Model 4

Lecture – 19

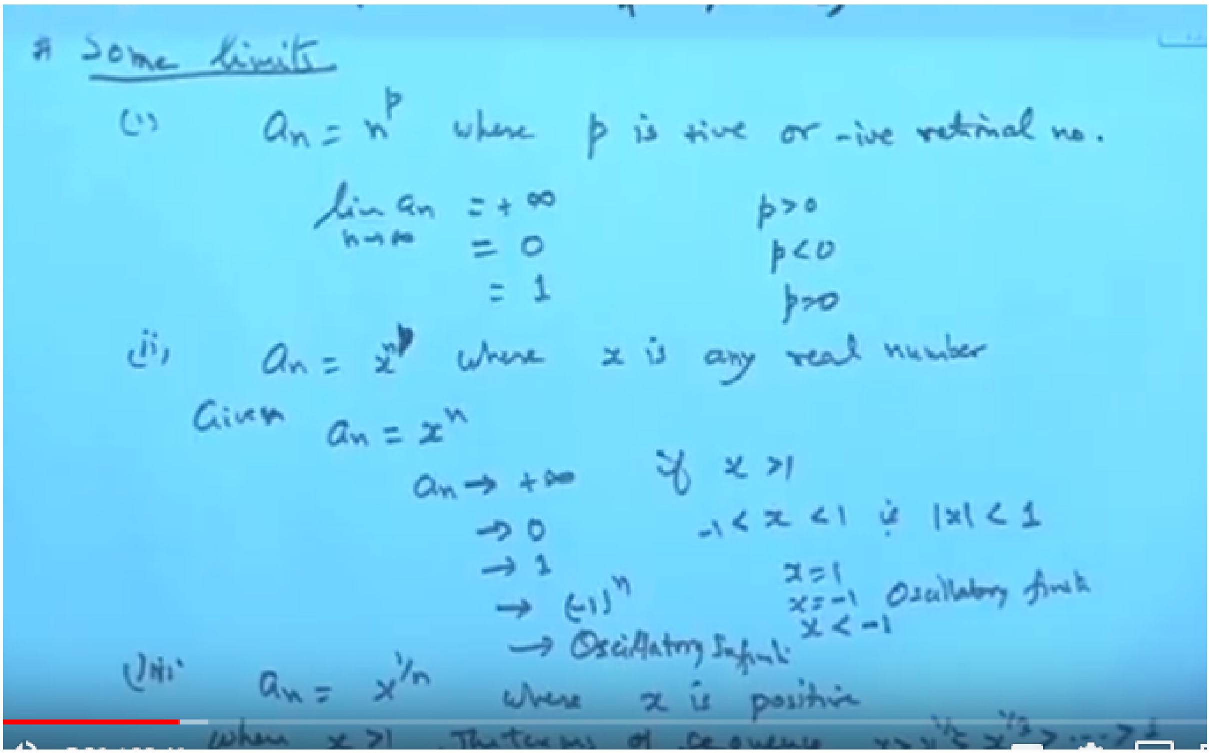
Some Important limits of sequences

Course

On

Introductory Course in Real Analysis

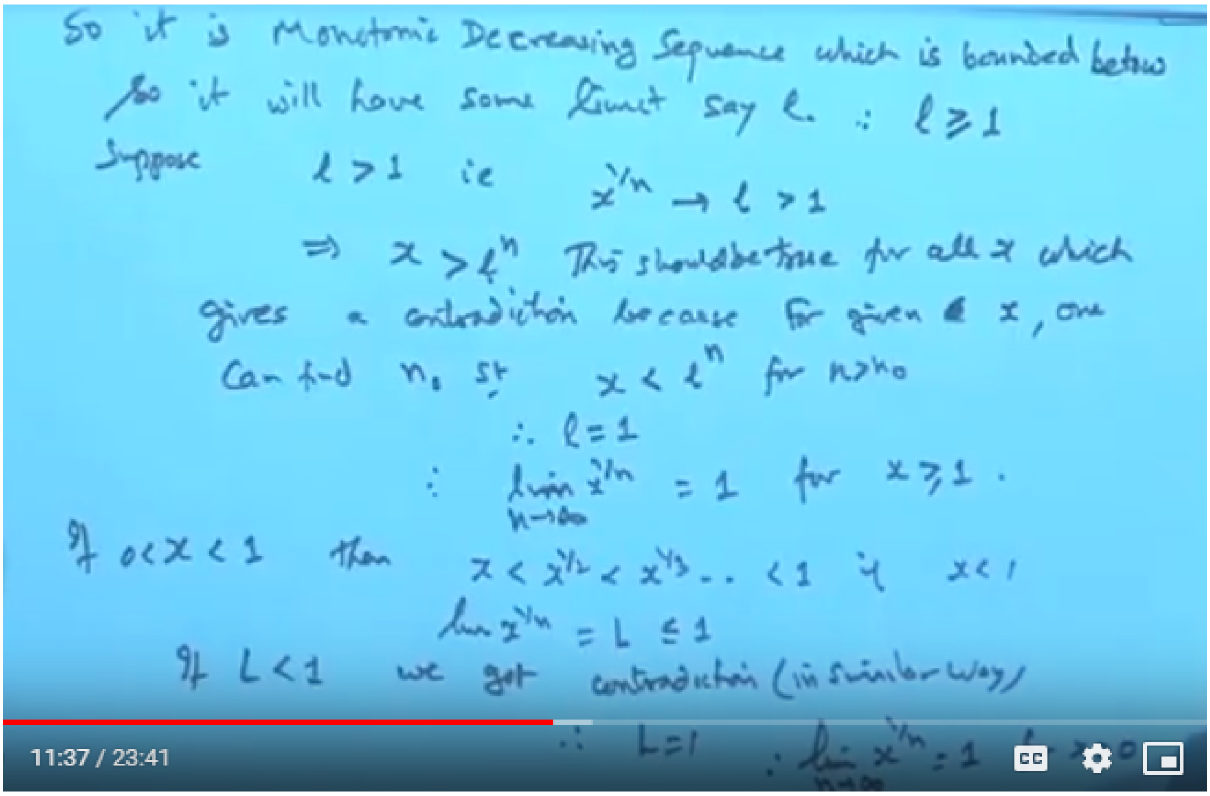
(Refer Slide Time: 00:17)



So let us see there is some limits, Okay? The first one very simple is, if suppose A n, is their sequence which is n to the power P a sequence of real number, where P is a positive or negative, rational numbers, rational numbers. Then limit of the sequence a n, L, n tends to infinity will be. If P is positive, the limit will come out to be plus infinity. Is it not? That is because as n increases, it is unbounded sequence and so it is plus infinity. When P is negative, then, the limit will come out to be 0, it will tends to 0. Okay? So when P is this and less than 0, we are getting them. And when P is zero, then the limit will come out to be 1. So it is simple, that it is nothing to Prove, just.

then, another sequence. Suppose n is X to the power P, where X is any real number, real number, X is any real number. X to the power n, so let it be n. Okay? X to the power n, yeah that is, is X to the power n given. Now this n will go to plus infinity, if X is greater than 1, the X is greater than 1, each term of the sequence is greater than 1 and in fact it keeps on increasing, so increases to infinity, so then. And when it X lying between minus 1 and plus 1, minus 1 and plus 1. It means, mode of X is less, strictly less than 1. That is, mode of X, is strictly less than 1, absolute value of this is less than 1. Then in that case this sequence will keep on decreasing and decreases to 0. And when X is equal to 1, the limit is tending to 1. What happened to that when X is strictly less than minus 1? When X is strictly less than minus 1 or when X is equal to 1, minus 1? When X is equal to minus 1, it is basically of the form, power n. So this is an Oscillatory Finite Series, oscillatory finite. Because the limit will, it will go to plus 1, minus 1, like this. So mode of a thing is bounded. But even X is strictly, less than minus 2, then it is, it is a oscillatory infinite, in it. Is not? It will go oscillatory infinity, plus infinity, minus infinity; it will go, so we get this. Okay? Then third limits. If an's be suppose, X to the power, one by n, where X is positive, x is positive. Now when X is positive and n is any integer, n is any integer, then, what happens that, when X is greater than one, each term of the sequence is greater than one. So when X is greater than one, the terms of the sequence, the terms of the sequence, terms of the sequence, sequences that is one. X. X to the power half. X to the power one-third, and so on. These are satisfying this my X is greater than this, X to the power half, is greater than this, is greater than

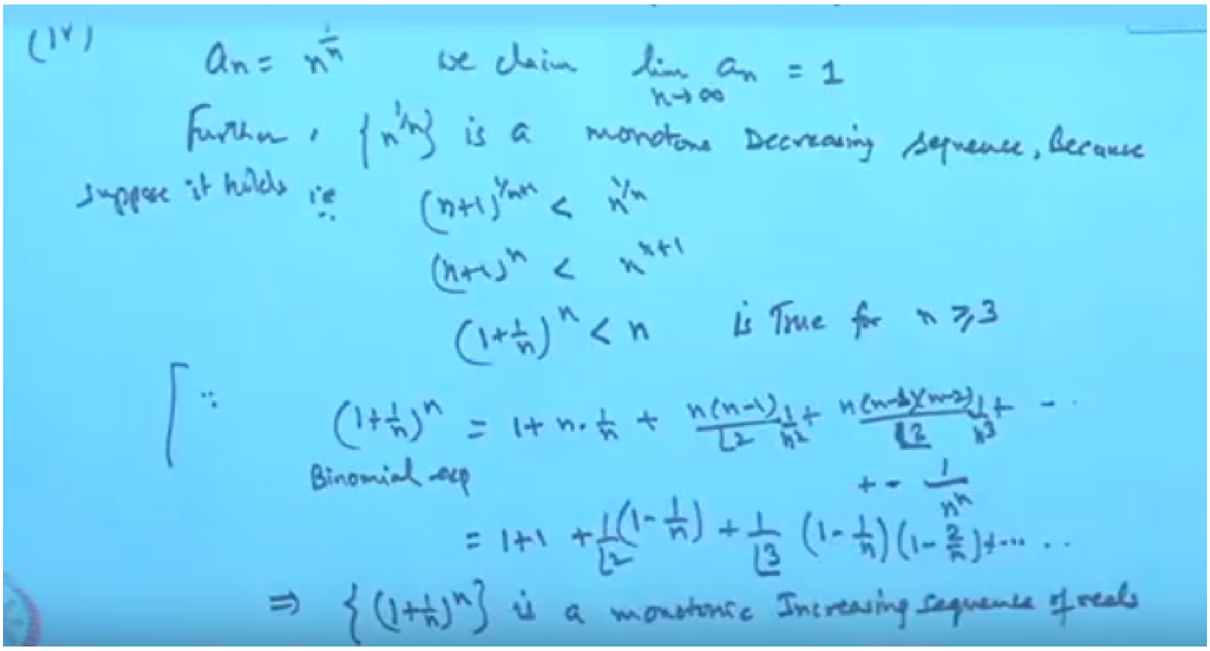
this, and all the terms are greater than one. It means it is a monotonic decreasing sequence, bounded below. Is it not? So it is a,
(Refer Slide Time: 05:27)



so it is monotonic decreasing, is strictly decreasing, decreasing sequence, which is bounded, below, bounded below. Monotonic decreasing sequence bounded below. So it must have a some limit? Because, every boundary sequence, is limit. So it will have it, will have the limit, it will have some limit, say L. And this limit, therefore L will be, either, either greater than 1 or equal to 1. Suppose L is, L is suppose, L is strictly greater than 1, suppose we say, Okay? Then what happens this? That is X to the power 1 by n, limit of this, X to the power by N, this limit, as n tends to infinity, tends to L, which is greater than 1? So that is only the possible when X to the power 1 by n, should be strictly greater than 1 for all n, ok now this shows that X must be X must be greater than L to the power n is not limited X to the and this is true for all X but when L is greater than 1, L is greater than 1, then this is not true for all X. This should be, this should be, true for all X, which gives you a contradiction, which gives a contradiction, Why contradiction? Because, for given X, for given X, one can find, n naught, such that X will be strictly less than L to the power n, for N greater than n naught. Because, this keeps on increasing. Right hand side it keeps on increasing, but the left-hand side is fixed. Once you choose the X, left hand side is fixed. And what we are saying that if limit of this is, suppose L which is strictly greater than 1, it means, for large values of N or when X, what is the behaviour of X? X will be exceeding L to the power n, Is it not? It will be, greater than L 2, then it would goes on decreasing and decreases say something like that. But here, is give leads to a contraction, once you fix up the X and since L is greater than 1, so when n is sufficiently large, this will go keep on increasing. So a number n naught, can be obtained, so that X can be made, as it can be made less than the term, L to the power 1, after certain stage, and its contradiction is because our wrong assumption, that L is greater than 1. Therefore L must be equal to 1. Is this Clear? So what we conclude is, the limit of this, therefore limit of X to the power 1 by n, when n is sufficiently large, is 1,

when X is greater than 1, for X greater than 1. Okay? And for X equal to 1 already true? For X equal to 1, also it is true? So let us. Now if X is strictly less than 1, Is it not?
Then xB are choosing positive, remember this is X is a positive, so we cannot choose the minus or something, X is positive, greater than 0, So, if 0 less than X. less than this. Then. Then. What happens to this? L limit is there. Is it not? So we can get a contradiction here, again. Because X is less than 1, X is less than 1 and we are choosing the limit L. Because if X is less than 1, what is the behaviour of the sequence? Yeah. If X is less than 1, the terms of the Sequence, this sequence, will be what? It keeps on increasing, this keeps on increasing. Is it not? So then X is less than X, raised to the power half, less than X to the point and so on, if X is less than 1. Okay? X is less than 1. So what happen is, increasing function of this, Okay? And then what happen. When X is equal to 1, but it is always be less than 1 and less than 1. So when you take the limit of this, limit of this, X raised to the 1 by N is supposed capital L, which is less than or equal to 1 But again contradiction. If L is strictly less than 1, we get a contradiction, in a similar way, in similar way. Therefore L must be 1. So what we conclude is, that limit of this, therefore, limit of this X to the power 1 by n, when n is tends to 1, is always 1, for X greater than 0. X negative is not defined, so that is why we are not choosing. Because X is negative, the roots are not reals, so we cannot get it, so that is what.

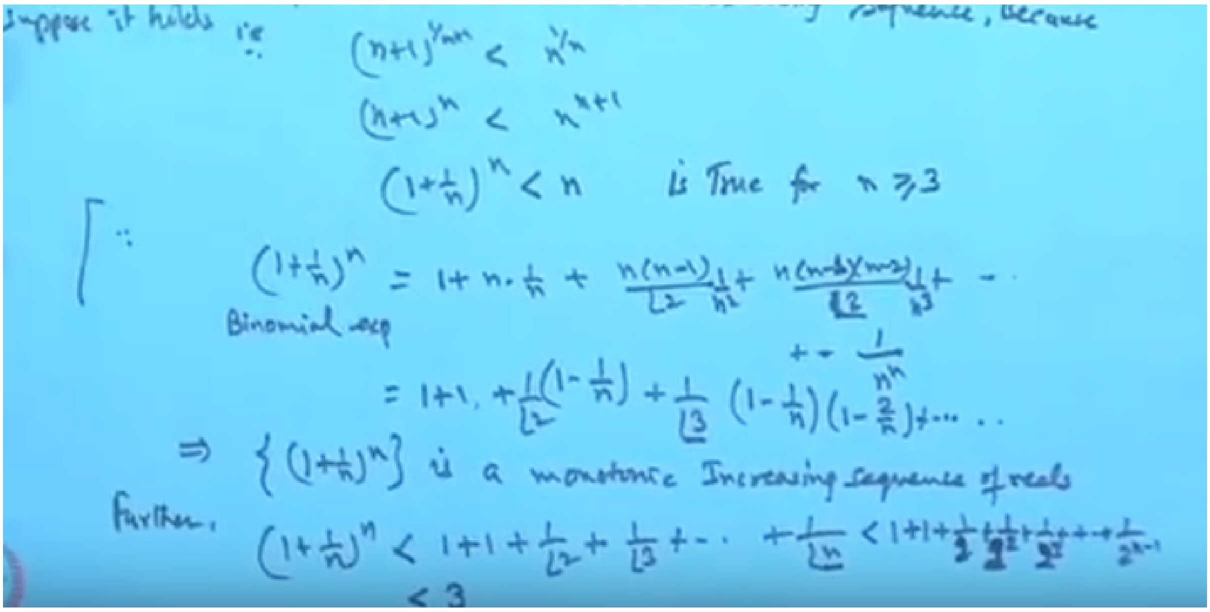
(Refer Slide Time: 11:45)



Now if we take, this is our fourth problem, limits. Suppose we take a n to be, n raised to the power 1 by n. Now we claim that limit of this sequence, is a. We claim that limit of this a n, as n tends to infinity is 1, as n is sufficiently large, this basically tends to like this, Okay? So, how to justify, limit to be 1? So first what we will show it, that this sequence, is a monotone sequence, I Okay? If we prove it is a monotonic decreasing or increasing, Okay? If increasing bounded above, it is decreasing, then it is bounded below, then limit will exist. So first is, we claim this. Further, the sequence n to the power1 by n, is a monotone decreasing sequence, is a monotone decreasing sequence, monotone decreasing sequence. Why? This is our. Because, suppose it is true, suppose true, suppose it hold, then it means that is, this is, should be less than n to the power 1 by n? As n increases, it keeps decreases, so we claim that this is a decreasing sequence, monotone decreasing. n plus 1, n plus 1, exactly. So we are getting this. So what we get is, n plus 1 to the power n, is less than n to the power n, plus 1. Okay? Now this is equal to, when you divide by n, then we get, 1 plus 1 by n, to the power n, is less than n, it is less than n. Now this is true, this is true, for N greater than or equal to 3, As n is sufficiently large. For n equal to 1 and 2, it will, it will not be true. But when n is greater than or equal to 3, it holds. The reason is, because, 1 plus 1 by n, to the power n, apply the binomial, binomial expansion. So 1 plus n X, plus n, n minus 1, over factorial 2 and plus n, n minus 1, n minus 2, over factorial 3, like this. So suppose I apply the factorial, then what will be this, say last term, up to n term? Then we get 1 by n to the power n. Okay? Clear, just I, 1 plus NX, plus n n, minus 1 and so on. Okay? just binomial. Okay? Now this will be equal to what? 1 plus 1, plus 1 by, No. 1 by N Square will also come. NN minus 1, factor 2, into X square, so 1 by X square, this is 1 by n q and so on. So n gets cancelled. We get 1 minus 1 by n, by factorial 2, then 1 by factorial 3, 1 minus n NN minus 1, 1 minus 1 by n, 1 minus 2 by n and like this, up to, so on. Okay?

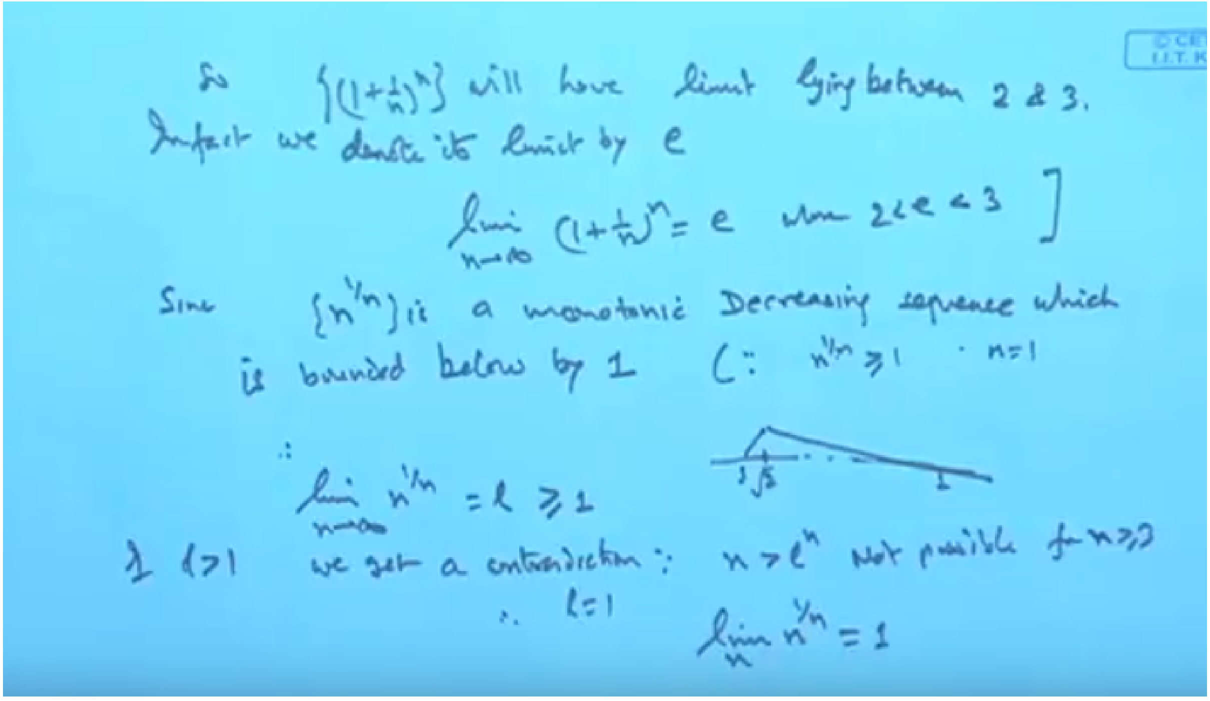
Now you see this, each term is positive, each term is positive. So the sequence, so the sequence, 1 plus 1 by n, to the power n, is a monotonic, monotonic, increasing sequence, of reals. Because when n increases, the terms increases here? So n, if this is positive, this positive, so you are getting some positive terms, here again? So it is a monotonic increasing sequence. Further, that this sequence or the, for all n this at least greater than two, because these are positive things, Okay?

(Refer Slide Time: 16:52)



But this sequence, further, 1 plus 1 by n, to the power n, is, is strictly less than 1 Plus 1, plus 1 by factorial 2, 1 by factorial 3, plus 1 by factorial n. Is it not? That will be and this is, last term n to the, okay. Clear? that will be, factorial n. So this one. Now this will be further less than, or this is further less than, 1 plus 1, plus 1 by 2, 1 by 3, 1 by 4 and so on? Sorry. 1 by 2 square, 1 by 2 cube, and so on? Like this, so it is 1 by 2 to the power, n minus 1, 1 by 2, to the power, n minus 1. Because factorial 2, is greater, is equal to 2 basically. Okay? So 1, so first term is additive, factorial 3, is greater than 2 square. So 1 by factorial 3, is less than 2, square 1 by factorial 4, is less than 1, by 2 cube, 1 by factorial less than 2 to the power, n minus 1, now this is the geometric series. So what is the sum of this geometric series is, 3 less than, a over 1 minus, so it is less than 3. Okay? Is it okay? So this is. It means this is a monotonic increasing sequence, of real number, which is dominated by three, bounded by three, so it must be a convergent sequence. And limit of this cannot exceed by three, and will always be strictly greater than two, at the most it may be three.

(Refer Slide Time: 18:57)



So limit of this n, so limit of this, so sequence 1 plus 1 by n, to the power n, will have, will have limit, will have limit, lying between, 2 & 3. Okay? And in fact, we denote its limits by E. And it was shown later, it was shown. That limit of this, as n tends to infinity, 1 plus 1 by n, to the power 1 e, where e, lies between 2 & 3, & it is irrational number, 2.7 something like that, so this is. So, this will be the limit. Now here we are discussing about this part. We are it is given the N to the power 1. We are interested in finding the limit of this. Now what we claim this sequence is a monotonic decreasing sequence. Which, is already proved. When n is sufficiently large, greater, this is a monotonic decreasing sequence, monotony. But each term of this sequence, is greater than 1 or equal to 1, because n is 1. So it is a monotonic decreasing sequence. Is it not? And each term, is greater than or equal to 1. So now, so since n to the power 1, by n, is a monotonic decreasing, monotonic n, sorry, it is a monotonic, yeah, this sequence is monotonic, in decreasing, Is it not? decreasing sequence, decreasing sequence, which is bounded below, by 1. Is it not? Because n to the power 1 by n, is greater than equal to 1, at least 4 when n is equal to 1, you get 1, okay? n is 1, you get 1, 2 it is greater than 1. Is it not? So it is monotonic decreasing sequence. After 3 it is there. Is it not? n to the power 1 by n, because this is 2 for n, is greater than equal to 3. So after 3 this is decreasing sequence. But, all the terms will definitely greater than one. Did you get it or not?

But you cannot say if n n is equal to 1 and 2, you cannot say that n is equal to 1, but rather than, n equal to 2, it is greater than 1, 1 for n, n equal to 3 onward, it decreases, for n is equal to 1 and 2 it increases. Clear? But when n is 3 onward, it decreases. So it is just, like this. It goes like this and then decreases. Okay? So this is 1, this is 1 and here is something, 2 to the power under root and then it keeps on decreasing and decreases to basically, again 1, like this. Okay? So this is, therefore the limit of this sequence is, when you say limit of n, exist and we get the limit to be 1. Limit of n to the power 1 by n, when n tends, to is suppose L. Okay? We get limit. Now this limit is greater than or equal to 1. Is it okay? Now we claim L cannot be greater than one. Now if L is greater than one, then again will read a contradiction? Then we get a contradiction. Why? Because n is greater than L to the power is n, L is greater than one, so not possible, not possible. For n greater than equal to three, okay? Three

cannot be greater than some number power three, okay? So that will be more. Suppose L is equal to two, then what happen? Three cannot be greater than 2 to the power 3, so it is contradiction. Therefore, L must be one. So limit of this, n to the power 1 by n, over n, is 1. That is also interesting limit.
Okay, then.