Model 4

Lecture – 20

Ratio Test Cauchy's theorems on limits of sequences of real numbers

Course

on

Introductory Course in Real Analysis

So, in the last lecture we have discussed the ratio test. So today we will discuss few problems, based on the ratio test. We will further see, the Cauchy's first and second theorem, on the limits will also be discussed and that is useful to evaluate the limits of the sequence, of the functions. Okay? So today we will discuss the few problems, based on the ratio test, using the ratio test, we can solve those problems. And also we will discuss the few of the results, which are also needed for further study. So let us see, we have gone through the ratio test,

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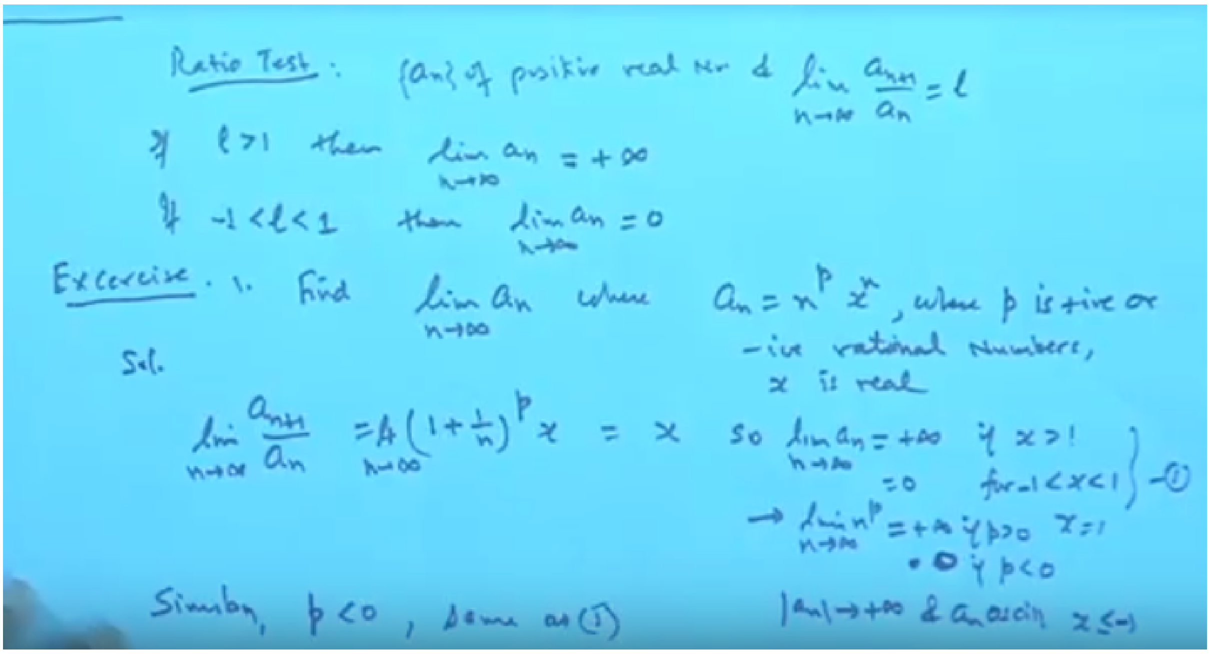

the ratio test says, that if n be a sequence of positive, if n's our sequence of the positive numbers, of positive real numbers, if limit of this, n plus 1 over N, and n tends to infinity, is a L, and when if L is greater than 1, then limit of the sequence a n, as n tends to infinity, will be diverging and diverges to infinity. And the second part of it shows that if, limit of this L, lies between minus 1 and plus 1, then the limit of a, as n tends to infinity, is 0. And particular cases is, when An's are, when L is less than one, then in that case, we are negative, then we will see, that all negative terms, then we can apply, take the minus sign and get the resource. Okay? So let us see the exercise based on this.
Suppose it is given, that. We have the sequence one is, find the, find limit of a n, as n tends to infinity, where an's are, is of the form, n to the power P, X to the power n, where P is positive or negative, rational number, where P is, P is, rational number, positive or negative rational number, rational numbers. Okay? And X is a real quantity X, X is real, X is real. Okay? Now if we look this apply the formula. So what is an plus 1? Over N, if we look this, it comes out to be 1 plus 1 by n, raised to the power P, X. Now when n tends to infinity, P is fixed. So this term will go to 1. And basically the limit of this, when n tends to infinity, is nothing, but X. So, when X is greater than 1, so according to the ratio test, if the limit of an plus 1 over n is L, where L is greater than 1, the limit of n will be plus infinity, when L lies between minus 1 to plus 1, the limit will be 0. So limit of this a n over n, infinity, if X is greater than 1, and 0 for X, lying between minus 1 and plus 1. Okay? And when X is equal to 1, then it reduces to the form n to the power P, and the behaviour of n to the power P, we have already there, if P is positive, it will go to plus infinity, when P is negative it will go to 0. So when x is 0, it is equivalent to the limit, n to the power P, as n tends to infinity, which will be plus infinity if P is positive and minus infinity and 0 and 0 if P is negative. And for P is equal to 1 also. Okay? So this is.

When X is less than equal to 1, if X is less than or equal to minus 1, then what happens? This sequence, mode xn, less than equal to 1, the sequence is minus 1 to the power n, minus 1 to the power n and then it will keep on oscillatory and oscillating. So the mode of an, will go to plus infinity and On's, an's are an's oscillatory infinite, oscillatory infinite. Okay?

Similarly, when P is negative, similarly, when P is negative, then in that case also, the same thing happened. It is again, this will be X greater than 1, it is infinity. X language n minus 1 is 0 and for other it, so the same results as 1, same as 1. Okay? That is one. So this will give that. Okay? Second exercise, let us see.

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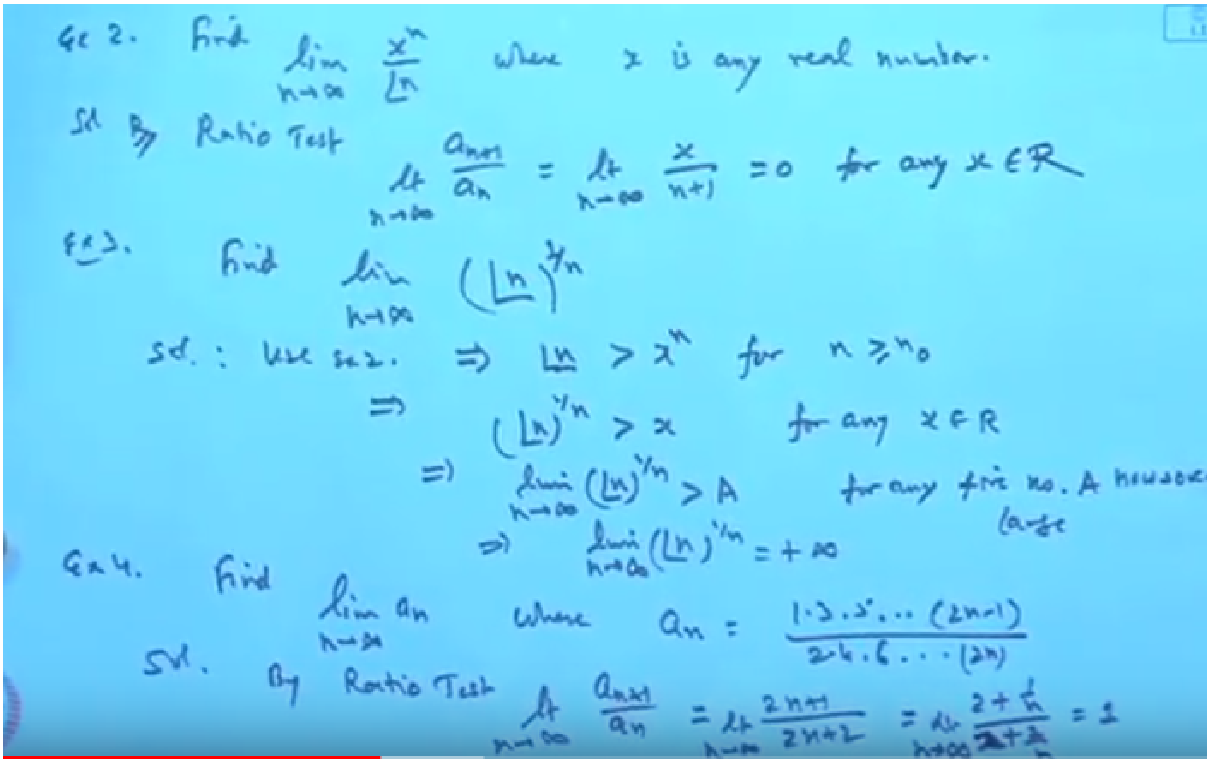

So exercise 2. Suppose, find the limit of, limit of X to the power n, factorial n, as n tends to infinity, Where X is, any real number, any real number. Now again apply the ratio test. So, why ratio test? what happens to this? a n plus 1 over a n, as n tends to infinity, it comes out to be, what? Limit of this, as n tends to infinity, x over n plus 1? So whatever the X may be, this limit will go to 0, for any X, real. It means this limit will always be 0 for all X. Okay? Third - find the limit of this limit of, factorial n, to the power 1 by n, as n tends to infinity. Factor N, 1 by n. Now, to sort this limit, let us use the previous one. We know the limit of X to the power n, factorial n, is tending to 0, its 0 in. So use the, use exercise 2, so exercise 2, it implies, that factorial n, goes much faster than comparative to X to the power n, that is, this is always be greater than n, for n sufficiently large, say n naught. Because then only it, because it is tending to zero, it means the denominator is going much faster to infinity, comparative to X to the power n. Therefore, it will, factorial n, will be greater than n extra after a certain stage, say n is greater than, n naught. Therefore X. factorial n to the power 1 by n. will be greater than X. And this is true for any X, for any X belongs to R.

It means the limit of this thing, limit of this factorial n, to the power 1 by n, as n tends to infinity, will exceed to any number a, will exceed to any number a, for any any positive number a, howsoever large, howsoever large. And this shows, this only possible the limit of this, is infinity, limit of this will be infinity, so it will be followed by this one. Okay? Now this one. Exercise 4 - Find the limit of, find limit of an where n tends to infinity, where n is 1, 3, 5, then 2 n minus 1, over 2, 4, 6, up to 2, 3. Suppose this is our n, okay? Now these are all positive terms? So apply the ratio set? By ratio test? a n plus one, over n, this comes out to what? 2 n plus 1, over 2 n, plus 2. N plus 1, is n, is n plus 1, 2 n plus 1, + 2 n plus 2. So the limit of this, as n tends to infinity, is the limit of this, as n tends to infinity? Divided, by n. So when you divide by n, it comes out to 2 plus 1 by n, over 2 plus 2 by N. And limit as n tends to infinity and that will come out to the 1. It means the limit of this, a n as n tends to 1 is, convergent and we goes to, limit is, 2 n plus 1, by 2 n. Okay? this limit. What is this? Is 1, sorry, is 1. Now, what is the ratio test says? The, ratio test if, you look the ratio test, the ratio test says, when an is a sequence of positive numbers, such that limit an plus 1 over an is L, when L is greater than 1 limit will be plus infinity, when L lying, between minus 1 to 1, limit is 0. But it does not say anything about L is 1. When L is 1, the series, the sequence a N, may converge, may not converge. We could, depends on the type of the sequence. We cannot say, the limit is 0 or limit is infinity or limit is finite. We have to compute that.

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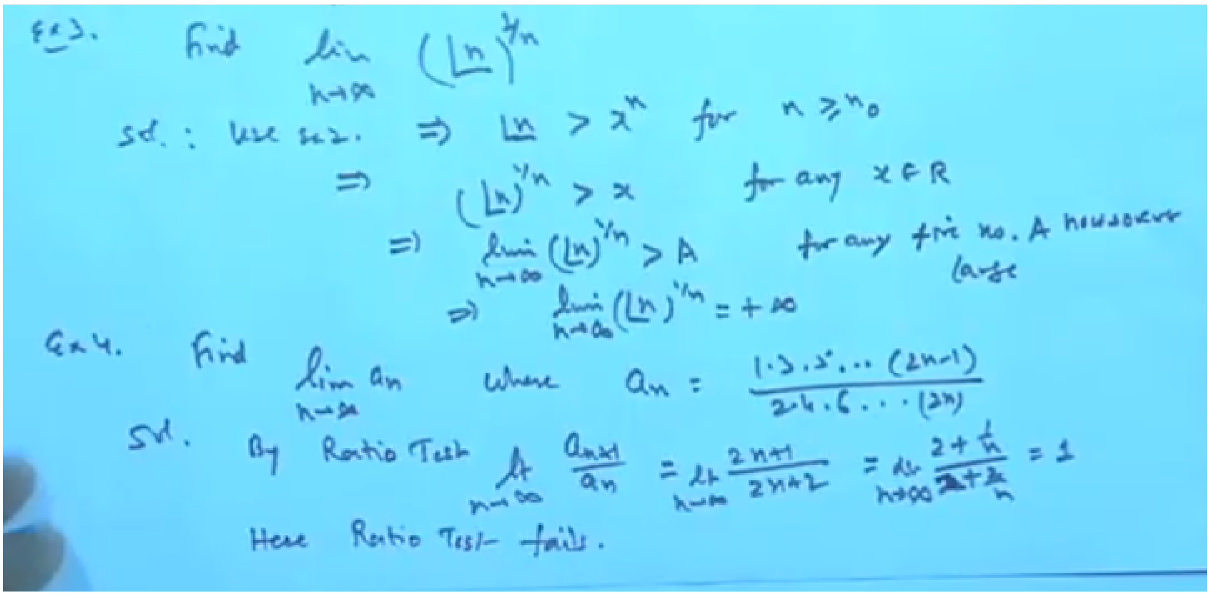

So here the ratio test fails. Here ratio test fails.

We are unable to get it. So what to do? Then in that case, we have to apply our previous knowledge. That is, we know, if a sequence which is a monotonically decreasing or monotonically increasing sequence and if it is bounded, monotonic increasing bounded above, monotonic decreasing bounded below, then such a sequence will definitely have a limit. So let us find out, whether this sequence is a monotonically sequence or not? Now here if we look, 2, an plus 1, over an, this comes out to be, 2 n plus 1, over 2 n plus 2. It means, that this is total thing; this total is less than 1?

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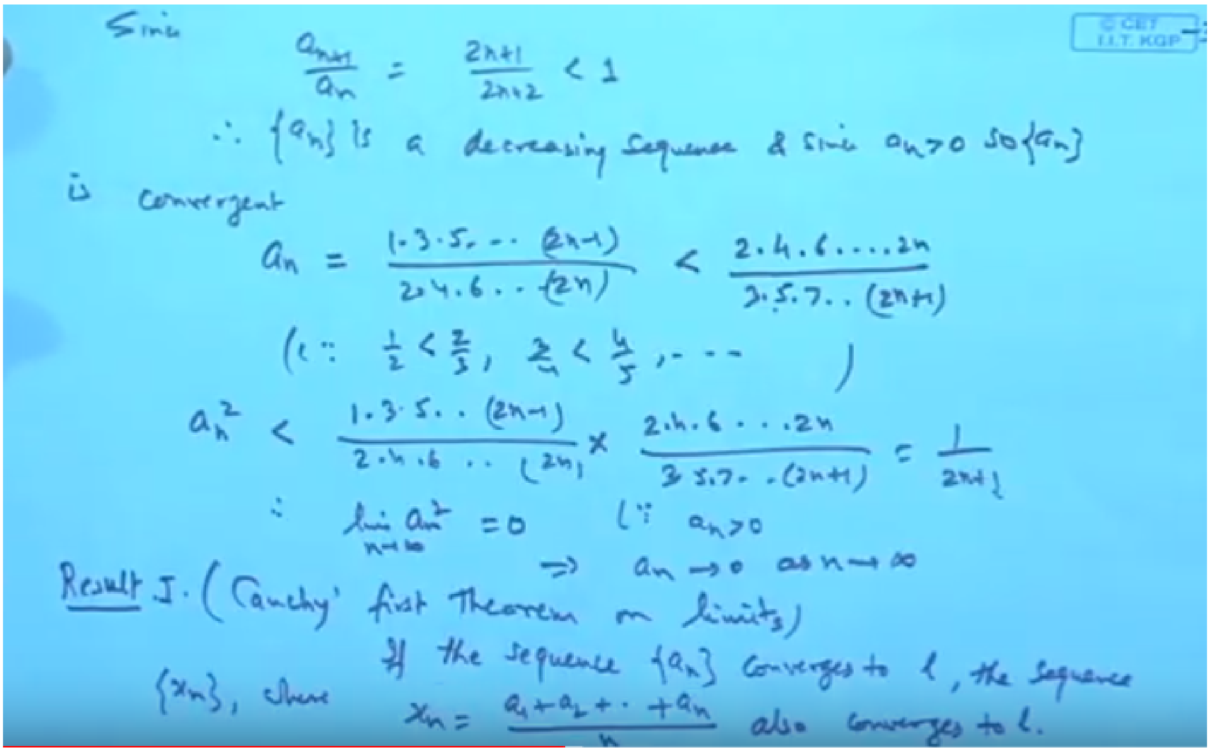

Since, since the ratio an plus 1, over n, is coming to be 2 n plus 1, over 2 n plus, denominator is higher than the numerator. So the ratio is strictly less than 1. So it means, the enitram is larger than, n plus 1. So when in n, 1 2 3 etcetera it keeps on decreasing. Therefore the sequence a n, is a monotone decreasing sequence, is a decreasing sequence. Okay? And since an's are all positive and since an's all positive, so it decreases and at the most, it will go to a positive number. So and you, so sequence an, is convergent. This is one thing which is clear, from this concept. Okay? We are interested in finding the limit now. So, it is, it is that convergence part is clear, that, the sequence has to converge. But what will be the limit? That, so let us see the an's again. What is the a N? A n was, 1 3 5, upto 2 n minus 1, over 2 4 6 up to 2 n. Now can you not say, it is less than, if I write, 2, 3 is 4, 5 6 up to 2 n, Okay? And this part, I am just increasing, 3, 5, 7, 2 n, plus 1. So this term. Because 1 is less than 2, or 1 by 2 is less than 2, by 3, you can say like that? So 1 by all, 2 is less than, so 1 by 2, is greater than 3, 1 by, 1 by 3 is greater than 1 by 2, and like that. So this will be 1 by 2, is less than 2 by 3, 3 by 4 is strictly less than the 4 by 5, 5 or 6, less than, say like this. So we can say this is less than. Because half is less than 2 by 3, 3 by 4, is strictly less than, the 4 by 5 and continue, so we can say this one. Okay? So an's is less than this, an is this, so n Square, will be less than, less than, the product of this two? 1 3 5, 2 n

minus 1, 2 4 6, 2 n, into product of this, 2 4 6, 2 n divided by 3 5 7, 2 n plus 1 and that comes out to be, 1 by, over 2 n, plus 1. So N Square, is coming to 1 by over 2n plus 1.
Therefore the limit of N Square, is square, is tending to 0. Because an's are positive, so limit cannot go negative, it could go at the most zero. And an square is less than this, as an is sufficiently large, the limit is tending to 0. So it means the an limit must go to 0, as n tends to infinity. So that will the answer for this. Okay? So this is what. Now, there are others also, but before going few, let us take one result, which is given by the cauchy. And, so before going this, let us see the result, which is known as the, 'Cauchy's first theorem on limit, on limits'. So what this theorem says, if, if the sequence an, if the sequence an, converges to L, then the sequence, the sequence, xn, where, xn is a 1, plus a 2, plus an, y n, also converges to L, also converges to L.
Let us will read at 1. Okay? Can we put it in exercise also? But because this is a extended result, so we can, say put it as a result one. So what it says is, if suppose n sequence are convergent then their mean, arithmetic, basically it is a arithmetic mean, mean of this, will also, converge to n, that is the first result.

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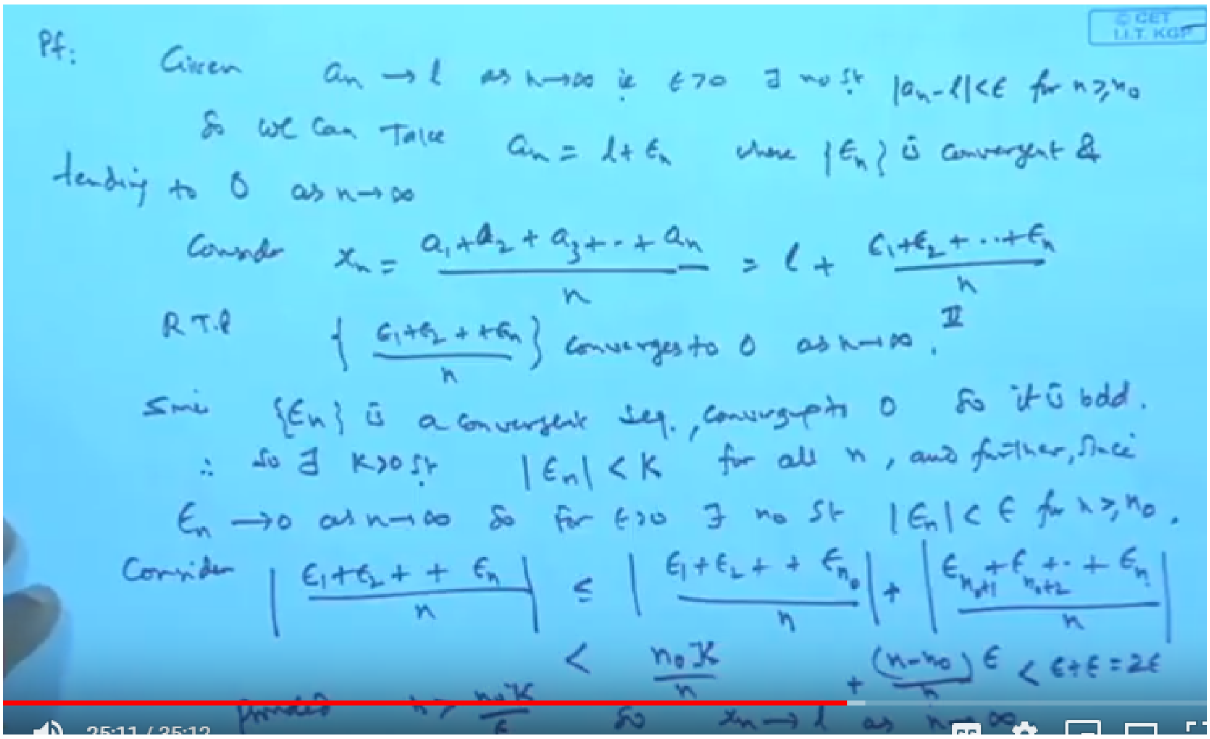

So let us see the proof of this result or solution of this. What is given is, a n converges to L, as n tends to infinity. It means. The difference between a N and L, that is, for a given epsilon, greater than 0, there exist an N naught, such that the mode of n minus L, is less than epsilon, for n sufficiently large. Okay? So what we can do, we can say, that so we can assume or we can take, an's is L plus epsilon n, where epsilon n, is convergent, this sequence, is convergent and tending to 0, 0, as n tends to infinity? So when n tends to infinity, the sequence epsilon epsilon to 0. It means n minus L, can be made as small as we please, after a certain stage, Okay? So let us suppose, now consider xn, so consider xn, xn is a 1, plus a 2, plus a 3, a n, divided by n. So substitute a 1, in terms of epsilon 1, so we, what we get is, if I substitute a 1, is L plus epsilon 1, a 2 is L plus epsilon 2 and so on this is n times L divided by L, so L plus epsilon L 1, epsilon 2, epsilon n, divided by n. Okay? We get this. Now we wanted the sequence xn goes to L. It means if I prove, that this second term, the second term goes to 0, as n tends to infinity, then it will. So required to prove is ,the sequence epsilon 1, epsilon 2, plus epsilon n, divided by n ,this sequence basically converges, to 0, as n tends to infinity, this is. Once we prove, then xn will go to L. Okay?

Now, since epsilon N is a convergent sequence, converging to 0, converging to 0, so it is a bounded sequence, so it is bounded sequence. Because every convergent sequence is Bounded, so there exists a K, such that mode of epsilon n, is less than or equal to K, for all n, for all n. There exists a positive K, just. Okay? And further epsilon tends to 0, so we get and further, since epsilon n, tends to 0, as n tends to infinity. So for a given epsilon, greater than 0, there exist an N naught, such that, mode of epsilon n, can be made less than epsilon, for N greater than equal to, say n naught. Okay? Now consider, now this one. Consider mode of epsilon 1, epsilon 2, epsilon n, divided by n. Now this will be less than equal to, mode, epsilon 1, epsilon 2, plus epsilon n naught, divided by n plus, epsilon n naught plus 1, epsilon n naught plus 2, up to epsilon n divided by n. Okay? I've choosen this. Now, when n is greater than n naught, the mode of epsilon is less than. It means each of this term is less than Epsilon. So total becomes n minus n naught. So this will be less than, n minus n naught, times, n minus n not times, and then, this is what, this is equal, less than epsilon, divided by n, plus. Now this term, each term is bounded by K, so this is bounded by K. It means this is less than, n naught times, K, divided by n. Is it not? Now n naught K, if I choose, n naught is fixed, K is fixed. So if I take epsilon such that, that n naught K, by n, is less than say, epsilon, then what happens? So if this is less than, epsilon plus epsilon, provided n is greater than, n is greater than, n naught K, by epsilon.

If I take n to be greater than this, then this part is less than epsilon, this is less than 1, so this is already less than epsilon, so totally less than, 2 epsilon. So this part when n is sufficiently large, goes to 0. Then once it goes to 0, then this sequence xn will go to L. So xn will go to L, as n tends to. Is it okay? That is what. This is the first Cauchy theorem on the limits. The second Cauchy theorem –

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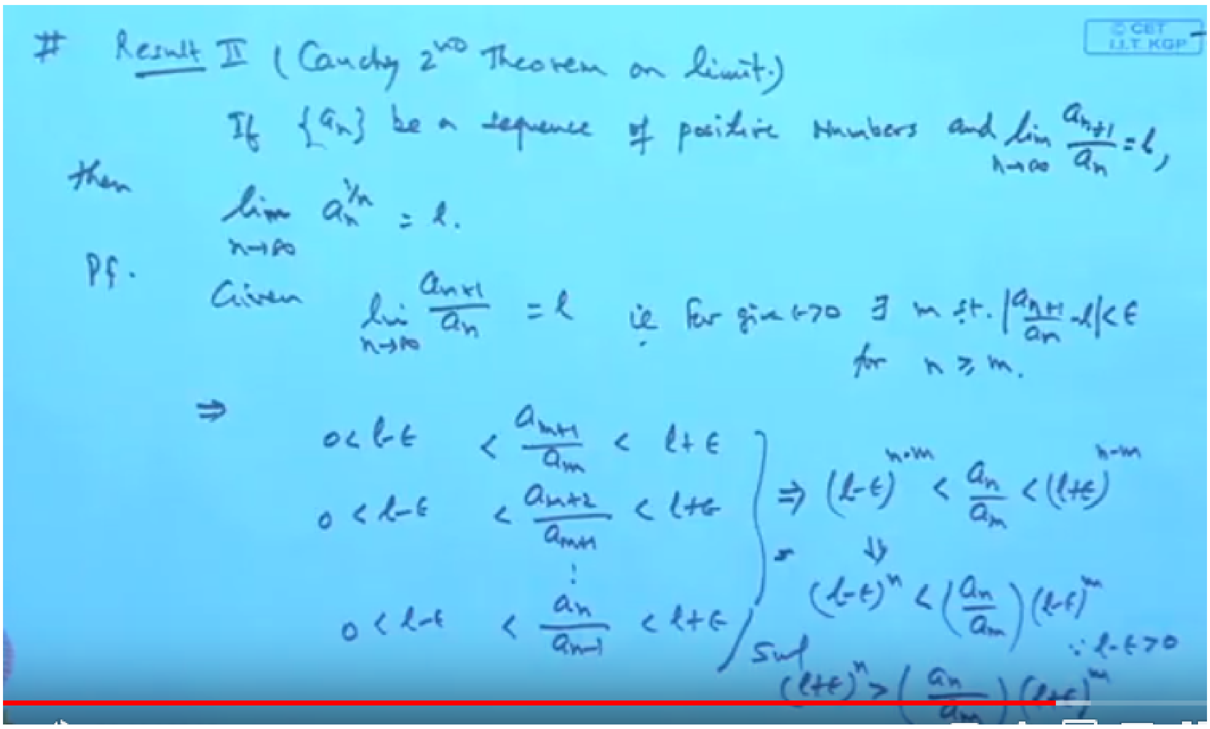

That is the result, to which is also we call it as a Cauchy's, Cauchy's, second theorem on limit, on limits. What this theorem says is, if, if sequence n, be a sequence of positive number, positive numbers, sequence of positive numbers, and limit and limit of a n, a n plus 1, divided by a n, a n tends to infinity, each L and limit of this is say, L. Then limit of n to the power, 1 by n, as n tends to infinity, will also L. So that gives me means, if, the issue of these sequences, is L, then the n'th root of this n, will also have the limit L. So let us see the proof of this. Okay, what is given is, that, this ratio is L, given a n, plus 1, over an, limit of this, as n tends to infinity, is L. It means, that is for, given epsilon, greater than 0, there exists say M, such that mode of n minus, a n plus 1, over n minus L, is less than epsilon, for all N, greater than equal to M. Okay? So, what you? It means, this implies that, a, m plus one, y an, this term, lies between L minus epsilon and L plus Epsilon? Okay?

Now if we continue this M plus 2, by M plus 1, lies between L plus epsilon, L minus epsilon, and like this, up to say any term. Which is a n, over, aN minus 1, lies between, L minus epsilon, L plus epsilon, just continue. Now find the product. Because these are all positive quantity, remember. Because an's is a sequence of positive. These are all positive, greater than 0, greater than 0, greater than, so this is also positive. So once they are positive, we can multiply, without getting the change in their inequality. Okay? So when you multiply this thing, what you get? These are total, n minus M terms? So here we get L minus M? So this implies that L minus epsilon, to the power, n minus M, is less than, if you multiply this, so the close, this telescopically change, getting cancelled, so n over m, is left only. So an over am, an over am and which is less than L plus epsilon, to the power n minus m. Okay? Where n is greater than M. Now divided by L minus epsilon, to the power minus M, so if I divide, because this is positive Quantity, so again when you divide, so this implies that, which implies that, L epsilon, to the power n, is less, L minus L is less than, is less than, if I divide by this, then what you are getting? a n over a m, a n over a m, into L minus, epsilon to the power M, because, this will come because, this is positive, because L minus Epsilon, is positive. So we can do like that. Now take the power 1 by n. So what we get from here is, and similarly when you take L plus epsilon to the power n, here also we get, this is greater than an over am, an over am, into L plus epsilon, to the power M. Okay? Now, this part L plus M, is greater than 1, sorry, this an by am, so now from here, again. So what we get basically?

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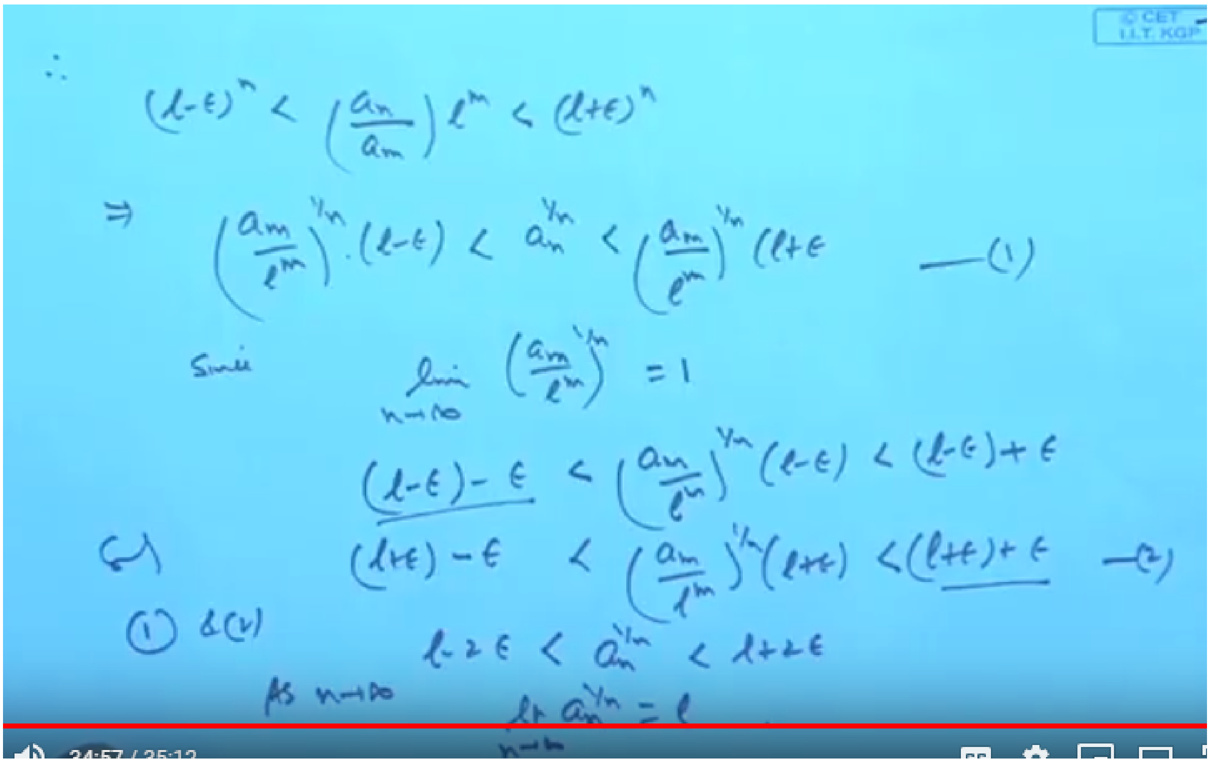

So we get therefore we get, L minus epsilon, to the power n, is less than an over am, Lm, is less than, is less than, L plus epsilon, to the power n, just like. Is it not? So that is not.
Now take this term, here. So we get, L minus epsilon, an am / LM, Okay? Raised to the power 1 by n, into L minus epsilon, into L minus epsilon, is less than n to the power 1 by n, which is less than, an and aM, over L to the power M, raised to the power 1 by n, L plus epsilon. Okay? Let it be 1. Clear? Now since, limit of this, am over L to the power M, by 1 by n, as n tends to infinity, is 1. Why? because, this is fixed. aM sorry, the aM for fixed point. L M is also fixed. So this is some constant, power 1 by n. So when n is sufficiently large, the limit will go to 1. It means this term, will lie between 1 minus epsilon and 1 plus Epsilon. Okay? So we can say, that this limit, entire Thing, L minus Epsilon, this part, can be, so we can say this, limit of n. So what we get is, therefore this entire thing lies between, yeah, so therefore we get, L minus epsilon, minus epsilon, is less than, less than, am by L M, a M by L M, raised to the power, 1 by n, L minus epsilon, L minus epsilon, which is less than, L minus epsilon, plus epsilon. Why? Okay? This aM minus 1 by L, because it is, this entire thing, this entire thing, lies between L minus epsilon and then minus 1. Okay? So this will be, this 1 tending to 1, basically. So this limit will go to L minus epsilon, only, basically, L minus s. So this total thing will lie, L minus epsilon, minus 1 and L minus 1 Epsilon. Similarly this term will also lie between the. Similarly we can say, L plus epsilon, L plus epsilon, minus epsilon, less than aM over L M, rest to the power 1 by M, L plus epsilon, which is less than L plus epsilon, plus epsilon. So what we conclude is, that, if I look this entire thing, an to the power 1 by N, n to the power L , the lower bound will L minus 2 epsilon, This will be the lower bound. And for n by 1 by n, upper bound will be this. So this 1 & 2, if we combine, then we get from here is, L minus 2 epsilon, is less than a n to the power 1 by n, bill is L plus 2 epsilon, as n tends to infinity, the limit of this, a n to the power 1 by n, limit of this, is nothing but L, and this proves the result. Okay? Now here when it is L is in, when L is greater than, sufficiently large, then this limit will go to infinity, also infinity, in fact.
Okay?