Model 4

Lecture – 23

Criteria for Convergent Sequence

Course

On

Introductory Course in Real Analysis

So today we will discuss the criteria for convergent and divergent sequences. we have already seen that if xn be a sequence of real numbers, then it is said to be a monotone if there is a

sequence of the positive integer in increasing order or decreasing order, such that the corresponding terms of the sequence xn 1, X 2, xn n then they form the monotonic sequence N 1 is less than X n 2 less than xn 3 a monotonic increasing or if it is a reverse, then we say monotonic decreasing and also if a monotone sequence which is bounded above or below it must be a convergent sequence. So based on this we have seen that sequences if you are monotone and bounded we can say the sequence will definitely converge, but every sequence need not be a monotone sequence. because there are these sequence, which are not at all monotone sequence, then how to find out whether the sequence it convergent or divergent. So for this we have certain results, which will directly tell without computing the limit whether the sequence is convergent or not and one of them which is very important result is given by the Cauchy which is known as the Cauchy convergence criteria. For a sequence of real or complex numbers,
ok. so let's see the first thing that if a sequence is convergent then all of its sub sequences will also converge to the limit, we know if a sequence is convergent then the limit is unique, this we know that as limit of a convergence sequences we know.

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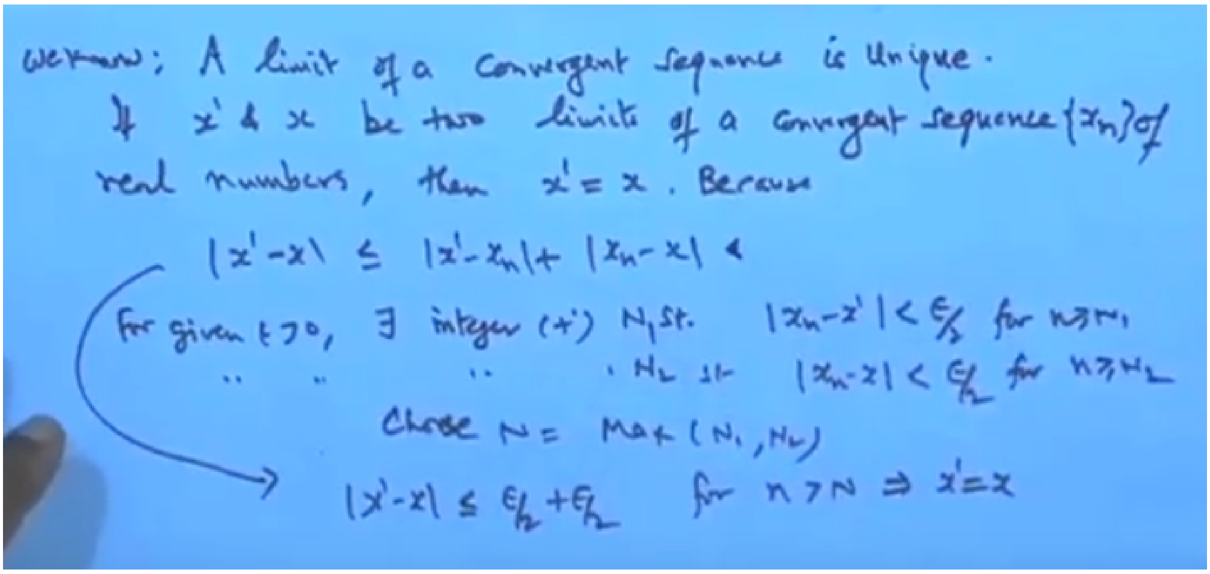

a limit of a convergent sequence is unique, that is if there are two limits if suppose if X Dash and X be the two limits of a convergent sequence, xn of real numbers, then X dash must be equal to X the reason is if we start with X dash minus X, then this can be written as X dash minus xn plus xn minus X. now since xn converges to X dash it is a limit so by definition of the limit the for a given epsilon greater than 0, for given epsilon greater than 0, there exists an integer positive integer capital N1 such that mod of xn minus X dash is less than epsilon by 2 say, for all N greater than equal to N 1. similarly for the same epsilon greater than 0 there exists the positive integer say n2 such that mod of xn minus X remains less than epsilon by 2 for all N greater than equal to n 2. so if we choose capital N to be the maximum of N 1 and N 2 then this result is also true for n greater than capital N, this is also true for greater than n capital n, therefore this thing can be made less than this thing can be made less than epsilon Y 2, plus epsilon Y 2,   for n greater than n, so this shows that X dash must be equal to X. that is a limit of the convergent sequence is always unique, so it means if a sequence which has a different limits it cannot be a convergent sequence. so this is the one of the criteria you can say that if a given sequence is there find out the limits if along various paths it has different limits then the sequence will not be a convergent sequence, then we call it the such a sequence of course a diverging. so you can which you all know, the another thing if a sequence is a convergent sequence then all of its sub sequences will also have the same limit.

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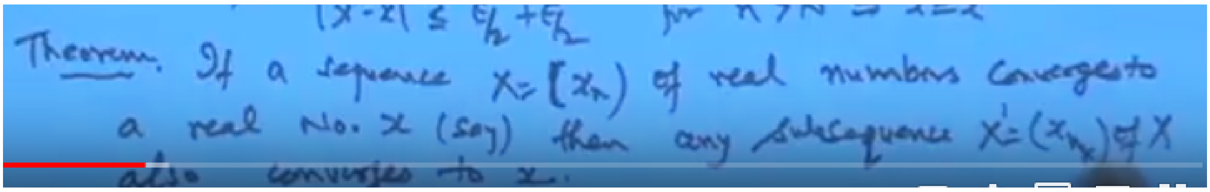

so that is very interesting result, which is not true in case of the divergent sequence, divergent sequence the sub sequences have different limits. ok, so if a sequence if a sequence X which is say xn of real numbers of real numbers converges to, converges to a real number, a real number say X, X then any subsequence any subsequence X dash which is a xnk of X also converges to X. so if xn is a convergent sequence of real number then all of its sub sequences will also converge to X sub sequences we mean we have discussed there that if xn is a convergence xn is any sequence and if we identify this integers n 1 into n n such that which are increasing or decreasing order then this sequence xnk it means increasing n 1 is less than into the 3 then this sequence will be a sub sequence. so this will now let's see proof, proof is very simple.

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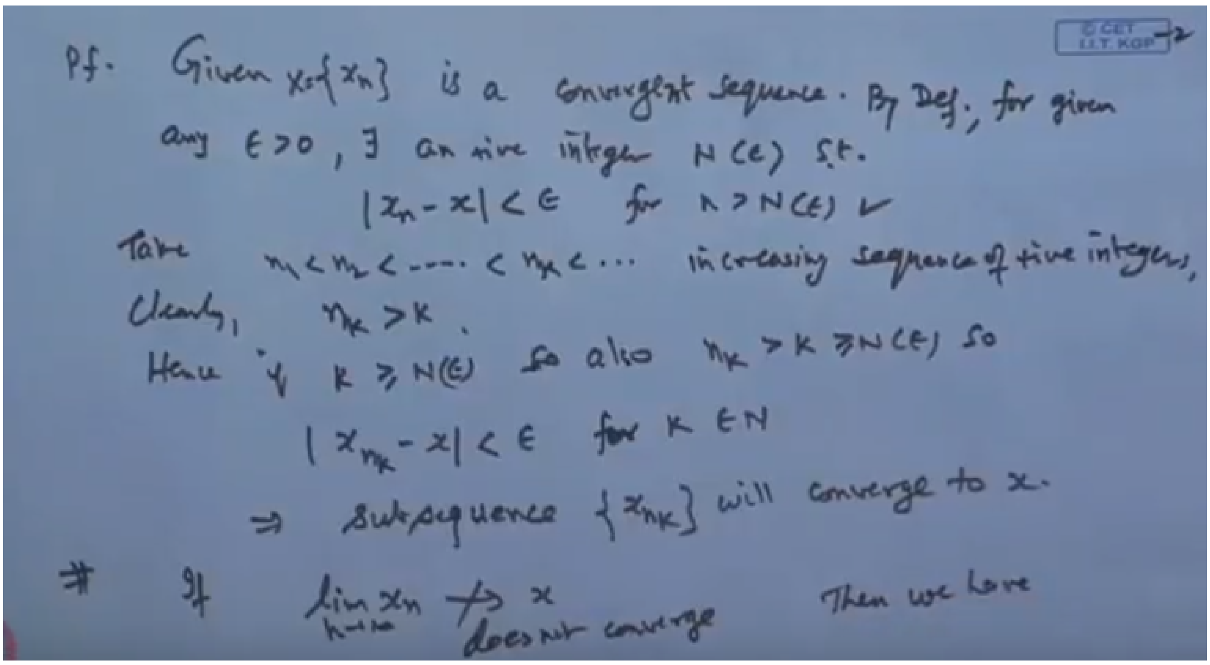

what is given is the sequence is a convergent sequence so it is given, given the sequence xn which is don't advise ax is a convergent sequence, so for the given epsilon by definition, so by definition for given, for given any epsilon greater than 0 a positive number there will exist there exist up and in positive integer, say capital N which depends on say epsilon such that, such that mod of xn minus X is less than say epsilon for all N greater than equal to n depends sort of signer, okay. this is what we did now we are wanted thus any sequence subsequent must converge so let us pick up the Thames, take n 1 less than 2, less than n 3, less than n K and like this, this is the sequence of increasing sequence of natural number, increasing sequence of positive integers positive integers, okay. positive integer now obviously, clearly this NK will be greater than K for any because K is 1 then you can identify N 1 which is greater than 1 then k is 2 you can identify another integer positive in to such that n1 is less than n2 which is greater than 2 n like this so it is very easy to verify that they say follow, ok.

Now if this K is greater than this number here capital K and if this number is capital N, so what we do hence if this K which we have taken is greater than equal to capital N, which depends on epsilon then obviously NK will also greater than K so for all N K's get us so NK will also be greater than K which is greater than equal to n, depends on epsilon, therefore for this these nK, this result is true, so from so we get mod of xn K minus X is less than epsilon for all n K, for all K belongs to the natural number n, for all K belongs to n. this is true. But n 1, n 2, n satisfy this conditions also which we achieve therefore, the sequence subsequence xnk will converge to X, so this implies that the subsequence xnk, this sub sequence will converge to X. And this is an arbitrary sequence we are choosing. so any subsequence of X will definitely converge. so if a sequence xn is a convergent sequence then all of its sub sequences will converge. so this is 1, now let us suppose we have a converse part of the criteria where the convergent and the limit fails, then it will lead to a sequence which is a diverging sequence. ok, so let us find out what we the corresponding, your equivalent condition when the sequence does not converge. ok, so let's say that criteria or the divergence of this suit.
if the sequence if sequence xn or limit of xn when n tends to infinity does not converge to X, does not converge to X all the limit fails, then what will be the then we have then we have the following equivalent criteria, then we have.

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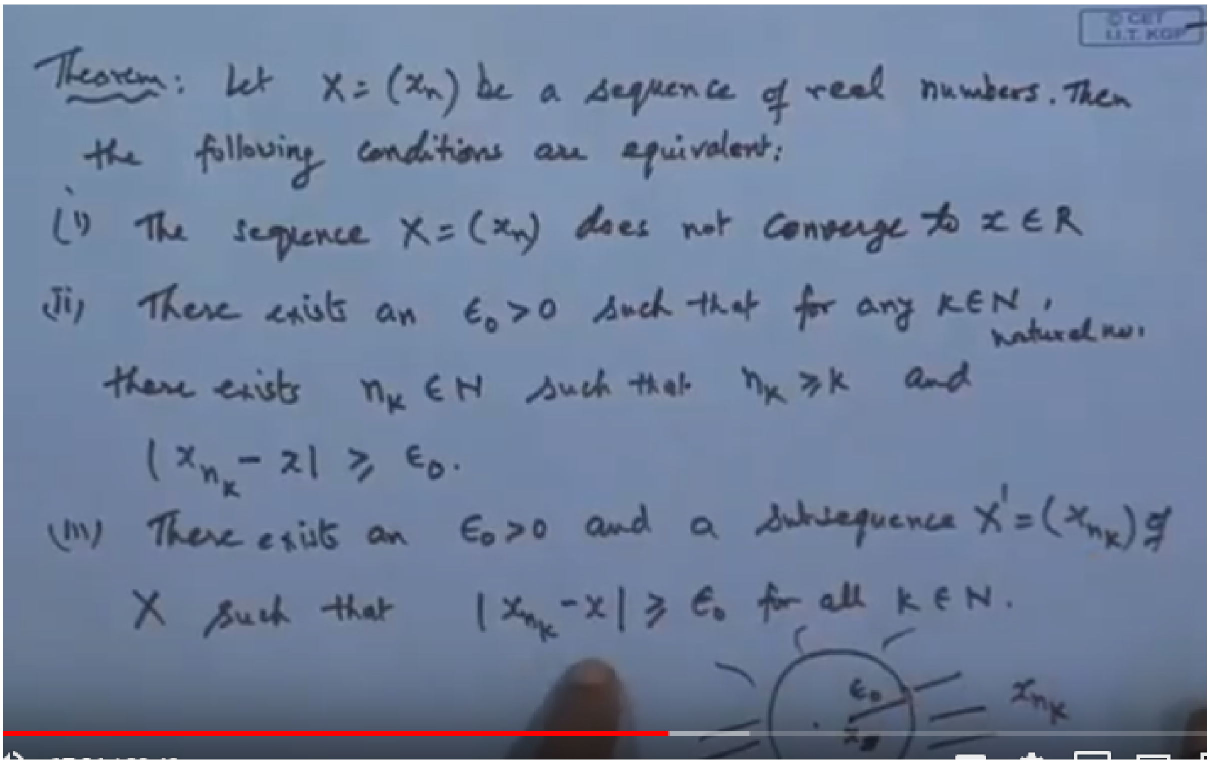

This theorem will help you in identifying the diverging sequence the criteria for dive diverging sequence. so let X which is xn be a sequence of real number, real numbers then the following conditions, then the following conditions are equivalent. the first condition is the sequence X which is denoted by X xn, does not converge to, converge to X in real, means does not converge to it number X then. second criteria is there exist n, there exist an positive epsilon, epsilon naught greater than 0, such that for any K belongs to n and is the natural number. this is the set of natural number n for any K belongs to n, there exist nK, a positive integer belonging to n such that, such that nK is greater than equal to K and mod of X and K minus X, is greater than equal to epsilon naught. And third condition is there exists an epsilon naught greater than 0 and a sub sequence and the subsequence X dash say X of X and K of X of X such that X and K minus X, is greater than equal to epsilon naught, for all K belongs to N, okay. so what this shows is the proof we will see, xn is a sequence of real number, then the following conditions are equivalent this sequence xn does not converge to X then it is equal into this equivalent to this. basically the difference between second and third race is says that, n is, which does not follow the increasing order, may or may not follow the increasing order that just there exists some NK where this is true but here there must be a sequence N 1 n 2 n m which should be a decreasing order and 1 less than n 2 less than anything and corresponding to this positive integer a sequence xn K can be obtained. so that it satisfy this condition. Now what is this condition? when we say the limit of this xn is x naught, or X it means if I draw a neighbourhood around the point a this with a suitable radius say epsilon naught, with a suitable radius epsilon or naught then the all the terms of the sequence must fall within this if the limit exists, but if the limit does not exist, it means the all the terms of the sequence after certain stage, will fall somewhere outside of this neighbourhood, because this also shows that if X is if you draw a neighbourhood around the point X, with a radius epsilon naught then all the terms of the sequence after this integer say N 1 is a n 1 X n 2 etcetera will not satisfy this will not fall within this season it will fall outside of it. so this shows the criteria for the this shows the sequence xn does not have a limit, okay. so we will go to the proof that how these three conditions are equivalent means we can if the sequence does not converge we can immediately say this condition horse that say there exists a epsilon or not and a subsequence such that the excellent most of the terms of infinitely many terms lies outside of the epsilon neighbourhood epsilon naught neighbourhood of X proof of this, okay.

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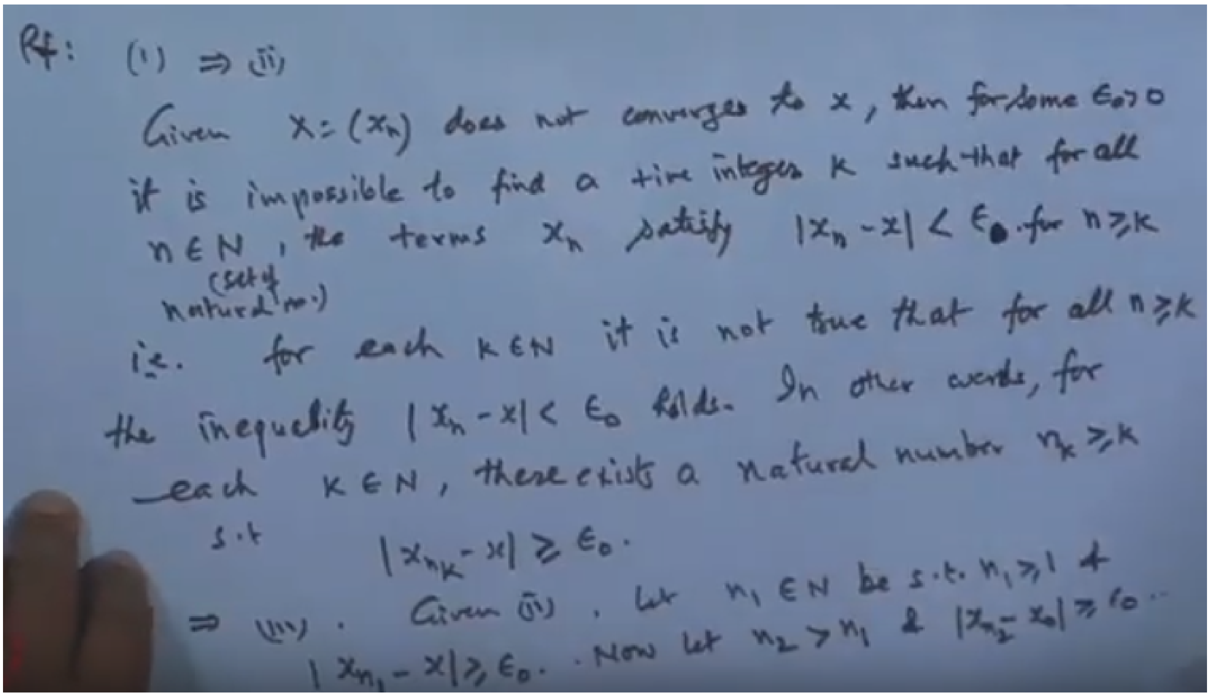

So first is very first implies to one implies two, let us see. what is the first sequence does not converge, then we have to find out this thing, ok. So if the sequence does not converge given the sequence xn does not converge, does not converge to X, it means what? it means for the convergence for any epsilon greater than 0, there will be n, such that all N greater than a difference in length but if does not converge so we can identify a epsilon naught and corresponding to the epsilon naught we can identify the sequence, well it violates the condition of the neighbourhood, that so then for some epsilon naught greater than 0 for some epsilon R naught greater than 0 it is impossible, it is impossible to find a natural number, find a positive integer or find a positive integer, say K such that for all n belongs to a natural number capital N, this is a set of natural number. set of natural number belongs to capital N, such that for all K, K such that for all N greater than here the term, the term xn, satisfy the condition satisfy mod of xn minus X is less than F so naught. so what he says is if support does not converge it means for at least for some epsilon naught, it is impossible to find an integer K, such that when you chose all n for all N greater than equal to K. this condition will

not satisfy it, it is impossible to satisfy this condition, that is to say, that is this means that for each K belongs to for each K belongs to n it is not true that for all n for all N greater than equal to K, for all N greater than equal to K the inequality, iniquity mod xn minus X less than epsilon naught hold for this holds, that is the same thing for each K it is not what do you mean? it means we can identify a sequence we can identify a natural number n K such that which n K is greater than K and this condition violates, that is in other words in other words, we can say that for each K belongs to capital N there exists a natural number, natural number or positive integer, you can say n K which is greater than or equal to K such that mod of xn K minus X is greater than or equal to epsilon naught. that is what you because if does not converge means this condition will satisfy now it is impossible to find out n K which will follow this condition, so there must be a in K some integer can be obtained which will violate this condition so that is what you get so 1 implies 2, now 2 implies 3, it follows immediately second condition said that they will exist and epsilon R naught such that for any K there exist n K such that n K is greater than equal to K visit now you can find out N 1 which is greater than 1 and satisfying this condition then n 2 which is greater than N 1 and greater than this will satisfy. so using this given 2, okay.
Then let n1 is a integer is a natural number be such that N 1 is greater than equal to 1 and it violates is satisfied this xn 1 minus X is greater than equal to epsilon naught then you choose n 2 which is greater than n 1 and again it satisfies that condition is greater than equal to Epsilon.

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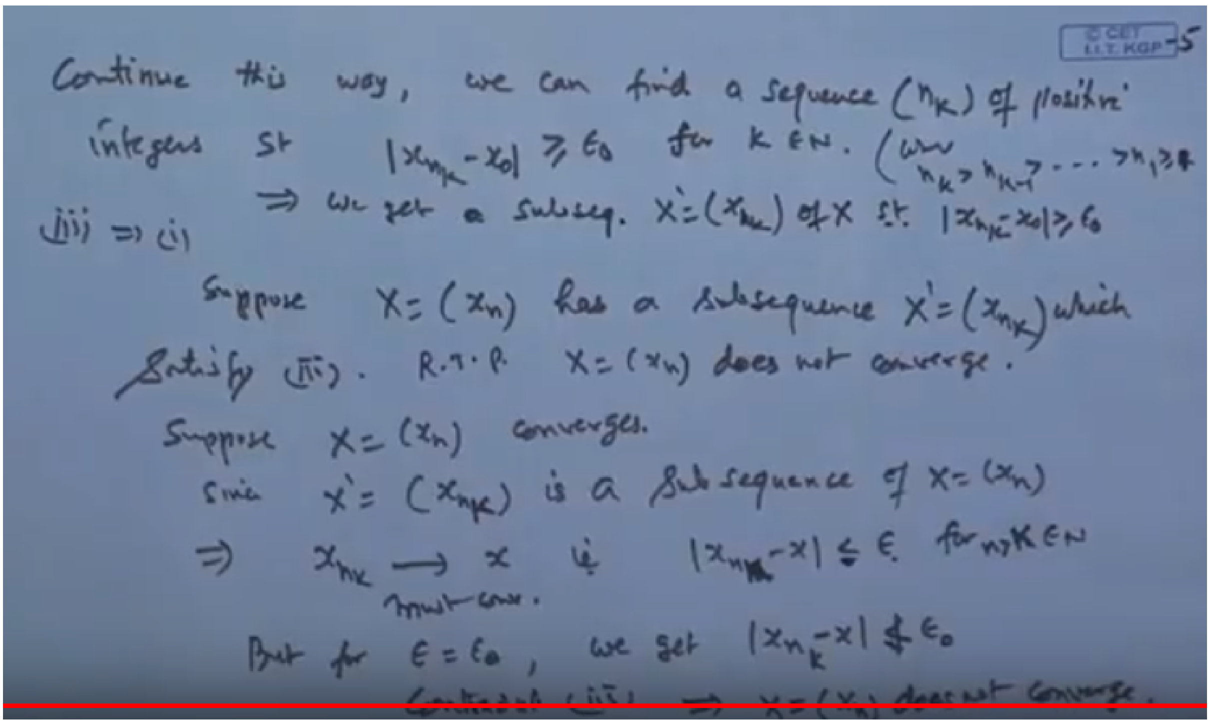

so continue this way, so continue once you continue you will get a sequence NK so continue, this continue positive integers or positive integer such that, such that x NK minus X naught, is greater than equal to epsilon naught, for all k belongs to N. Here what is NK? we are this condition is satisfied where n K is greater than NK minus 1 and so on is greater than N 1 which is greater than equal to k 1, ok. like this, so we can identify the sequence clear! And this shows what this shows that the condition third is valid equallent okay. So third now third implies first given this condition holds, that is there will exists an epsilon naught and a subsequence x dash n, so the and a subsequent same X dash, so we can say subsequence xn so they right this way yeah. This, was the third condition. there exists X naught and a subsequence X dash n such that this condition holds so here this is given we want the xn does not converge, okay. so suppose the sequence X here you can say in this the subsequence we can say that there exists a week of 10 we get a subsequence X dash which is xn k, xn k of x such that this condition violates that's all okay. so this condition is suppose a sequence xn converges, suppose the sequence X which is xn has a subsequence, has a subsequence X dash n, xn K satisfies, which satisfies we satisfy condition 3, okay. So once it is there then we have to show required to so it required to prove the sequence X which is xn does not converge. This we wanted to solve suppose it is not true suppose the sequence X which is xn converges, okay. Suppose it converges now X dash, since X dash which is xn k it is subsequence sub sequence of X and this sequence we have assumed to be convergent so all of its subsequent must be convergent, okay. by the just previous show, so this shows this implies the sequence xnk this sequence must converge to X, must converge to X or converges to X. it means that that is mod of xn K minus X is less than equal to epsilon for all N or for all K belongs to n, if it is convergent or for all n you just say xn minus F for all N greater than equal to capital K belongs to okay. but that will violate this condition because the condition set x NK minus X naught is greater than epsilon naught so for this particular epsilon equal to epsilon naught, but for epsilon equal to epsilon naught, we get xn minus X is not equal to less than epsilon naught, it is greater than or equal to Epsilon. so this is that violates that condition, therefore this is a contradiction of our given condition three so contradicts condition three. because if it is supposed to assume convergent then it must satisfy this condition, that X minus xn K for all K belongs to you this must satisfy condition, but for this particular epsilon naught this condition is violated because this is this condition is not satisfied because it is greater than epsilon. Therefore contradicts the condition 3 and contradiction is because our wrong assumption so you can converge. Therefore, this implies the sequence X which is xn does not converge and that proves that is not, OK.