Model 3

Lecture – 18

Tutorial III

Course on

Introductory Course in Real Analysis

Okay so this is the tutorial class based on the lectures, in 11 to 15. Here we will discuss few problems.

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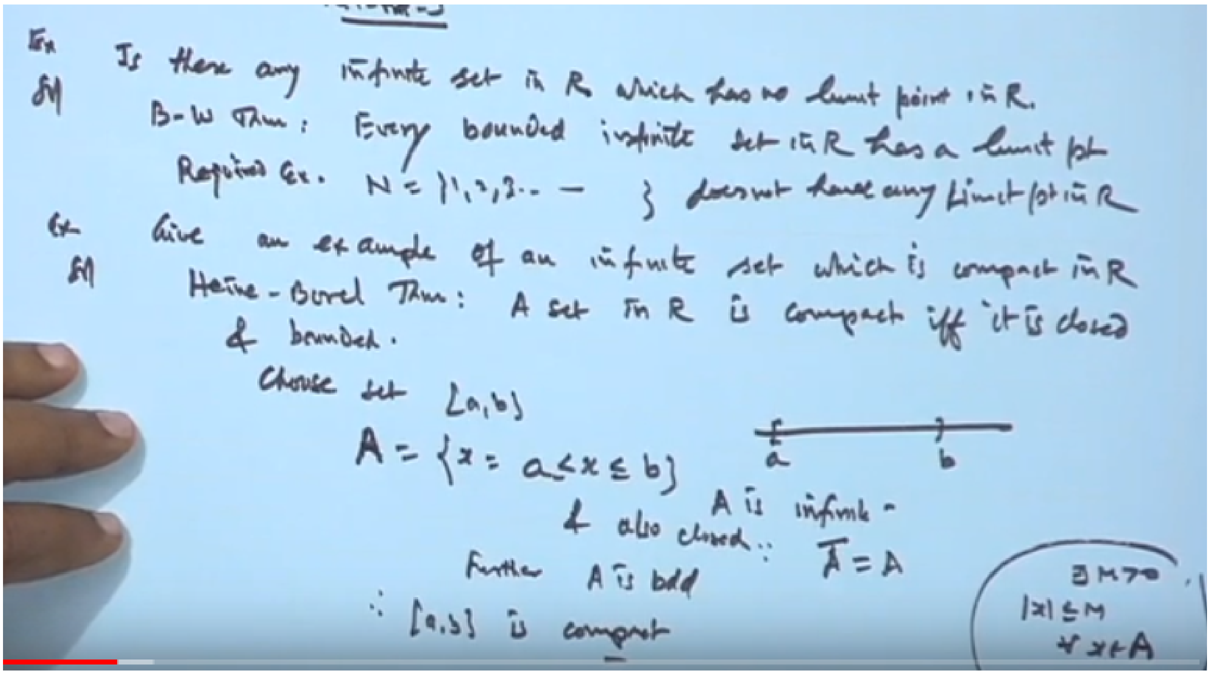

The tutorial 3 Okay? The first problem is if there exists is there any in finite set, in R which has no limit point, point in R. And this is very very simple problem. What the Bolzano Weierstrass theorem says? The Bolzano Weierstrass theorem says that, every infinite bounded set, in R, every bounded infinite set in R, has a limit point. So infact, if you want that example, of infinite set, which does not have any limit point, so it must be taken to be unbounded. Therefore the example required example will be; set of natural numbers, 1, 2, 3, it is infinite set, but unbounded. So it does not have any limit point in R that is all. So this is exercise we have been based on the Bolzano Weierstrass Theorem. Next example also based on the, some Heine Borel Theorem. Given example, given Example, of an infinite set, of an infinite set, which is compact, in R. So what is the Heine Borel Theorem says? But the Heine Borel Theorem says, that a set in R is compact, if and only if, it is closed and bounded. So we wanted an example of infinite set, which is compact in R.

So an infinite set if it is closed and bounded, then it is compact, if you know the entire set is closed and bounded, compact. So for example, choose the set [a, b]. Set of all points, lying between the closed interval, a, b. So this is R, set of X, such that this happens, now this set a, this set a is infinite. The number of real numbers between a, b is infinite and also closed. Closed, because ab, all the limits points of the set ab, are the a B itself, because, the limit point of this, because the closure of a bar, coincide with a. And further it is bounded. A set is said to be bounded, if there exist a constant M, such that mod of X, is less than equal to M, for every X belongs to the set a. Then Xa is said to be bounded, if there exists M, greater than 0, such that this condition is satisfied. So here it is bounded, because the interval, all the line points does not exceed by B. or this one.

Therefore a set is bounded and all the limits points, like a closed and bounded. So this is an example, of an infinite set which compared. Therefore a b is compact. In finite set which is, Okay?

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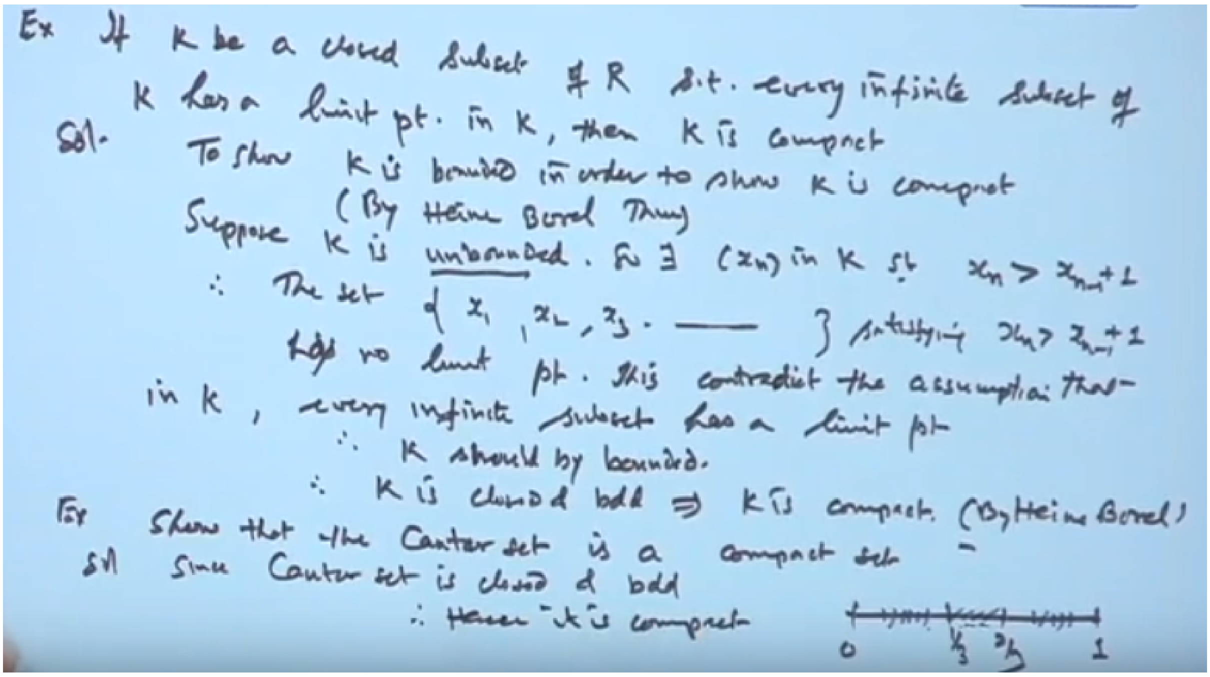

Next example. If a, be a closed subset, closed subset of r? Such that, every infinite subset, subset of k, every infinite subset of k, has a limit point, has a limit point in k, then K is compact. So we will use the Heine Borel Theorem, gave me a closed subset over, such that every infinite subset of K, has a limit point in k, then k is compact. So what we wanted to show, to show? K is bounded. In order to show K is compact, by Heine Borel Theorem. Because closed and bounded set, all Compact sets, Okay? Now, what we given k is it closed subset, of such that every infinite subset of k, has a limit point in n. So we wanted to show, this K is bounded. Let us resume by contradictions. Suppose k is not bounded, unbounded. So once it is bounded, so they will exist a sequence xn in k, such that say, xn is greater than, X n minus 1, plus 1. It keep on increasing, its keep on increasing, so it is unbounded, okay? Since, it is number. Therefore the set of these points, x1, x2, x3 and so on, satisfying this condition, satisfying the condition, X n is greater than X n, minus 1 1, each unbounded set, is has no limit point, no limit point. Because, say if I takes x naught, then x1 becomes x naught, plus 1. X 2 becomes x, x 1, plus 1, that is x naught plus 2, and xn becomes x naught, plus n. So as n tends to infinity, it will go to infinity. Therefore it has no limit point by construction. Now this contradicts this contradicts, contradicts our assumption, assumption that K, in K, the every infinite subset, subset has a limit point. This contradicts.

Therefore this contradiction is, because our wrong assumption. Therefore K should be bounded. So k is already giving close, k is bounded. Hence K is closed and bounded, implies KH compact by Heine Borel Theorem. So therefore this completes the results for. Next is very simple, what is the fundamental question, so that the canter set is a compact set, is a compact set. What is canter set? If we take the interval 0, 1 and divide into three part, one-third and two-thirds and remove this one, remove this portion, leaving one third and two-third, again you further divided into, again and remove this part, remove this part and remain these points, so the remaining set, is a canter set, Okay? Now this canter set, obviously, is a closed and bounded and bounded. Bounded, because all the points lying between zero and one, so it is a bounded set and since it is closed, why it is closed is? Because the one-third is the limiting point, of these sequence, of the point, which are dropped. So one third is the limit point of all these point here, as well as, here also, Okay?

So these are the limit point of the schedule, and this limit point, belongs to the set, Cantor set.

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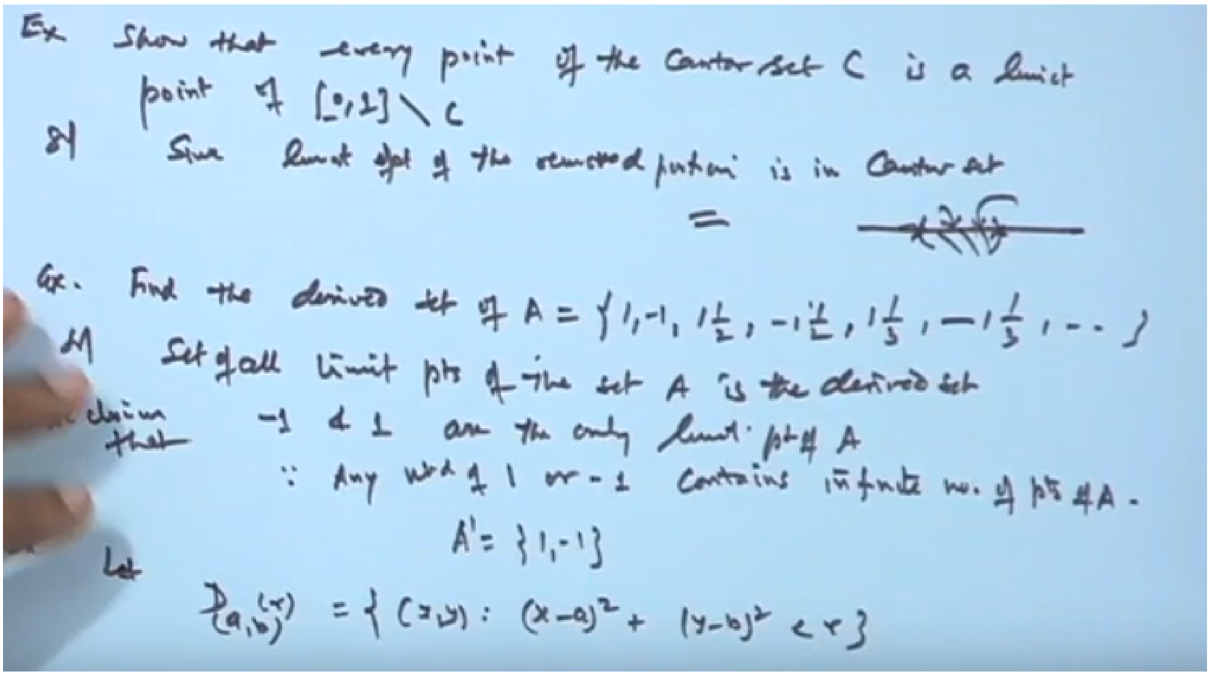

Therefore this canter set is a closed and bounded. Hence it is compact. And as a corollary of this, we can say That, we can say that, so that every point, next, very point, so that every point, of the Cantor set, C is a limit point, is a limit point, of the closed interval 0 1 minus C. And then solution is very obvious. So if we remove, as explained our, we are removing the middle for third and the endpoints of the removed portion, remain fixed in there, therefore and then the same limit point, Okay. Since the limit point, of the removed portion, removed portion, is these points, C is the limit point of this portion, Okay? C is the limit point of the removed portion, is in the Cantor set. Item Okay? So not this we removed. So this is the limit point of these sequences, these is the limit point of these sequences. Therefore limit point of the removed intervals, if you take any sequence in the removed interval and sequence, whose limit point, will be the point of this thing. Therefore this is justify, Okay, so nothing much to prove it.

Now, suppose we have, what is that drive sets? Find the drive set of A, which is one minus 1,1 and a half, minus 1,1 and a half, 1,1 one-third, minus 1,1 third and so on. Okay? Drive set. Solution, drive set means, set of all the limit points, of the set a, is called the drive set, so here, the, if I take minus 1 and 1. We claim that minus 1 and 1, are the only limit points, of the set a, because any neighbourhood of 1 or minus 1, any neighbourhood of 1 and minus 1, contains in finite number of points, of a. Therefore 1 and minus 1, is the only limit point and nowhere else. So drive set a days' will be 1 and minus 1, that will drive set for this. Okay? Okay, next to example is let d, ab, r is the set of all, point x y, such that, x minus, a whole square, y minus, v whole square, is strictly less than R. Then illustrate the closeness and openness of the folding.

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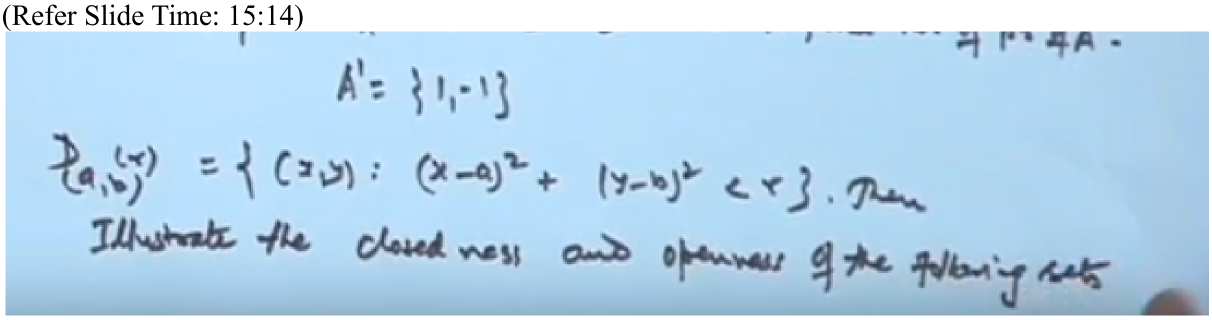

Then illustrate the closeness and openness and openness of following, following sets. The sets are 1,x, which is d 1, X, D, 0 0, 1, Union 10, Union d201, V is w, d, 0, 0, 1 union b 20,1. Okay, so solution is, what is our dab? DAB r means, set of those point X Y, such that, X minus A whole square, Y minus B whole square, is less than equal to R. So basically it is, less than R not equal, less than R, Okay? So if we take the d 0 0 1. Then d 0 0 1, is the set of all X Y, such that X square, plus y square is strictly less than 1, well the d20 one, is the set of 4 X Y, such that X minus 2 whole square, plus y square is strictly less than 1. So we plot these things, then this is our 0 0 1 and this one is, so this is 0 0 this is 20 and this is the set of these elements and this element excluding this part, this part is excluded, this portion are not there, this portion is not there. But in the set a, in this example a, we have added the point 10. So this point is also added, this point is also added, so this is our 10. So when we take the a, a part, the x is the union of these points, including this. But what is this is?

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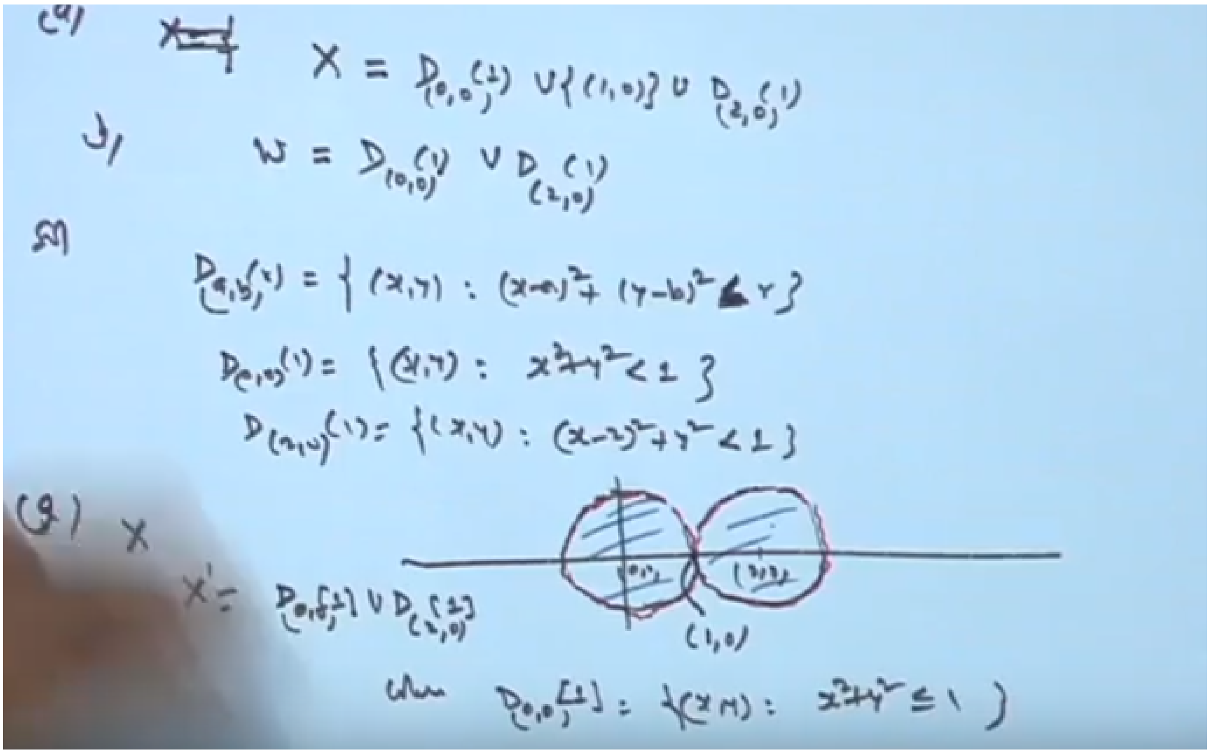

The drive set X days, is the complete X, drive set is d 0 0 1 and union of d201, where this close sign shows, where d 0 0 1, we mean, the set of all X Y, such that, X square plus y square less than equal to 1, similarly d20 means. So we have taken the point over the boundary. So set of all point inside, together with the boundary, is the closed set. So the limiting point of the set X, is the point, which are lying inside the disk, as well as on the boundary. But what happens to this x days'? So X days', Is it a subset of X? No, because X does not include the point on the boundary, so it is not the subset of X. Therefore all the limits point of this set, together is, not contained inside it and totally. Therefore this X days', and sorry, the drive set xbar, is not subset of this. Therefore is not closed. It means X does not include all of its limit point. So X is not closed. While on the point 1 0 1, is not an interior point, so it is not closed.

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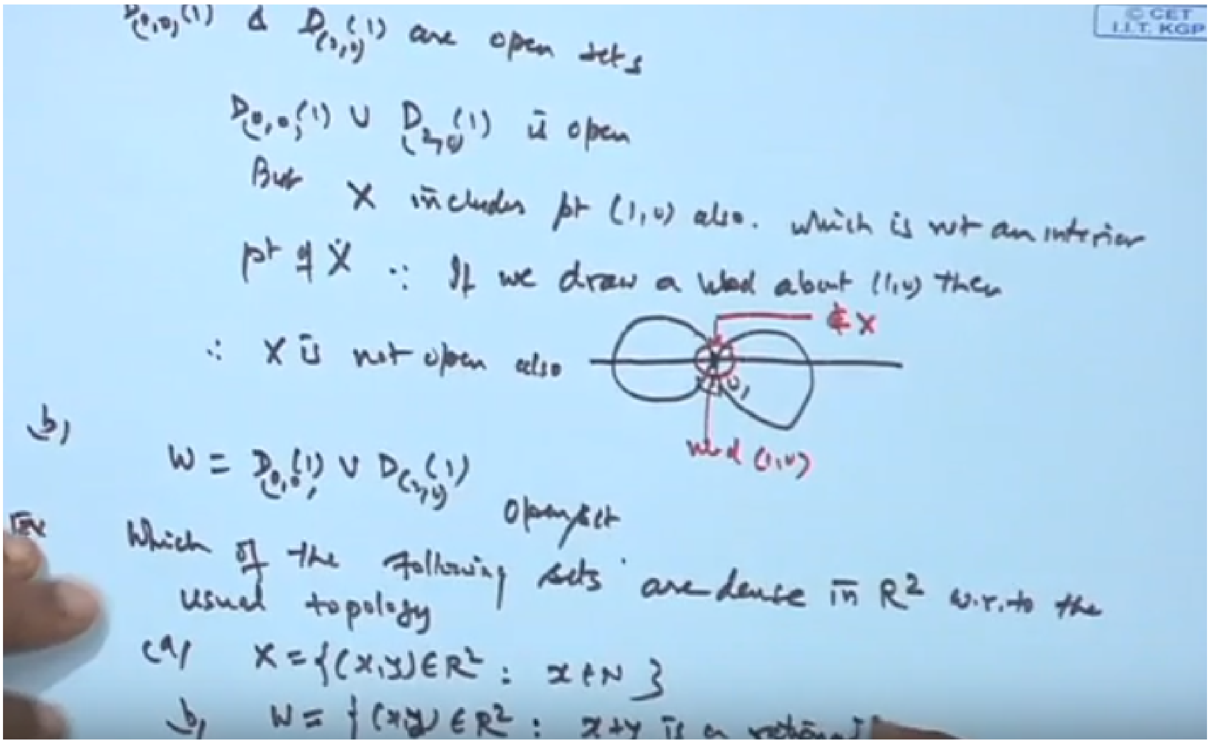

but what about the d 0 0 1? d 0 0 1 and d201 are open sets, are open sets. So their union is open. So d 0 0 1 Union d201 is open. But what is our X? X in crews point 10 also, 10 also so which is not an interior point, interior point of X. Because if we draw any neighbourhood, because if we draw a neighbourhood about the point 10, then the number is lying outside of it. This is 100, this is one, this is another and if I draw the neighbourhood, the neighbourhood will be, this one, this is the neighbourhood of 10. Now here these points are not there does not belong to X. So 10 is not a integer point. Therefore X is not open also. So it is neither closed nor open. While in case of second, what is the W? w is d 0 0 1, Union DG 2,01, that is one. So obviously it is an open set. Because it is a union of the two open sets, so it is open. Therefore w is the open sets.
Next example is, which of the following sets are dense? or dense in r2, with respect to, the usual topology, topology? The first set is set X, ordered pair X Y, belongs to R 2, such that X belongs to n. B is w, XY belongs to R2, such that X plus y, is a rational number, rational number, xy is rational number.

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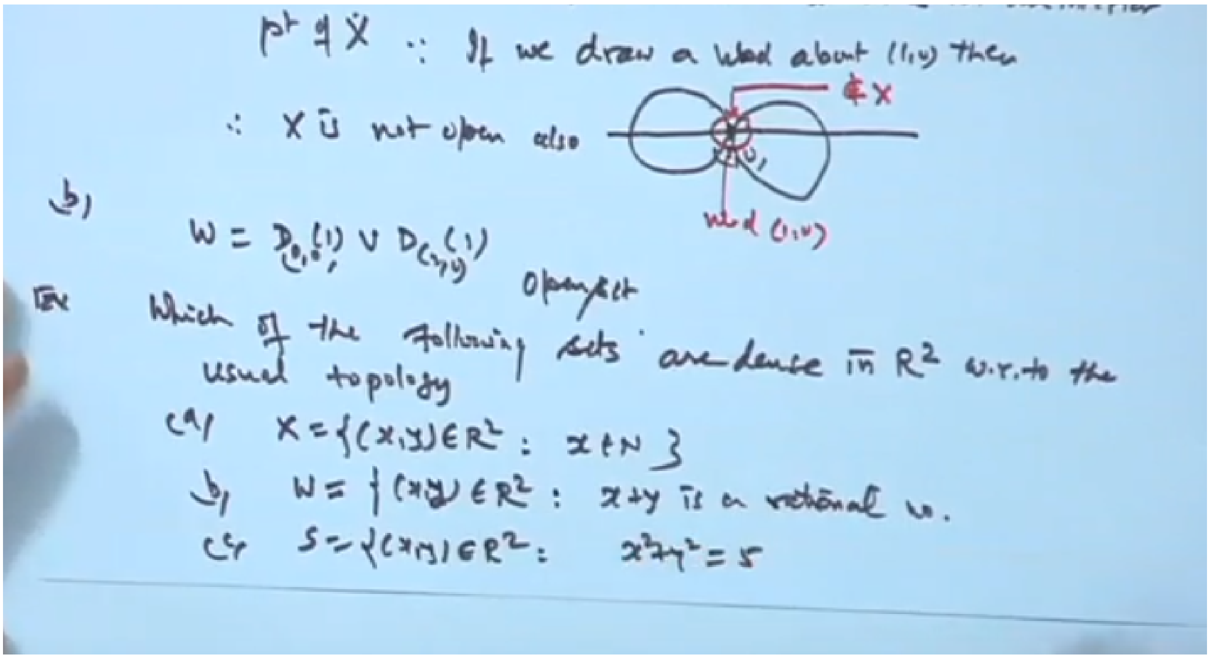

And C is s XY belongs toR 2, such that X a square plus y square, 5. Well the d is T; XY belongs to r 2, such that x y is not equal to 0. Okay, so let us see the solution. Before going to the solution let us define the dense set, dense set. What is the dense set?

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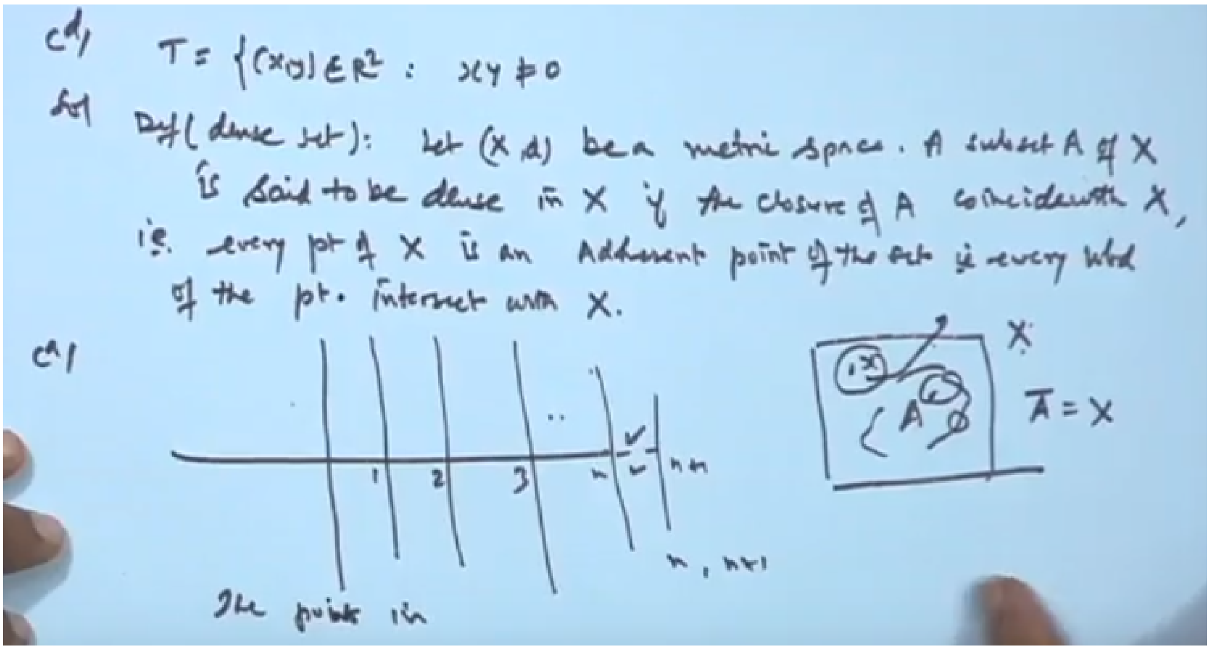

Let XD be a metric space, be a metric space, a subset a of X, a of X, is said to be dense. In X if the closure of a, a, coincides with X, coincides with X. The meaning of this is, that is, every point of X, X, is an adherent point of this set, point of the set. That is every neighbourhood, every neighbourhood of the poin,t of the point, intersects with X. So the meaning of this is, meaning of this is, if we take this is X and this is ra, then x subset a of X is said to be dense, if the closure of this is coincide AX. Or in other words, if we picked up any arbitrary point of X, and draw the neighbourhood around that

point, neighbourhood around that point, then it must include some point of a, X, so every inherited point of A, where every neighbourhood of the point, intersect with a, so whatever the point of A, will intersect with a. Then such a set is point is said to be, adherent point. And if the closure of a, coincide Ax then a is dense. Now the problem that.
What is the first problem is, xy belongs to rx2, where x is a natural number. So let us see. Suppose these are the natural number, one, two, three, these things are there, Okay? Now x is a natural numbers, a 1 2 3 and so on, now, all the points in between, n and n plus 1. If we take this is suppose, n and this is say n plus 1. So this point and this point in between this are neither the limit point.

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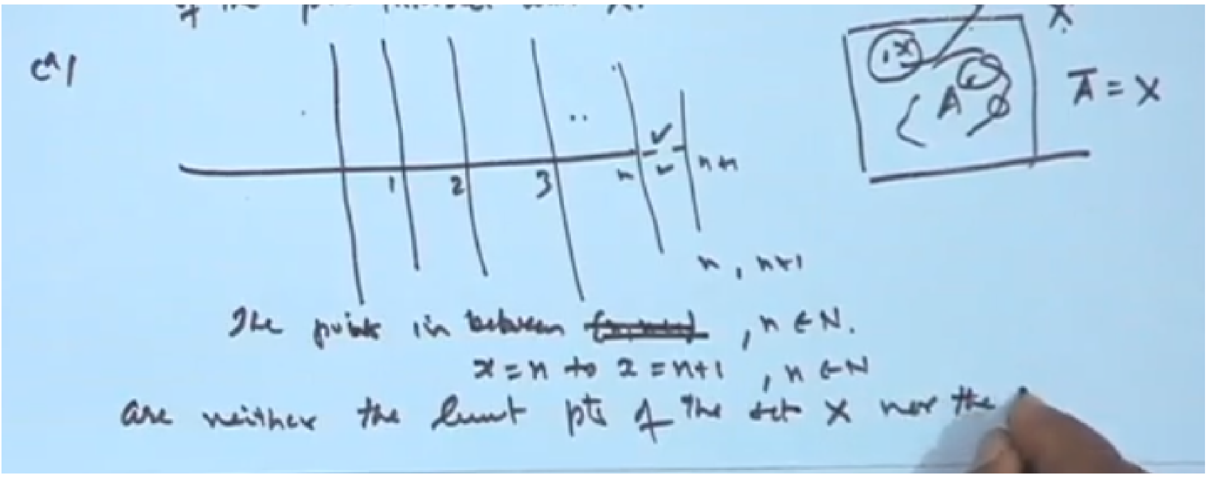

So the points in between and n plus 1, where n is natural number, in between n to n plus 1, that is in between, X equal to n to x equal to n plus 1, where n Is a natural number, so in between these steps all neither, all neither, the limit points, of the set. Of the set X, nor the points of X, point of X, is not the point of x.

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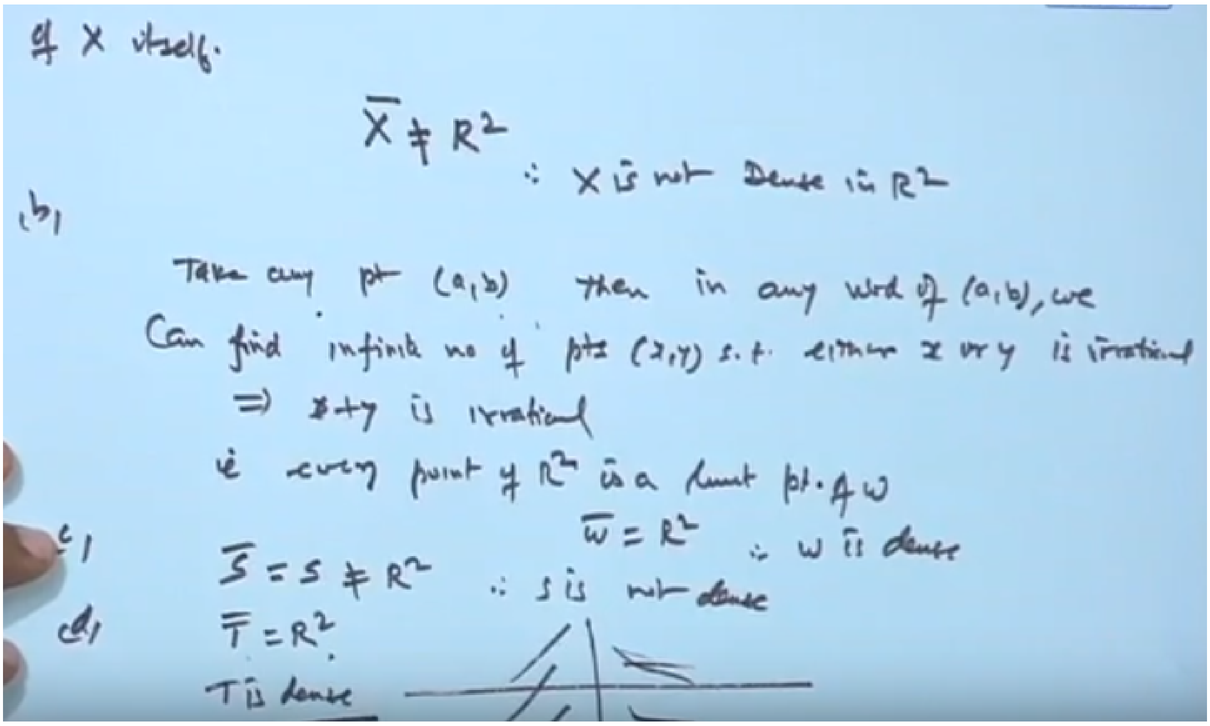

Therefore when you take the closure of this, set of all limit points of this, it does not cover the entire R. Therefore it is not equal to r2. Therefore X is not dense in R2.
The second example shows, that when we take the X Y, such that X plus y is rational number. So if we take any point AB, so take any point AB, then in any neighbourhood, in any neighbourhood of the point AB, we can find, we can find in finite number of points, number of points XY, such that either x or y is irrational. Because rational and irrational number they are very close to each other, Okay? So this implies that, x plus y, is a irrational number, always. And once we have we are having the irrational point, then we. So we get every neighbourhood, therefore, every point of R2, is a limit point, limit point of W. Because W consists of those points, which are rational points. So every point of R 2, is a limit point of the W. Therefore W bar is R2. Hence W is dense. Is it not? because, if you take any point, XY, in neighbourhood, it contains both the point, rational, irrational. So when you take the X plus Y, it becomes irrational. So these points X plus y becomes the limit point of these irrational points. Therefore it is a it is W bar becomes R2, Okay?
The third case see, in that case, the closure of S bar, in which is basically, the X plus, X square plus y square, is 5. So closure of S, which is S, is not equal to r2, therefore s is not dense. Because it does not have any point of this, is not the, is the limit point of itself, but not the sequence. Any neighbourhood does not belong to them, Okay? So it is not them. D, when you take the d the t-bar equal to r2. What is the XY? If we take this, then X Y is get not equal to 0. So basically we are getting this, except this line, where it is 0. So it is closure of this all to. Therefore D is dense.
That is all.
Thank you very much.