Model 4

Lecture – 21

Fundamental theorems on limits

Course

on

Introductory Course in Real Analysis

so in the last lecture we have discussed the few Theorem, the Cauchy's theorem of first time and cost time of the second. Where we have developed from results which will help in getting the limit of the frequency and

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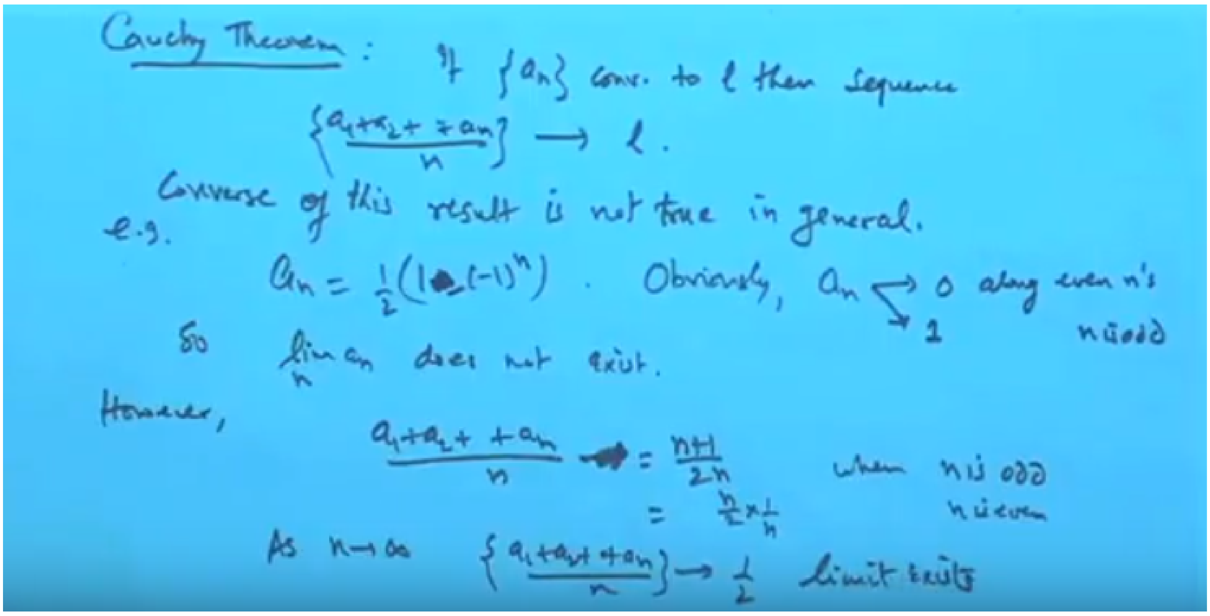

remember that Cauchy's theorem east Cauchy's theorem that is if the sequence of [a,n], the sequence in of real numbers converges to converges to l, then the sequence a1 plus a2 plus n by n, this sequence will also converse to the same point l and we have seen the proof for it. That is the sequence converges to l. will imply the convergence of the sequence of their means automatic means. The converse of this is not to converse of this result is not true in general. That is a sequence hence for which this a1 plus, a2 plus, n by n, will go to l but sequence n may not go to l. For example, if we consider the sequence an is 1 plus, minus 1 to the power n divided by 2, we take this sequence obviously the sequence an tends to when n is let us take this minus as Minus sign, let us take this is minus sign or minus take the minus sign, 1 minus 1 to the power n, so if the sequence and if you choose the sequence an is 1 minus, minus 1 to the power n then as n is even, then this term becomes positive and this minus so an will go to 0 along even ends, when n tends to infinity then along the even direction and when n is even and test me we then that will go to 0. While when n is odd, then this becomes minus plus 2 and this will go to one. So limit of the sequence does not exist. So limit of the a sentence in fear does not exist. we don't have a single bit limited, however if we consider the a1 plus a2 + an + by n, then this comes out to be, say when n is odd, so when you take n is odd then a1 becomes this is +1 so 1a3 becomes again one because a3 is this minus plus 1, so 1 plus 1 this will go and total values or when n is odd n plus 1 so it will go to the n plus 1 by 2 and then divided by n, so it will go To the to dense to the as when n is odd when n is odd you are getting this value and when n is even then it will be n by 2 into 1 by n. so in both the cases so as n tends to infinity, this sequence a1 plus a2 plus n by n. this will go to half because here is clearly half and when you divide by n then you are getting 1 plus 1 by N by 2 because n tends to infinity to go to up. So limit exists. H ok so here the Cauchy theorem one side is true that if you n converge clear then the sequence of their mean values a 1 plus a 2 plus in a sequence of the means will go to the same limited.

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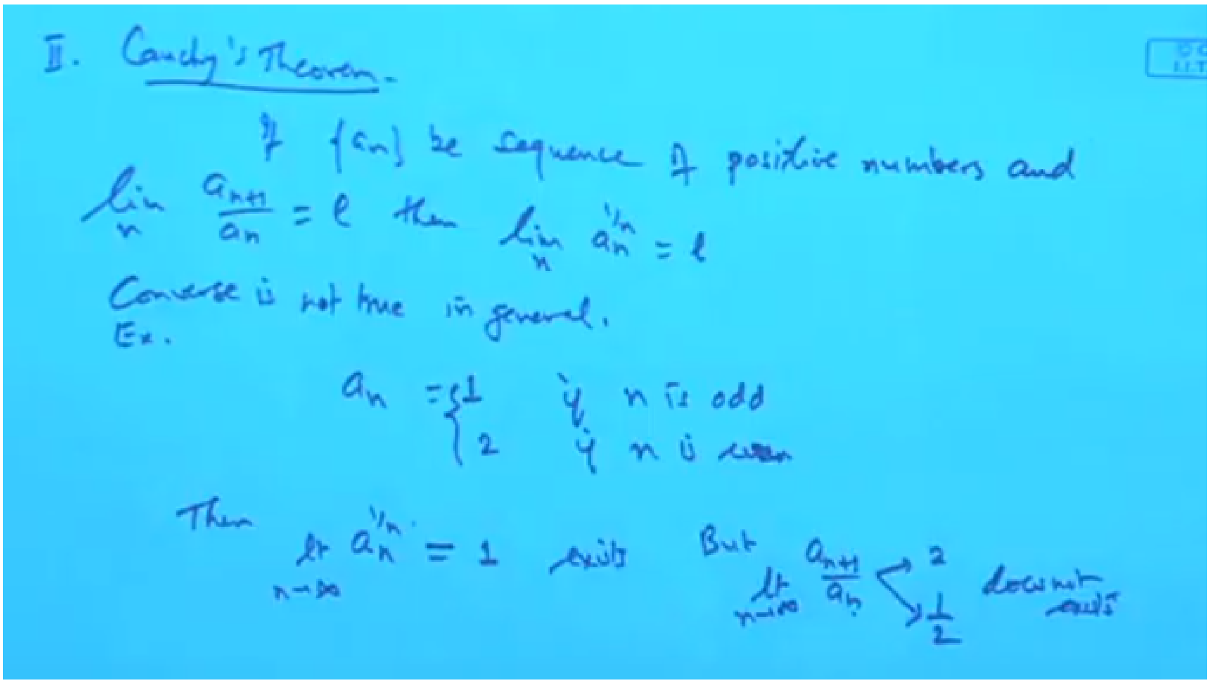

similarly when you go for the second result of the Cauchy's theorems, the second result of Cauchy say that, if a1, a2, and this is sequence if n be the sequence of the positive number, where if an sequence of positive number and limit of and limit of n plus 1 Y n as n tends to infinity exists n equal to L, then limit of the a N to the power 1 by n when n is sufficiently large will also be l. The converse is again converse is not true in general. For example suppose I take the sequence an is equal to 1 if n is odd and equal to 2, if n is even. then n to the power 1 by n, this limit is as n tends to infinity basically a constant to the power 1 by n and n is sufficiently large it will go to one, so limit will go to one exists, but the limit of this an plus 1 by an if I picked up this thing then what happen is then when n is odd the value is this will be even, so this will go to 2 by 1 and if n is here even then it is odd so this is even so it will go to half. So basically the limit will go to as n tends to infinity it the ratio limit does not exist because it keeps on jumping does not exist. And this shows the converse is not to true. Okay, so here these two examples which Are which we have discussed.

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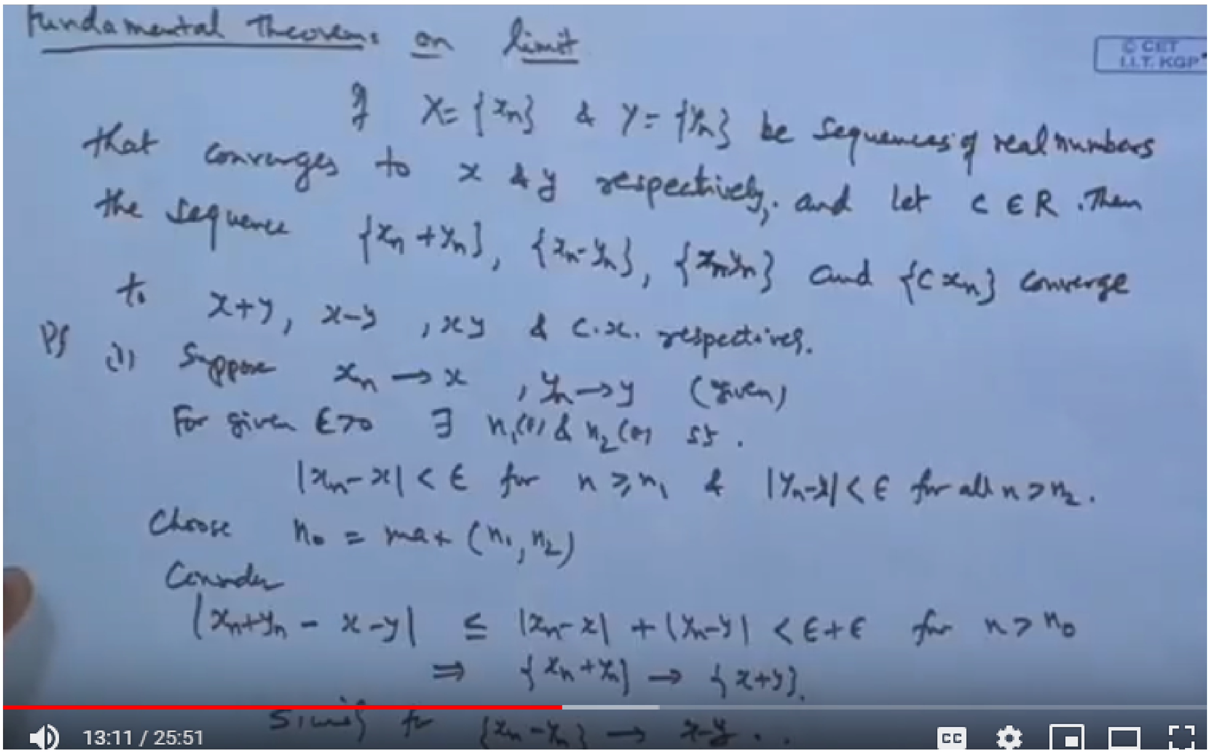

Now we will give few more results which are useful for that so let us see the first Fundamental theorems on limit. limit of the sequence of real numbers, okay so what this theorem says if xn, yn be the sequences of real numbers if X, which is a sequence xn and Y which is the sequence yn, if the xn, yn with the sequences of real numbers that converges to, to the real point x and y respectively. Respectively and let C be any arbit with constant or C belongs to R, real number. then the sequence xn plus yn, xn minus yn, xn into yn and CX C of xn will converge or converges to X plus y x minus y X into y and c into X respectively. So let us see the proof of proof is very simple we have already discussed the proof while considering the sequence of rational numbers and in terms of the cantor s and dedication scale. so let us see one of the proof of this. suppose xn and yn are convergent, given suppose xn converges to X, yn converges to y this is given, so for a given epsilon greater than 0, there exists an n1 and n2 which depends on epsilon, such that mod of xn minus X is less than epsilon, for all n greater than equal to n1 and as Y raise more mode of yn minus y is less than epsilon for all n greater than n2. now choose the n chose say n naught which is the maximum of n1 and n2 okay so there is n1 and n2, so consider mod of xn plus yn, minus X minus y, now this is less than equal to xn minus X, plus yn minus y. now when n is sufficiently large, greater than n1 then this part is less than Epsilon, when n is sufficiently large from n2, this part is less than .so when you choose n to be greater than n naught then this is less than Epsilon plus Epsilon for all n greater than n naught. Therefore the sequence xn plus yn will converge to x plus y and this proves the things. Okay now if we take other parts a second is similarly X minus similarly for xn minus yn this sequence will go to X minus y proof will be the same.

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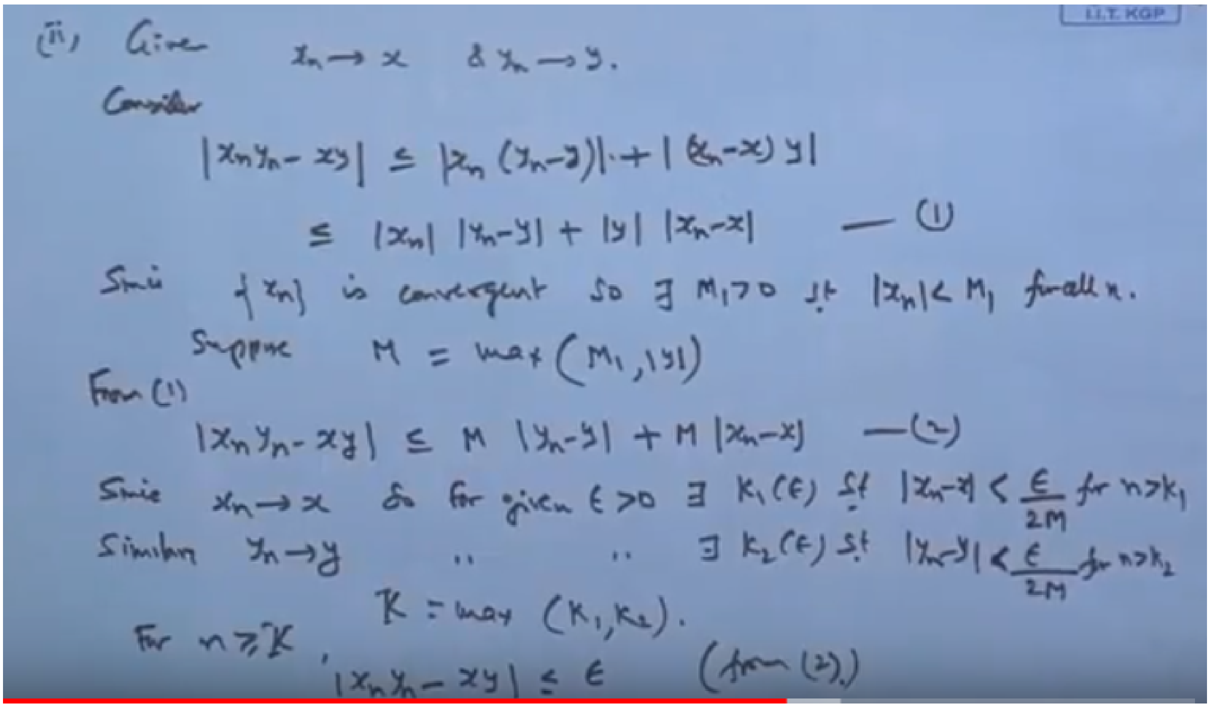

then for xn yn, let us take this you can let xn given xn converges to x and yn converges to y. Xn conversion yn converges to Y. so we wanted to show that this is less than equal. Okay, so consider xn, yn minus XY mode of this, now this will be less than equal to xn, yn minus y, mode of this plus mod of xn minus x into y. just by adding and subtracting and applying the triangle inequality, which is less than equal to mod xn mod yn minus y, plus mod y mod of xn minus X, let it be one. okay now since our x1 since the sequence xn each convergent sequence, so it is bounded so there exists M1 greater than 0, such that all the terms of the sequence is less than equal M1 for all n because it is a convergence chicken so it must be bounded for all n. let us suppose Capital M is the maximum of M1 and mod y, so we can choose this thing like this, now further this part from one we get mod of xn by n, minus xy is less than equal to they say less than M into mod yn minus y, plus M into mod xn minus x, now y n + xn both are convergent sequence so for a given epsilon since xn converges to X so for given epsilon greater than said 0 there exists some k1 depends on epsilon, such that mod of xn minus X can we made as small and equal as, so suppose I take choose the smaller number M epsilon by 2M. Similarly yn converges to Y, so for the given epsilon or get at there exists K2 which also depends on epsilon, such that yn minus y is less than epsilon y M2. Ok. now whence you have a then you choose that k as the maximum of k1 and k2, let's take a to be given here, then if x1 so for n greater than equal to K, for this number what happens use the form 2, get mod xn, yn minus xy is less than equal to. now this is less than equal to epsilon over 2m so basically this is epsilon L by 2 this is less than epsilon L by 2 but this is 2 for M and there exist K1, such that this is true for all n for all n greater than K 1 and this is true for all N greater than K 2. So if I choose the K as a maximum of k1 and K2 then both the results are true for n greater than K. So when through n greater than k then this is less than epsilon by 2 this is also less than so this is less than epsilon. From 2.

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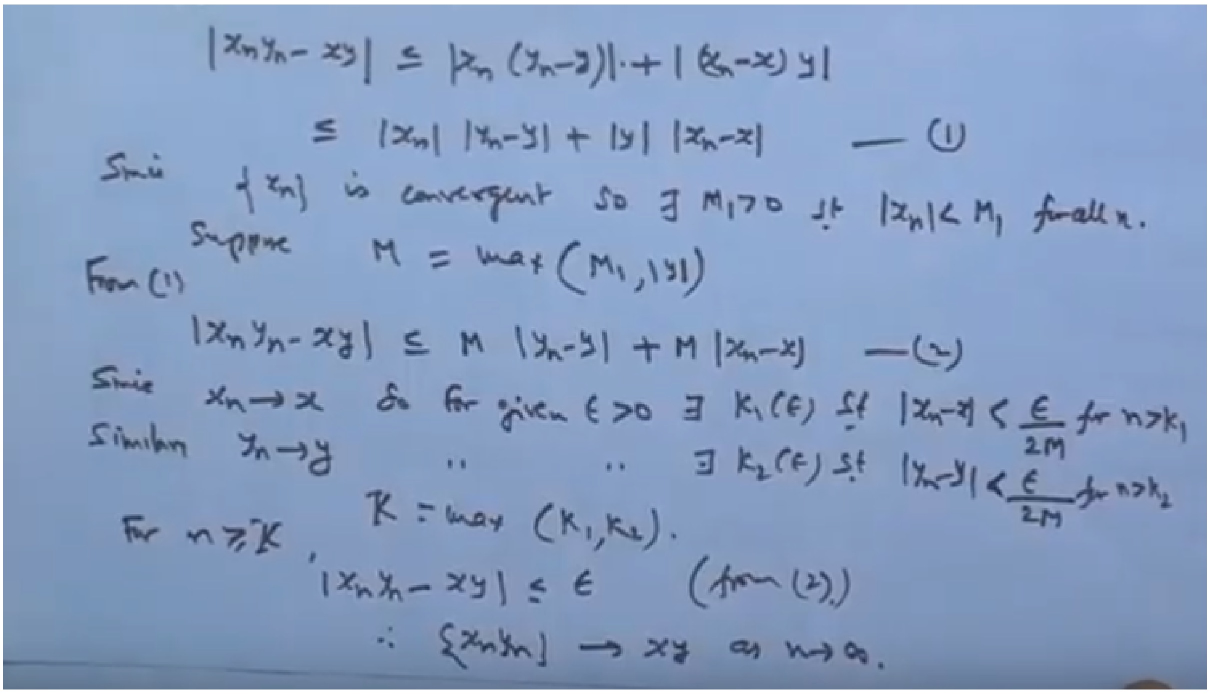

Okay so this shows that xn, yn converges to this. Okay so this implies xn, yn this sequence goes to X Y as n tends to L n tends to infinity okay so this will be okay. similarly when you take the third, c of xn, in a similar way you can do otherwise y n you can consider as the constant c. so y n a sequence efficiency and it will converges to cx, x of this. Okay now if the sequence another result if xn and yn this results let it be a).

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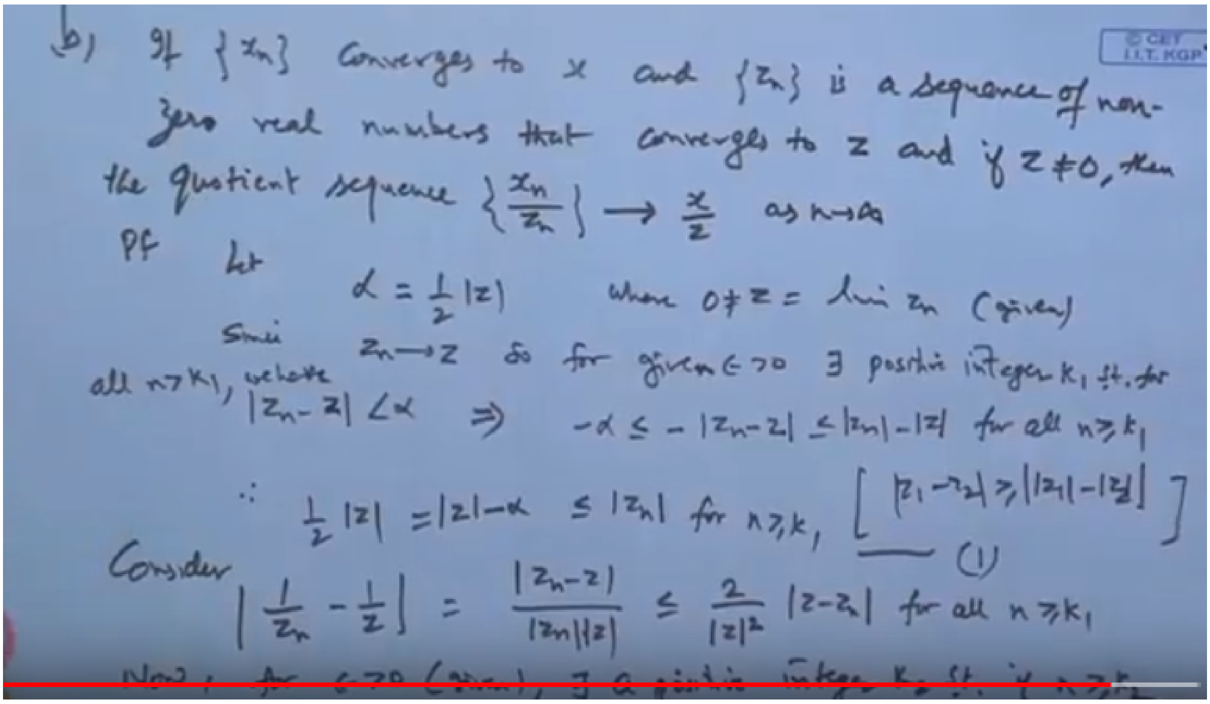

 This is the fundamental a), so let's next result is b). if sequence xn converges to X, if sequence xn converges to X, and zn is the sequence of non zero, real numbers, real numbers that converges to Z, say and if Z is not equal to 0 and if that is not then the quotient then the quotient sequence xn over zn will go to x over z as n tends to infinity. That is a limit of the sequence of what do it. Again the proof is can be followed with the epsilon Delta definition and the proof ends as it follows. Let us suppose zn sequence of number which is given okay non zero. Let us suppose let Alpha is half of mojit where z is the limit of zn, which is given nonzero. This is given, so let us pick up the Alpha like this okay now since zn converges to z, so for a given epsilon greater than zero there exists at natural number or integer there exists in natural number of positive integer K 1, such that mod of zn minus z, mod of zn minus z is less than alpha. ok it less than alpha now this implies that minus alpha is less than equal to minus zn minus z, which is less than equal to mod Z minus mod Z for all n greater than equal to k1. Why? Because this is true this is to such that for all N greater than K 1 we have this thing. now if I take minus n that minus alpha is less than this fine now apply the triangle inequality, so what you get is mod of z minus because z1 minus z2, mod of this is greater than equal to mod z 1 minus moderate this result is true. So using this result we get this part ok. now half of this z so what we are therefore what will be the mod Z minus alpha bring it here, this is greater than equal to mod Z and less than equal to mod Zn for n greater than equal to k1 but z minus alpha, alpha is mode z by two. So it is basically half of mod z. so when it is sufficiently love the sequence of the term mod zn is greater than equal to half of mod z. Ok, now use this one this is say one ok now consider mod of 1 by Z n, minus 1 by z, now this can be written as more zn minus z, over mod Zn, mod Z. but mod zn is greater than this number, so it is less than equal to 2 / mod Z square into mod z minus zn and this is 2 for all n greater than equal to k1. now as, z minus z n goes to 0 ,because the zn sequence converges to z, so for a given epsilon up now for given epsilon greater than 0, given we can find there exists a natural number of positive integer K2, such that if n is greater than equal to k2 then,

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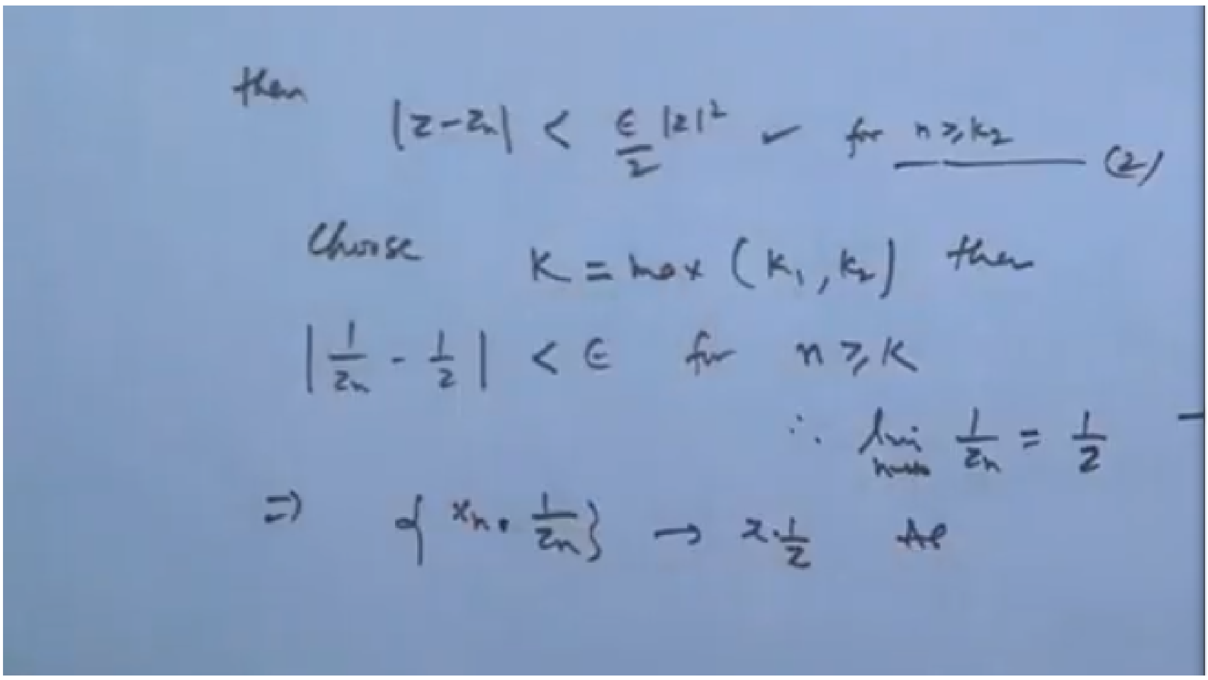

then mod Z minus Zn this equal less than the number epsilon mod Z square by 2, this is a smaller number so since zn converges to z, so we can identify this thing for all N greater than K1. ok now this is true for all N greater than K2 this is 2 for all and solely on and greater than K 2 this is true for all N greater than K 1 so if I pick this is too so let it be too so if we want to one known to combine then what happens is that this NK. if I take K a integer which is maximum of k1 and k2, then this defence can we made less than epsilon for all n greater than equal to K because what this is true and this shows the frequency a 1 by zn goes to therefore limit of the sequence 1 by zn, as n tends to infinity is 1 by z. ok let it be three. Now we are interested in xn by ZN. so it is the as good as, xn into 1 by zn. So the sequence xn into 1 x ZN will go to x into 1 by z that is the answer because of the product of the two sequence this is product of the two sequence will go to this. ok