Model 3

Lecture – 15

Cantor Sets and it's Properties

Okay so we will take a few problems, which are itself an interesting problem in itself is a very interesting problem and also it involves some other concepts in connection with these sets.

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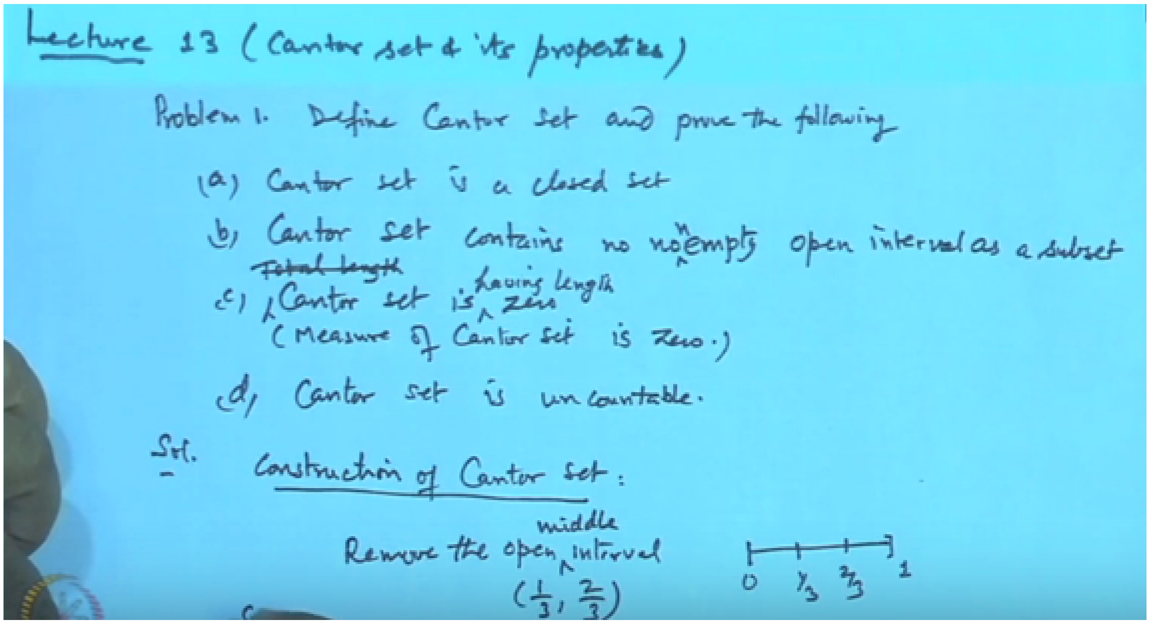

so let's see the first problem define the Cantor set and so that and prove the following and prove the following, say no one or a part the Cantor set is closed set ,Cantor set is a closed set. second we wanted to saw that Cantor set contains Cantor set contains no non-empty open Interval, intercept contents or contains no non-empty no non-empty no non empty anyway non-empty open interval as subset, is a subset. Third point which you want to prove that Cantor set is an uncountable set. Cantor set the length of the Cantor set you can say that the Cantor set has a length the total length of the Cantor set total length total length of the Cantor set you can say Cantor set is 0, in fact the measure of the cantor is is 0. we wanted to show this is and that length or we will say the measure is 0. The measure of the Cantor set instead of saying total length, we will just say the measure of Cantor set is 0, Cantor set is having length 0. Length 0. Okay and D the measure of the Cantor set I will because we have not discussed the measure that is why wanted to avoid this term measure. Okay and then fourth one is Cantor set. Cantor set is an uncountable set, is Uncountable. So we wanted to establish this following four things about the cantor sets. Let's see the source so first let us define the Cantor set. So what is the Cantor set? The construction of Cantor set. let us consider a closed interval 0 -1 divide this interval into 3 parts 1/3, 2/3 and then remove the opening interval and remove the open interval open interval 1 middle one open interval one third open middle interval, 1/3, 2/3, so closed interval 0 1 we are dividing first into three parts and removing this middle portion or the middle open interval we are removing,

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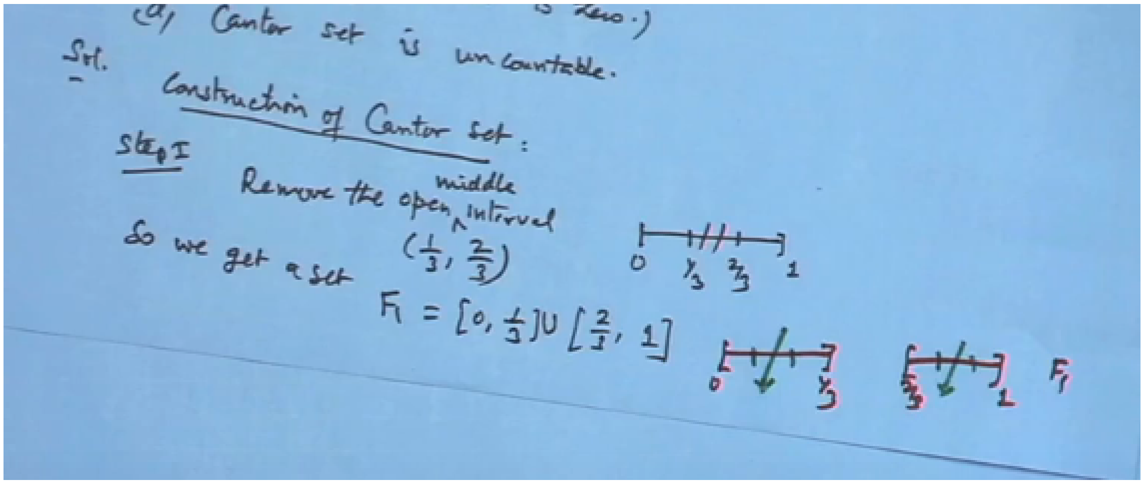

so the set which is obtained, so we get a set let it be denoted by F 2 or say F1 as the remaining set will be 0, 1 third closed interval union of two third one. so in the first step what we are doing we are taking a closed interval 0 1 and from there we are dropping the middle portion this portions we have dropped so the remaining one will be this interval 0, 1 third and then two third one ok this is the world remaining interval and this we said denoted by F 1 now what we are doing is now the next again subdivide this intervals the river into 3 parts and let it be divided by this interval again in three parts. This interval also again in three parts and then from here we drop this one we drop this portion this middle open four cells.

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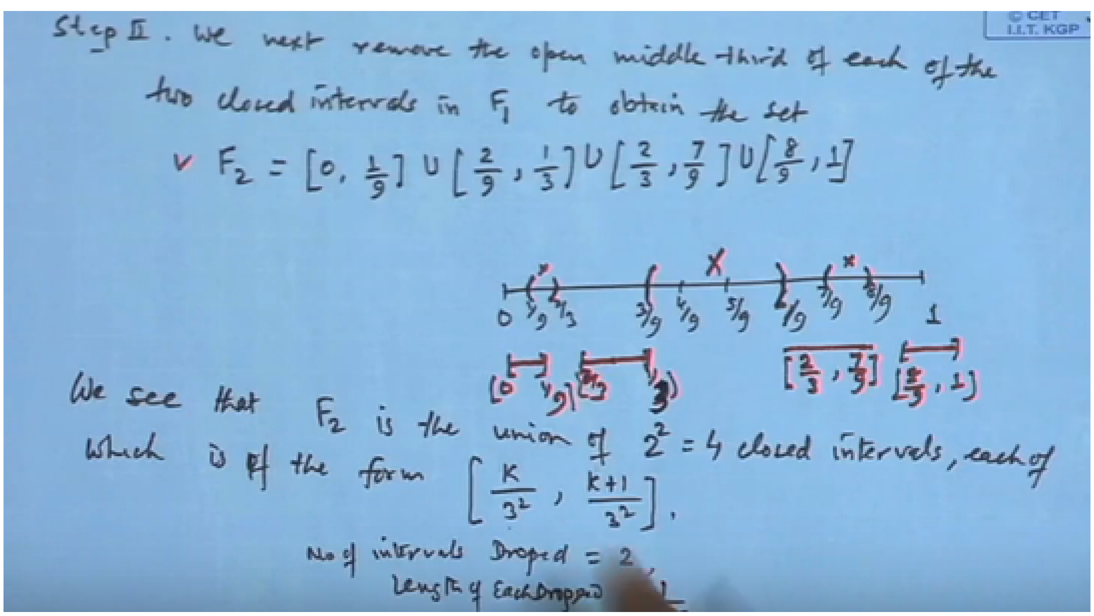

so what we do is we next remove the second step is we next remove the open middle interval remove the open middle third a middle third of each of middle third we open middle open middle third of each of the two closed each of the two closed intervals in f1, means we are dividing further of these remaining intervals into three parts and then out of these three of intervals we are removing the middle one open intervals and to get that set to obtain this set, say which we denoted by F2,[0, 1 by 9] Union [2y9, 1/3] Union, [2/3, 7x9] Union, [8 by 9, 1] okay? That's what. so basically this is if we have it this structure suppose I have 0 1, then what we do first we had removing this is say 0, 1 by 9, 2 by 9, 3 by 9, that is one third 4 by 9, 5 by 9, 6 by 9, 7 by 9, 8 by 9 so what the remaining persons was earlier we have already removed one-third two-third this portion was already dropped you know this portion was already dope this was dropped. Then in this interval we are dropping this portion we are dropping from here this portion so this portions are dropped so remaining portions will be this 0 1 by 9, okay then this portion is dropped so 2 by 3 and then 1 by 9, this portion then this portion is dropped so here 6by 9 means 2 by 3, then 7 by 9 and this portion is dropped so this one is then this one is 8 by 9 and 1 so basically these intervals are their closed intervals are left, Union of these we denote by F 2 okay. now each of this F2 we see that we see that F 2 is the union of you know 2 to the power 2, that is 4 closed intervals, close intervals each of this each of which each of the form is of the form K over 3 square, K plus 1 over 3 squared, you know because this is also K 0, so you are getting 0 & 1 by 3 square when you are taking K is 2 you are getting this one K is equal to I think this is 1 by 1 by 3 here is 1 by 3 ok so 1 by 3 so you are getting this one and similarly when you take this K is equal to 2 you are getting k equal to 6 you are getting this one k equal to 80 you are getting this one like this so all these of this form and then the length of this if you look the basically the length of the first one draft 1 which you have dropped what is the length of the dropped interval is 1/3 and here and only one number of intervals we are dropping the number of interval which is dropped is 1 now here the number of intervals number of length of these intervals the number of intervals dropped is how many? one and two and length of each interval dropped interval will what length of each dropped interval will be 1/3 squared so each one will be of length say this one 1 by 3 square you know so that was the first interval 2 intervals each length of 1 by 3 square 9 length we are dumping for this okay like this so continue this way so if we continue we continue this way.

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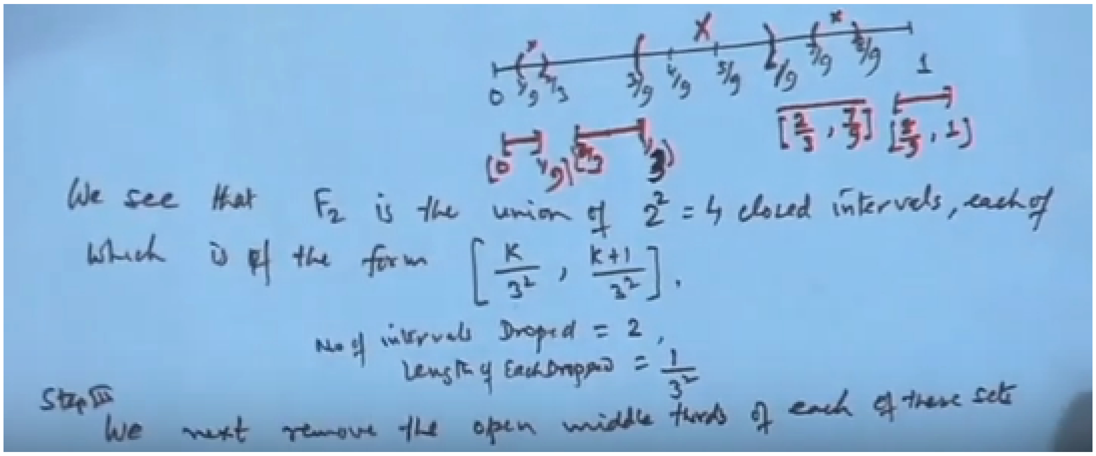

Then what may be next remove step thought we next remove we next remove the open middle third, middle third, thirds of each of each of these sets to get F3.

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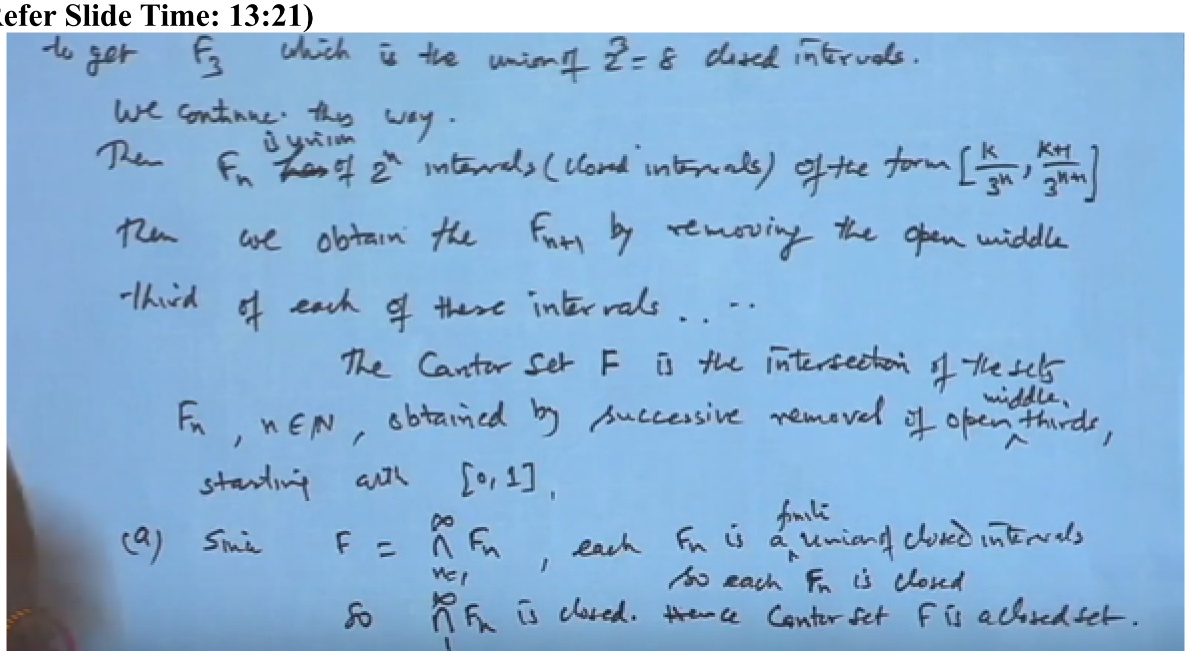

Which is, the union of 2 to the power 3 that is 8 close intervals close intervals. And like this we continue this way continue this. so what we get that in the Nth Step if we take then an Nth step then Fn at the Nth step had constructive has 2 to the power n into 2 to the power and intervals, close intervals of course, to the close intervals of the form, of the form K over 3 to the power n, k plus 1 over 3 to the power n plus 1, like this and then we obtain then we obtain the set or then FN is the union of 2 to the power n intervals, intervals of the form this one. Then we obtained the set F3 set Fn plus 1 by removing the middle again when in the open middle of in middle third, open middle third of each of these intervals, intervals and like this see. Then what is Cantor set? the Cantor set Cantor set denoted by say capital F is the intersection is the intersection of these sets of this Fn, when N is a integer set of natural number of positive integers obtained by successive by successive removal of open third intervals open thirds open middle third open middle thirds middle curves starting with the closed interval 0 1 so this set F will be that intersect. so basically the Cantor set will be the collection of all the endpoints of these removed intervals, so in fact we get since the closure so we know so basically this set is the collection of all the points which are the which are the corner points of these deleted I intervals .okay, now we wanted to establish these results the first result says the Cantor set is a closed set. which we want to show so now first is since Cantor set F is the intersection of Fn, is 1 to infinity a countable intersection of Fn and since each Fn is closed is a closed interval is a union of closed intervals is it not and close intervals finite union of is a finite union of closed intervals so each Fn is closed is closed. And when you take the countable intersection of the closed set then it is closed, so the countable intersection of Fn, 1 to infinity is closed hence the Cantor set F is a closed set. So this is the first we wanted to prove.

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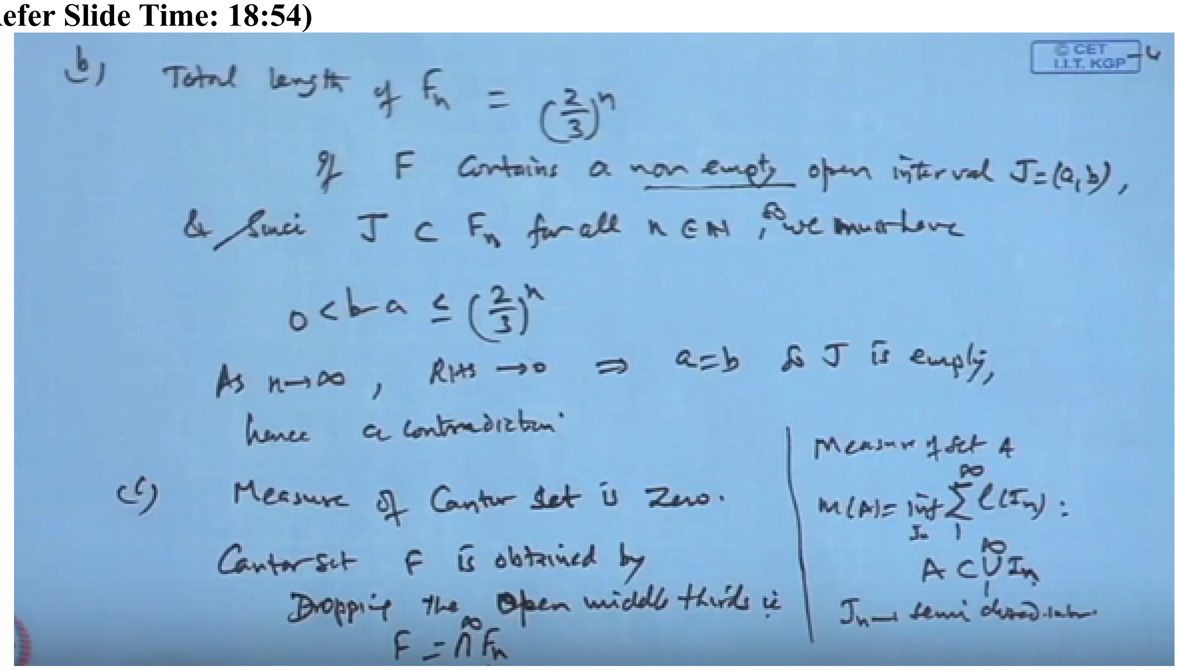

Second one, we wanted so that Cantor set contains non, non empty intervals as a subset and then length of these intercept method of the set 0. Ok so suppose what is our open B part the total length of Fn, the total length of Fn is basically what? is nothing but the 2 by 3 because if we look that one the first one interval is of length when it chooses the f1 the f1 this length is 1/3 plus 1/3, so it is equal to 1/3 plus 1/3 that is 2by3 when n is 1 it is 2 by thing when n is 2 the total length is coming to be this n is 2 so what is this is this is 1 by 9, 1 by 9, 1 by 9, 1 by 9, the 4 intervals of length 1 by 9, so it is two square by three square that is when n is 2 it is 2 square by 3 squares of

4 by 9 so similarly when FN come it length will be 2 by 3 and power n. so if suppose if suppose F contains, if F contains a suppose F contains, a non-empty, non-empty open intervals say J of say (a, B) then F is the intersection of all these opens intervals so obviously F1 covers F2 cover FN and so on. So basically we said this interval (a, B) will be so since and since J this is this open twice is contained in Fn, for all n, this is 2. Is it not? so the corresponding so we must have must have the length of this is (b minus a), which is positive, must be less than equal to the length of this Fn, but as n tends to infinity, this right hand side of this goes to 0, the right hand side goes to 0. So this implies that a is equal to b. it means we don't get any integers, so, so J is empty. J is an empty set empty, hence a contradiction hence a contradiction. Because what we assume that it f contains a none empty of an interval so which is not true contradiction. Hence the Cantor set will not contain any open nonempty open interval as a subset. Now third part is that measure of this Cantor set, measure of Cantor set is zero. in fact I don't want to interview major what is the may suppose I take the interval a, b, then when we say a,b is interval open interval closed interval the length of the interval is b minus a. so that is called the measure of the interval okay similarly when we say other set, when the set is there then it is difficult to find out the interval because if it's a set Sol is scattered, then and lying in between the interval a,b then if you take b minus a then the length of the set or the measure length is not exactly same because it is more what which will contain some more point which is not available in the set. so in order to get the measure of this set we introduced the concept of the measure in terms of the length and in fact when we say measure of the set then this is the infimum of measure of a set A means it is denoted by A, it is the infimum value of Sigma, the length of the nterval In, 1 to infinity, such that the countable union of In covers A where, In is a intervals open interval or semi closed intervals which covers In. are semi closed all open intervals intervals containing this. So when you take the length of the interval is possible find the sum and take the infimum over all such In’s, we get this infimum exists then we say it is a measure. so in nutshell or in the rough sense we say measure of the interval is nothing but the length of the interval. Since the set a F which is a Cantor set contains the points basically the points endpoints of the removed interval, so what we want so the length of the total set is 0. The measure of this set at 0. so let us see the total length of the movement okay. the Cantor set F is basically obtained, is obtained by dropping the middle the, open middle thirds, when you divide the whole closed interval to the three parts and opening and then middle third you are dropping. so is obtained by dropping this and so remaining one is nothing but that and Cantor set. When n is sufficiently large, is obtained by this and for any sufficiently large mid third n is successive middle third of this is it not. So that is what intersection of Fn. that is the intersection of Fn will be intersection of Fn1 to infinity, that will be the Cantor set. Is it not? That is F, where Fn’s are the remaining one and then we take the intercept. So when you drop the middle point, then what happens to this?

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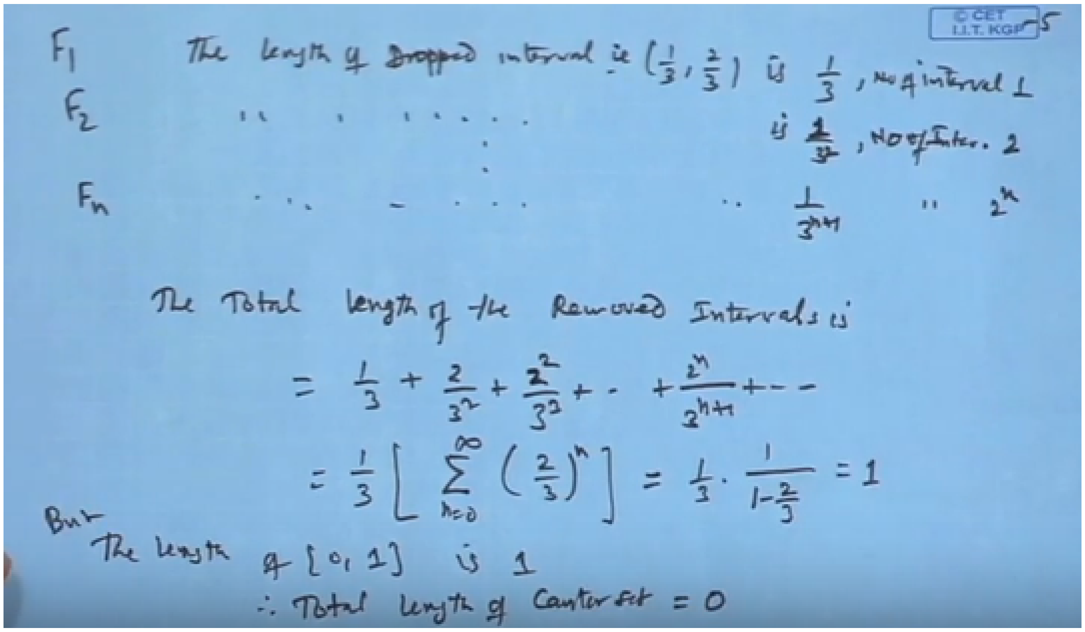

in the first case F1, the length of the dropped interval that is one-third, two-third is one-third. The in case of F2 the length of the draft interval is 2/3 and number of the intervals and number of the interval is 1. here the number of the intervals is 2, then continue this so in case of Fn when you are taking the Fn then basically it is the length of the draft interval will be 1 by 3 to the power n plus 1, 3 to the power n this in number is 1 sorry this one and then each one is 1 and but the length will be F2 3 square sorry squad number is 2 so here when it is said in that second case the number of the interval will be 2 to the power n. like this so it is 1. so the total length the total length of the removed interval, intervals is nothing but what 1/3 plus 2 over 3 square, plus 2 square by 3 cube, and so on plus 2 to the power n by 3, to the power n plus 1, and so on basically this is equal to 1/3 Sigma n is 0 to infinity 2 by 3 power n. now if I add this is a geometric series the geometric series the first term is 1 so it is 1 third, 1 over a minus R and the length is coming to be 1. so it means when the closed interval is there from the closed interval the length of the closed interval from where we I started is 1, and the dropped interval is also having length 1 and the remaining one is the Cantor set. Therefore the total length of the Cantor set is 0.