by definition of cushiness, for given epsilon, greater than 0, there is a natural number, there is a natural number, say n1, depends on epsilon, such that for all N and M, greater than or equal to n1, say epsilon Y2, let us take the epsilon by 2 because we will need further epsilon by 2, N 1 we have such that, for all is that we have the difference between xn, minus XM is less than epsilon let it one. Okay? Now the subsequence xnk is a convergent sub sequence, converges to X star, so by definition again apply the definition of convergence sequence which is a limit x star, it will limit of X and K when K tends to infinity, will is nothing but X x star. So since xn, since the limit of this xn K, when K tends to infinity is giving to be x star. Okay? This is given we have got it, is it not? So by definition of the limit we can say, for given for the same epsilon, for epsilon greater than 0 which is the same epsilon, there exists there is a natural number, say n 2, which is suppose greater than or equal to N 1, which depends on epsilon y 2, such that n 2 and such that this n 2, where, where, and together such that this n2 belongs to the set, N 1, n 2 n etc. This set means we are choosing the positive integer, from a sequence n case so there exists natural number n 2, from this set which is greater than equal to n 1, such that, mod of X n 2, minus X star is less than epsilon by 2, because this is difference of this can be made as small as we please. So when K is sufficiently large it goes, so we can choose the N 2 which is in this is greater than N 1 so that this condition is satisfied. Okay? Now consider, and this is 2 for what? This is Okay.

This less than is, now further, further this n 2 is greater than equal to N 1 which depends on epsilon by 2, so from 1 the result is also valid when you replace M by n 2, so from 1 from 1 we can say, we can say the result mod of X n 2, minus xn is less than epsilon by 2, for all N, greater than n 2 depends on epsilon by 2, Okay? For all M because M I can choose to be n, now consider, now consider

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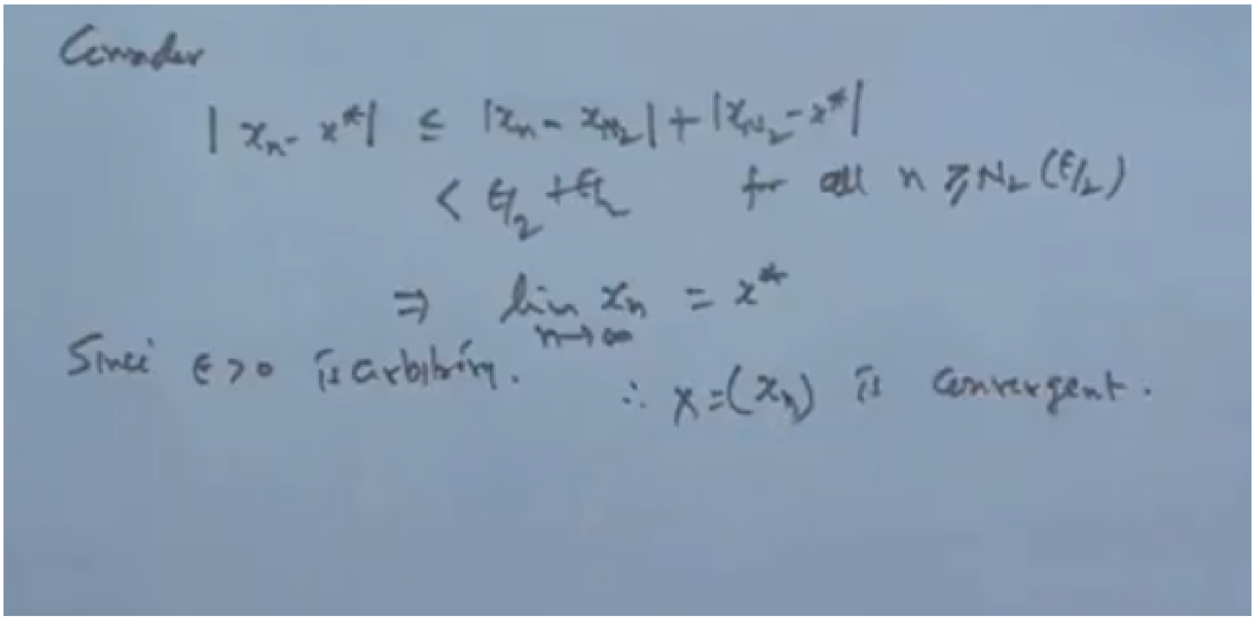


mod of xn minus X star, now this is less than equal to mod of xn minus X n 2, plus mod of xn 2, minus X star and then this will remain less than epsilon by 2, plus epsilon by 2 for all n which are greater than or equal to n 2 depends on epsilon. Because both the conditions, 1 & 2 are satisfied by this, so what is that the limit of xn is also n, limit of xn and n tends to infinity is also X star, this shows the Cauchy sequence is convergent, because epsilon is arbitrary is since epsilon is arbitrary. So once it is already be say limit of this, Ok? Therefore, the sequence xn which is, X is convergent.