Modell 4

Lecture – 22

Some Results on Limits and Bolzano – Weierstrass Theorem

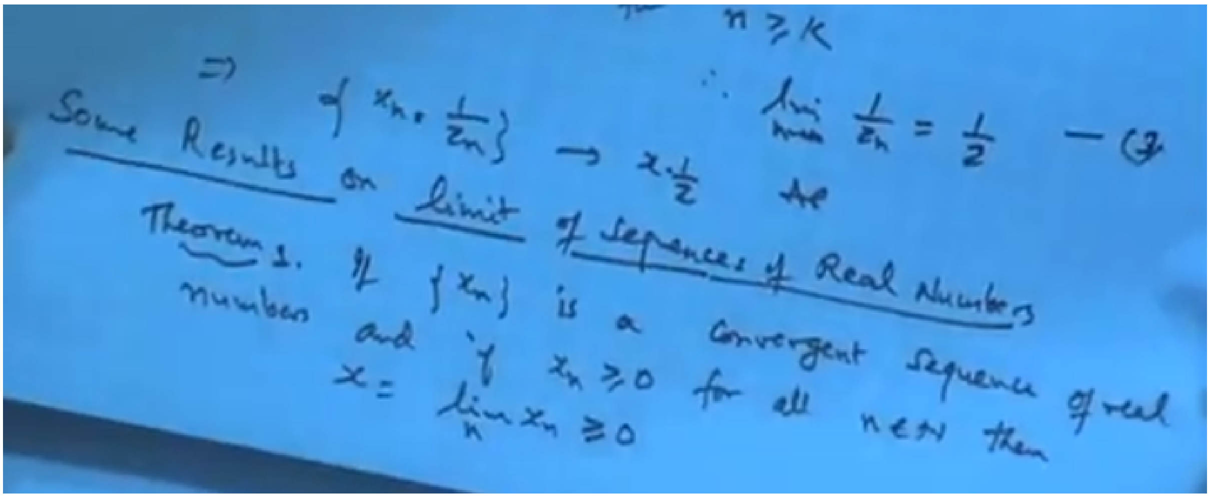
Course

on

Introductory Course in Real Analysis

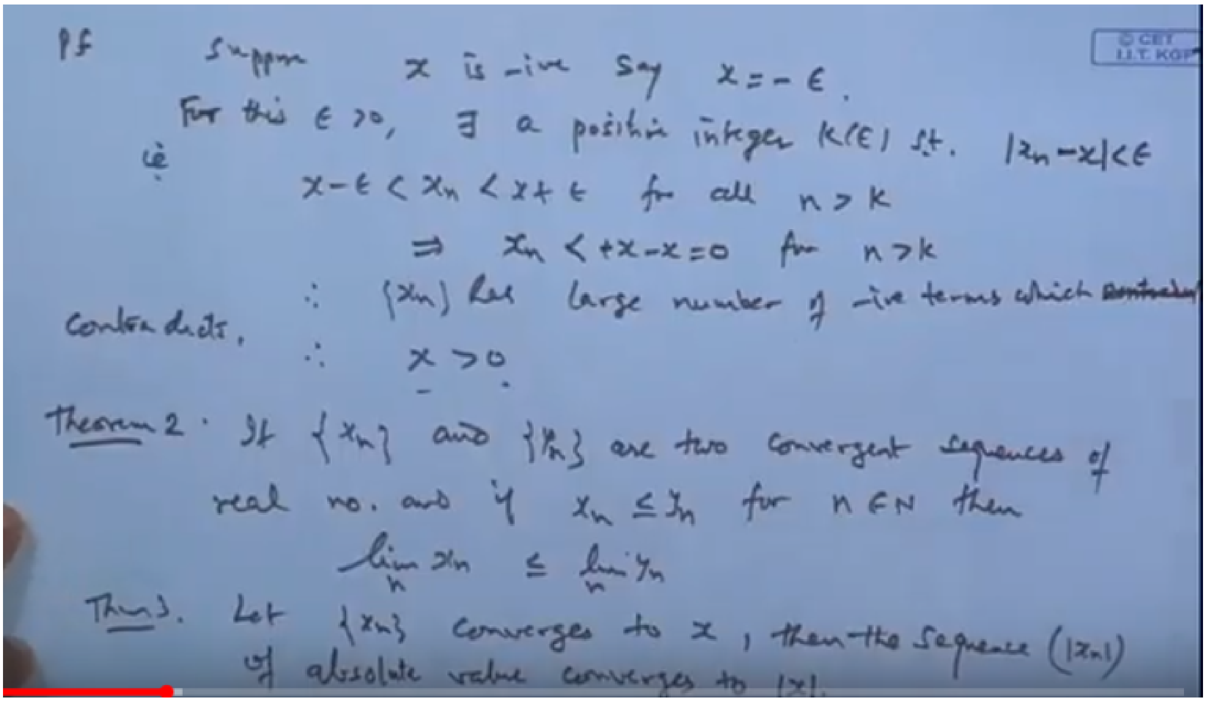
In the last lecture, we have discussed the limits concept. So here in this lecture, we will prove few results, related to the limits, of the sequences and functions, and then finally we will discuss, 'The Bolsano Weierstrass Theorem'.

(Refer Slide Time: 00:36)



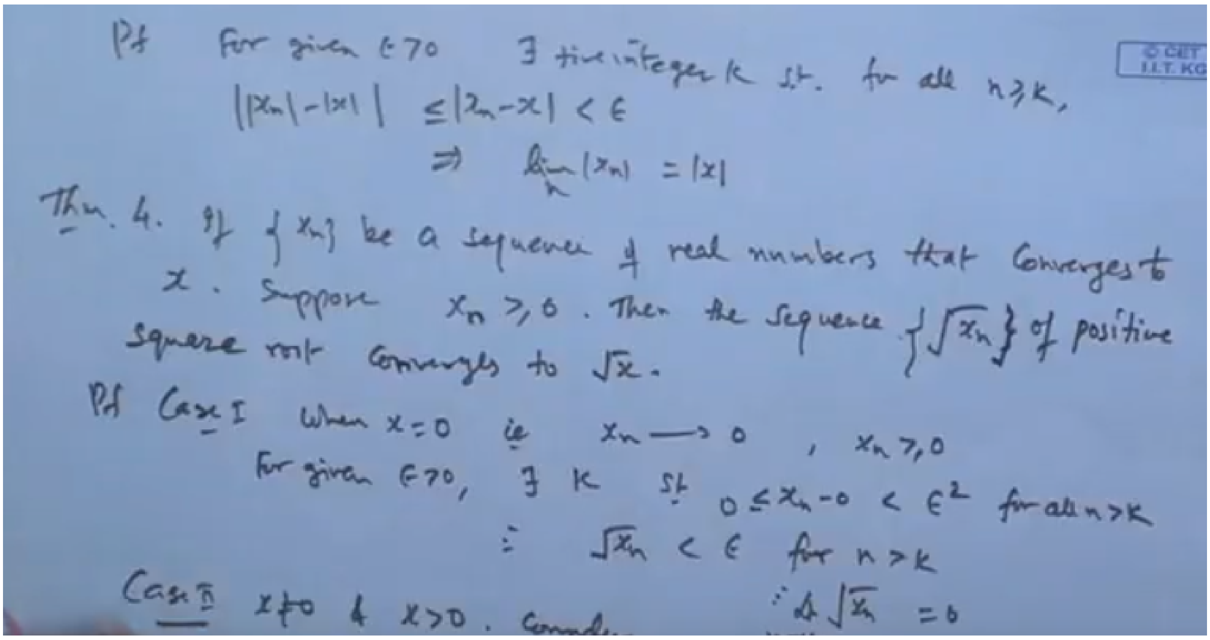
Some results on limits, of sequences, of real numbers, few more results on this. The first result which we call is a theorem. The theorem one says; if xn is a convergent sequence, is a convergent sequence, of real numbers, of real numbers, and if, all the terms of the sequence xn, are positive, non-negative, greater than equal to 0, for all n, belongs to capital N. If all the terms of the sequence are non-negative, then the limit cannot be negative. Then the limit of this X, which is the limit of xn, over n, will also be greater than or equal to 0. It cannot be a negative limit. The limit of the sequence of non negative term will always be non-negative, the proof of this reason.

(Refer Slide Time: 02:00)



Let us prove by contradiction, suppose the limit is negative and then apply the condition for this. So let us suppose, suppose, limit X, X is negative, say, X is equal to minus epsilon. Okay? Then for this epsilon for this epsilon, for this epsilon, which is greater than 0, there exist, for this epsilon, there exist a positive integer, say K, depends on epsilon, such that, such that, mode of xn minus xn, because this sequence is a convergent sequence, so basically the xn will lie, between the two term xn minus x, union less than, epsilon. That is the xn will lie between these two terms. Less than X plus epsilon and greater than X minus epsilon, for all N, greater than K. Is it not? Now epsilon is given, is already chosen minus K, minus X,? So from here this shows that xn is less than, minus X, plus X, minus X, that is 0 and this is true for all N, greater than K. It means the large number of the terms xn, are negative. This source the sequence xn, has large number of terms, as large number of negative terms, which contradicts, which is not, which contradicts, contradicts. Why contradicts?
Because, xn already are positive. Therefore our assumption is wrong. So X must be positive, that is the answer. The another results; 2 says, if xn and yn are two Convergent sequences, are two convergent sequences, of real numbers, of real numbers and if, xn is less than equal to Y n, for all n belongs to capital N, natural number, then the limit will also follow the same inequality. Limit of this is also less than equal to limit of yn. And similarly for the sandwich theorem, which we have shown earlier, that if xn and YN are the two sequences and Zn is a sequence with lies between xn and yN and if the limit of xn and yN converges to the same limit, then Zn will also. So that is the is Sandwich theorem, which also we have shown earlier. Okay? So this is what. Now next is result is divergence, sequence we had discussed. Now theorem third. Let the sequence xn, converges to X, X. Then the sequence, mode of xn, this sequence of absolute values, absolute values, converges to mode X. That is if the limit of the sequence is X, then limit of the mode xn will also be, will be, mode of X.

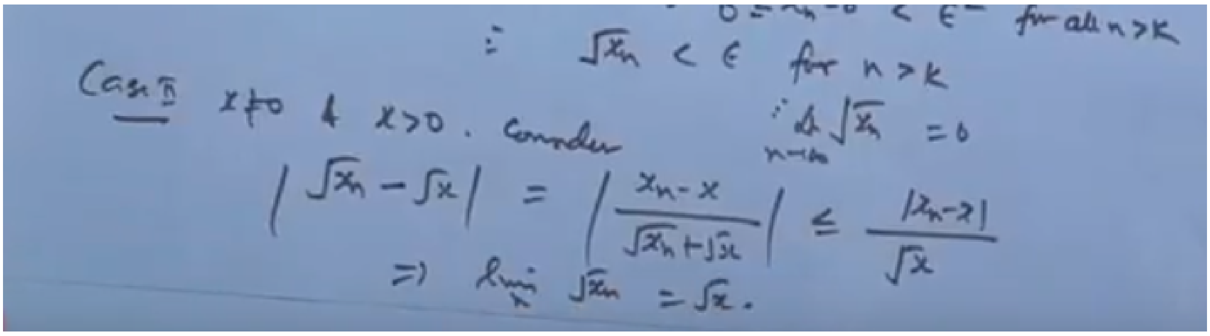
(Refer Slide Time: 06:28)



The proof is very simple. This follows from this inequality. xn minus X, is greater than equal to, mode xn, minus mode X, triangular inequality. We know this result. So if xn converges to X, means, for a given epsilon greater than 0, there exists and positive integer, positive integer, K, such that for all

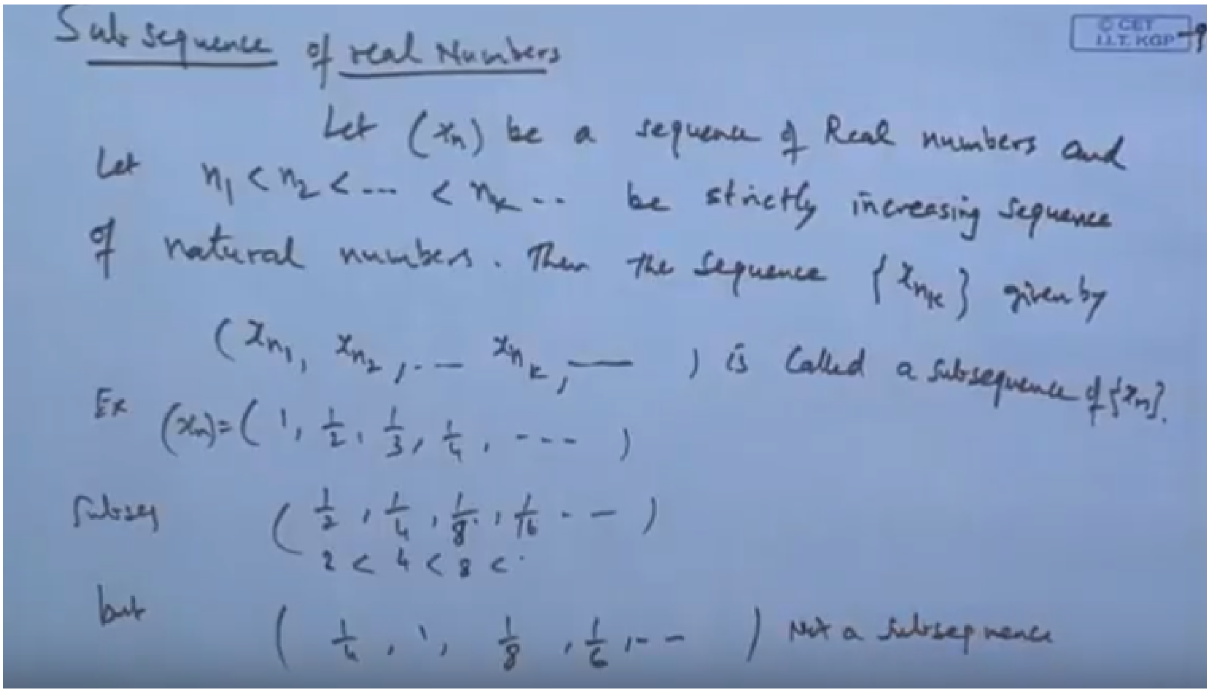
N, greater than equal to K, we have this part is less than epsilon. So this part less than epsilon means, is this part. So this shows the limit of mode xn over N, is mode X. So that is what, is so. Okay?
Then if xn be a, if xn be a sequence of real numbers, a sequence of real numbers, that converges, that converges, to say X, to X, and suppose that xn is greater than equal to 0. Then the Sequence, under root of X n, this sequence of positive square root, of positive square root, converges, to under root X. Okay? The proof is, since xn be a sequence of real number, that converges to J and we are assuming xn, to be greater than 0? So limit point cannot be negative? So we are considering only two case, when this is X is equal to 0 and X is greater than 0. So case 1 - When X is equal to 0. So when X is equal to 0, xn sequence converging to 0. That is xn is tending to 0, where Xn's are all positive, greater than or equal to 0. Okay? They are zero. So for a given epsilon, so by Epsilon, a for given epsilon, greater than 0, there exists an k, in positive integer K, such that xn, xn, which is greater than equal to 0, will less than xn minus 0, is less than say epsilon square, for all N, greater than for all N, greater than k. Therefore under root of xn, will less than, epsilon, for all N greater than K. So the limit of the sequence, xn will, so limit of the xn, when n tends to n root of this, will go to 0.

(Refer Slide Time: 09:50)



Similarly if we take X to be nonzero, then in that case, non 0 and greater than 0, because it's greater than 0, then we consider, consider, under root Xn, minus, under root X. The next rationalize it, so multiply and divide by and n root xn plus, we get xn, minus x, over root of xn, plus root X. Okay? Now this is given under root of xn and both are positive. So this will be, this will be, less than, equal to, so if I take mode of this, mode of this, then this is less than equal to, mode xn minus X. And some, this part, under root xn plus X, is get greater than 0? So this will be greater than equal to root X. So it will be less than, equal to root X. This is positive, this positive. So this will be greater than root n, so 1 upon, this will be less than.
Therefore this, limit of this, is that same as the limit of this, under root x. But this limit is tending to 0. So this implies that limit will go to the. So this implies, limit of under root xn, over n, is under root xn, So, nothing, not much to. Okay?

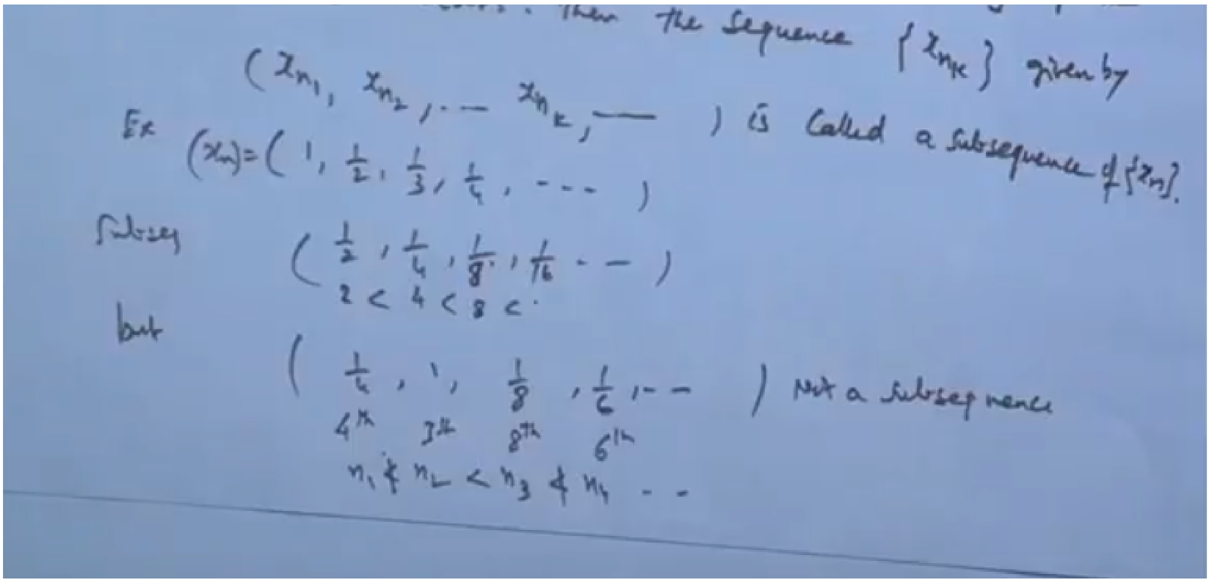
(Refer Slide Time: 11:47)



Then we Get, go for, monotonic sequence, we have discussed already? Now, yes. Sub sequence, that is and that part is Left, so let us see concept of the subsequence. Subsequence a of a real number, of real numbers, we are sequencing. Okay? So let xn be a sequence of, V. A sequence of real numbers r,al numbers. And let N 1, less than n 2 and so on, be less NK, be strictly, strictly, increasing sequence, strictly increasing sequence, of natural numbers, of natural numbers, then the sequence, then the sequence, X and K, then this sequence xnK given by, X n 1, xn 2, xn K and so on. This given by this, is called is called, 'A subsequence of X, subsequence of X N, subsequence of XN'. Okay? For example, if we take this, say, say, sequence xn is, 1, 1 by 2, 1 by 3, 1 by 4 and so on, this is the sequence X n.

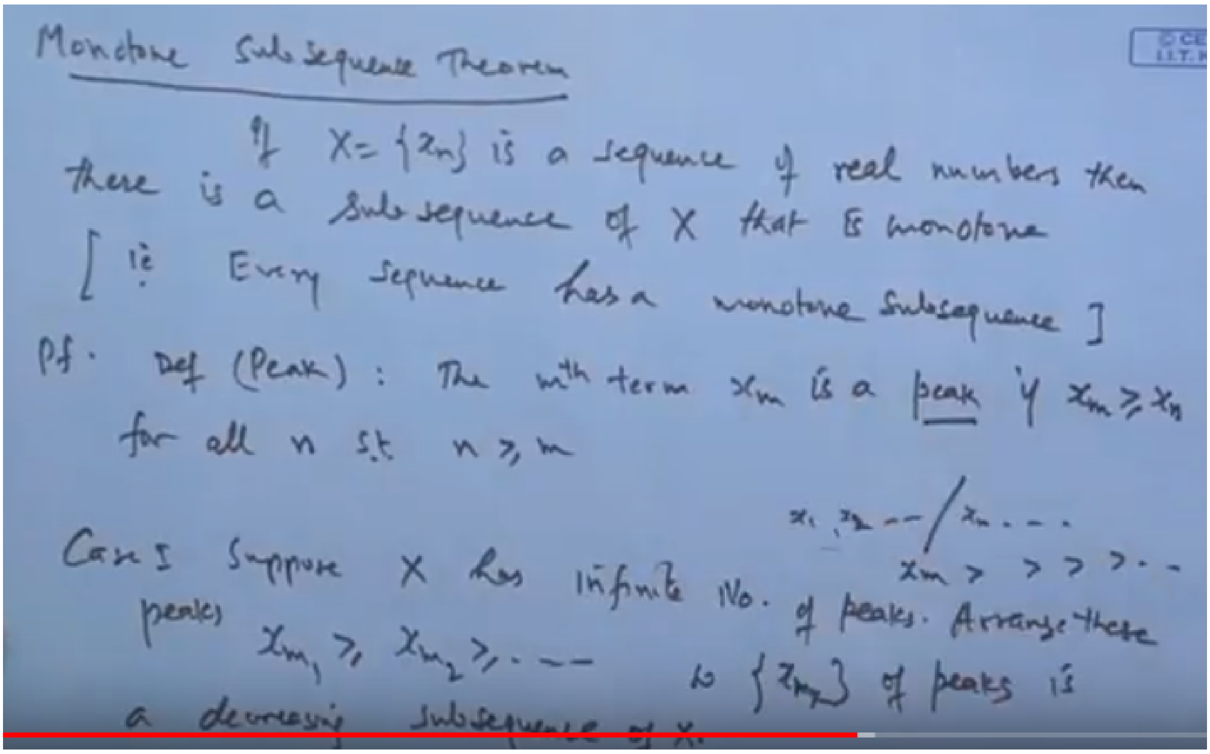
Now if we take the sequence ½, 1 by 4, 1 by 16, 1 by 8, 1 by 16 and so on, then this is a subsequence. Because this is the second place, point, of this. So this is the second term, this is the fourth term; this is the eighth term, is it not? And like this. So we are getting n 1, n 2, n, n1 is less than n2, and less than n 3 and so on, we are seeking. But if we take a sequence, like this, but if we take a sequence like this, say 1 fourth, one, then, let it be one eighth, then one sixth and so on. Then this is not a sequence, not a subsequence. Why? Why it is not a subsequence? It is a Sequence, but it is not a subsequence.

(Refer Slide Time: 14:52)



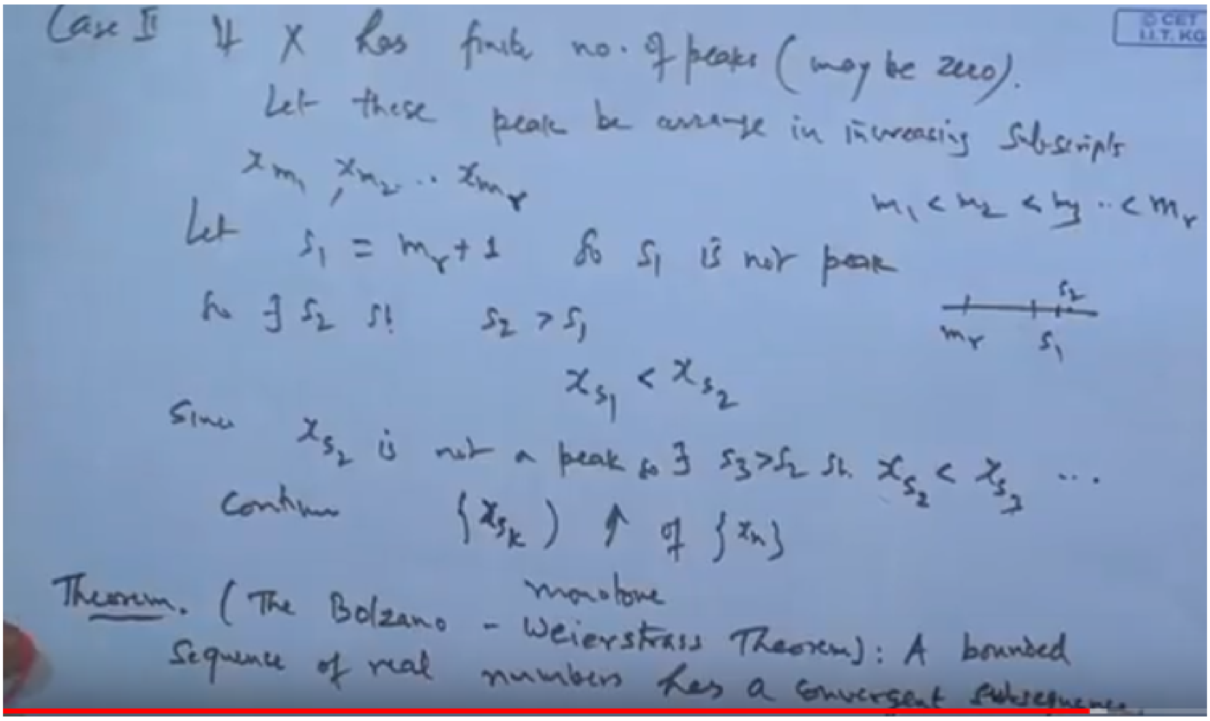
Why? because, the order is not retained. Because here n1, this is the fourth term, this is the first term, this is the first term, this is the eighth term, this is the sixth term. So there is no such order like this, N 1 less than n 2, less than, N 1, is not less than n 2, n 2 is less than n 3, but n 3 is not less than n 4, like this. So it does not satisfy, the criteria, of the subsequence, like this Richard, is it okay? So that will be. Now, this is very interesting results here. The results say, Okay.

(Refer Slide Time: 15:40)



Monotone subsequence theorem, monotone subsequence theorem, The theorem says, if X equal to xn, is a sequence of real numbers, is a sequence of real numbers, then they are there is a subsequence, there is a subsequence of X, that is monotone. So this is very interesting result, that, every sequence, has a monotone subsequence, every sequence that is; the result is very intense, that every sequence, every sequence has a monotone subsequence, subsequence. So that is the proof.
Let us see, suppose we have a, first be defined term, say X 1, X 2, Xn, let us define this term, which we call it, the the peak, peak, Okay? The M'th term, XM, is a peak, the M'th term, M'th term, XM, in the sequence xn, is a peak, is a peak, if X m, XM, is greater than equal to X n, for all N, for all n, such that, n is greater than equal to m, means like this. If suppose we are having a sequence, X 1, X 2, xn and so on, and if we identify some term X M here. Such that, all after this, all the terms of the sequence are of decreasing nature, are of decreasing nature. And above it is bounded by XM. This term X 1, X 2, xn, may behave abruptly, may be X 1 is greater than X 2, X 2 may be less than X 3 and so on, we do not care. But, if we find out some M, such that X M is greater than after a certain stage, all the terms of the sequence, then we say the peak, ma, XM is the peak of the sequence xn. Okay? So what we should do here, that every sequence, using this concept we will prove that every sequence, will have a convergent, will have a monotone subsequence.
So let us see the two cases, When the sequence has, an infinite number of peaks and the case when the sequence have only finite number of them. So suppose, xn, sequence X, heads infinite number of Peaks, infinite number of Peaks, let them arrange this thing. Arrange these Peaks, arrange these Peaks. Suppose the peak is X m1, okay? We are, then X m1, then X m1 is greater than equal to XM2, - greater than and so on. Because m1, m2, m3, these are the terms. M1 is less than M 2, less than m3. But the peak X, m1 is the peak. After this, all the terms of the sequence are less than. Then XM 2 is another term, Such that after this, all the terms, are less than this and like this. So arrange this. So this sequence will have a decreasing. So the sequence of the peaks, this sequence of peaks, Peaks, is a decreasing subsequence, is a decreasing subsequence, decreasing subsequence, of X. So in case of the infinite, what we, we identify the points, first peak, which is such that after this, all the terms of the sequence, are less than this. Identify x1, then after certain stage, again you will find out some term, for which the condition that X M is greater than xn, satisfied for all N greater than m,m. and like this. So arrange this thing, so you get a monotonic decreasing sequence, for them.

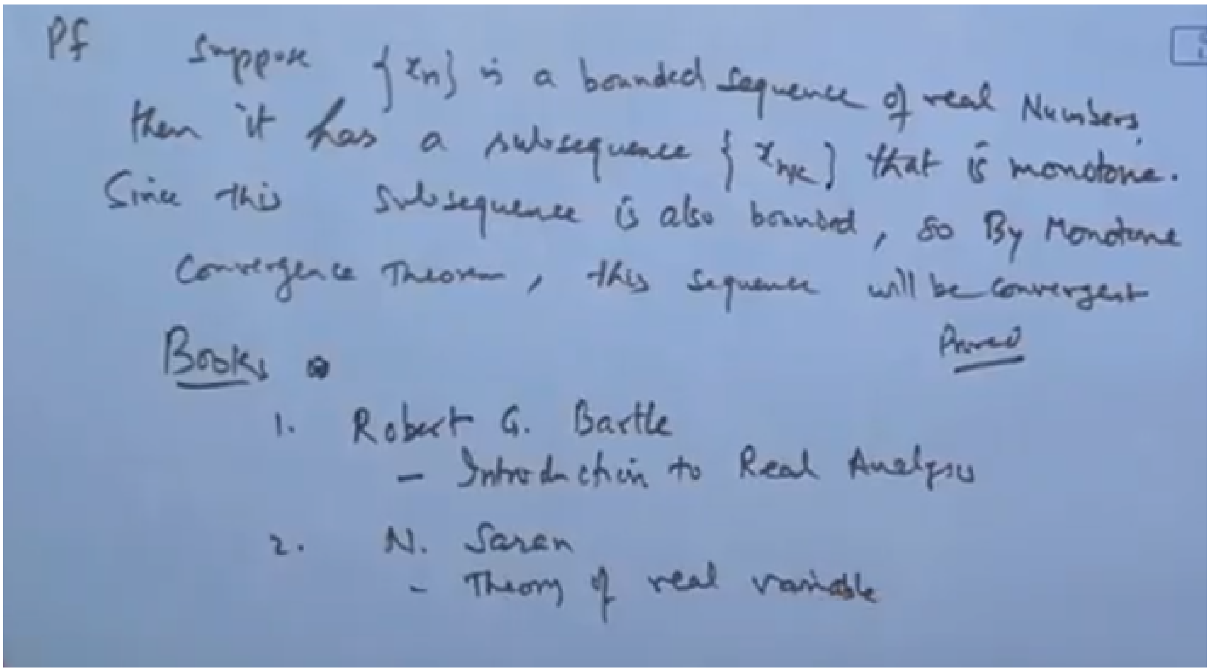
(Refer Slide Time: 21:05)



Now, if X is a finite number of, if X has a finite number, finite number of Peaks, may be 0 also, may be 0, no peak, this is also possible. So if the final number of 0, let this Peaks be, in increasing, this will list it by increasing order. Okay, so let this peak be arranged, in increasing order. Let us equate in increasing, increasing subscripts, subscripts. That is, if M 1 is less than M 2, that XM 1, XM 2, these are the peaks. XM, say all. And arrange in the increasing circuit. That is M 1 is less than M 2, less than M 3 and this is M r, arranging this.

Now let us pick up this term, let pick up the sum. Let s 1, is the term, which is, after this peak or let m r peak, in this first index, become you own this. Then, this s 1 cannot be a peak? Because this is the peak and this is the last peak. So we cannot take any number, which is greater than this, can be a peak. So s 1 is not a peak. Okay? This X is a finite, then s1 we have. Okay. Once s1 is not a peak, it means, there will be some point, here is M r. So s1 is this point something. Now s1 is not a peak, it means there may be a some number s 2, which is, greater than s 1. Then only it is, a s2 we can identify. So there exist in s 2, such that s 2 is greater than s 1, because s 1 is not a peak. So once it is greater than. Then the corresponding sequence X s 1, obviously this peak, is less than X s 2. This term will be less than this. Okay? Corresponding. Again, since X, 3 2, is not a peak, is not a peak? So we can. There exists, so there exists, another term S 3 greater than s 2, such that, the term of the sequence X s 2, is less than X s 3. Because this is not a peak, yes this is not a peak. So a term will be there, which is greater than this. And continue this. So once you continue, then we get a sequence X, X, s, K, a sequence, increasing sequence, of X. Machine monotone, monotone increasing sequence. So this holds that, every sequence will have a monotone subsequence, it has a subsequence of X, that is monotone. Following these two, we have an important result, which is known as, ‘The Bolsano Weierstrass Theorem’, the Bolsano Weierstrass theorem. What this theorem says, a bounded sequence, a bounded sequence, a bounded sequence of real numbers, has a convergent subsequence, has a convergent subsequence, has a convergent subsequence. So that is very important result, because it shows, that every bounded sequence of real numbers. May not be convergence, like minus 1 to the power, is not a convergence. But it has a bounded convergence subsequence, at least one convergence, bounded convergent subsequence, will be there.

(Refer Slide Time: 25:50)



The proof is, based on the previous result. Suppose xn is a bounded sequence, is a bounded sequence, of real numbers, of bounded sequence of real number. Then it has a subsequence extended that is monotone. Now since it is a bounded sequence of real number, so it will have, then it has, suppose a bounded sequence of real number, okay. Then it has a subsequence, every sequence has a subsequence, had a subsequence, xn K, that is monotone? By the previous, result? That is monotone by the previous result. Okay? Now since subsequence is also bounded, since this subsequence, this subsequence is also bounded, so it follows from the monotone, so, so every now, one thing. This is a subsequence, which is monotone and is bounded. So by bounded, by the monotone theorem, monotone convergence theorem, monotone convergence theorem says, that if a sequence which is monotone, or bounded above or below, must be convergent. By monotone convergence theorem, this sequence will be convergent. So this proves that, a bounded sequence of real number has a convergent subsequence. That proves the result. Now this completes your first module. That is, Cantor sets, Dedekinds and Theory of Sequences. Now here the books which, I have followed, books which are used. Reference - The first book which I take as a Introductory, by Robert G Bartle, Introductory, introduction to Real Analysis. Second book, which I followed, is, by, N Saran, that is Theory of real, theory of real variables. I think that is the theory of variables or something. I will give the name exactly. Theory of real variables or theory of real functions, like that. So mainly these two books, I followed, for this section.

Thank you very much.

Thanks.