Model 3

Lecture - 14

Course

On

Introductory Course in Real Analysis

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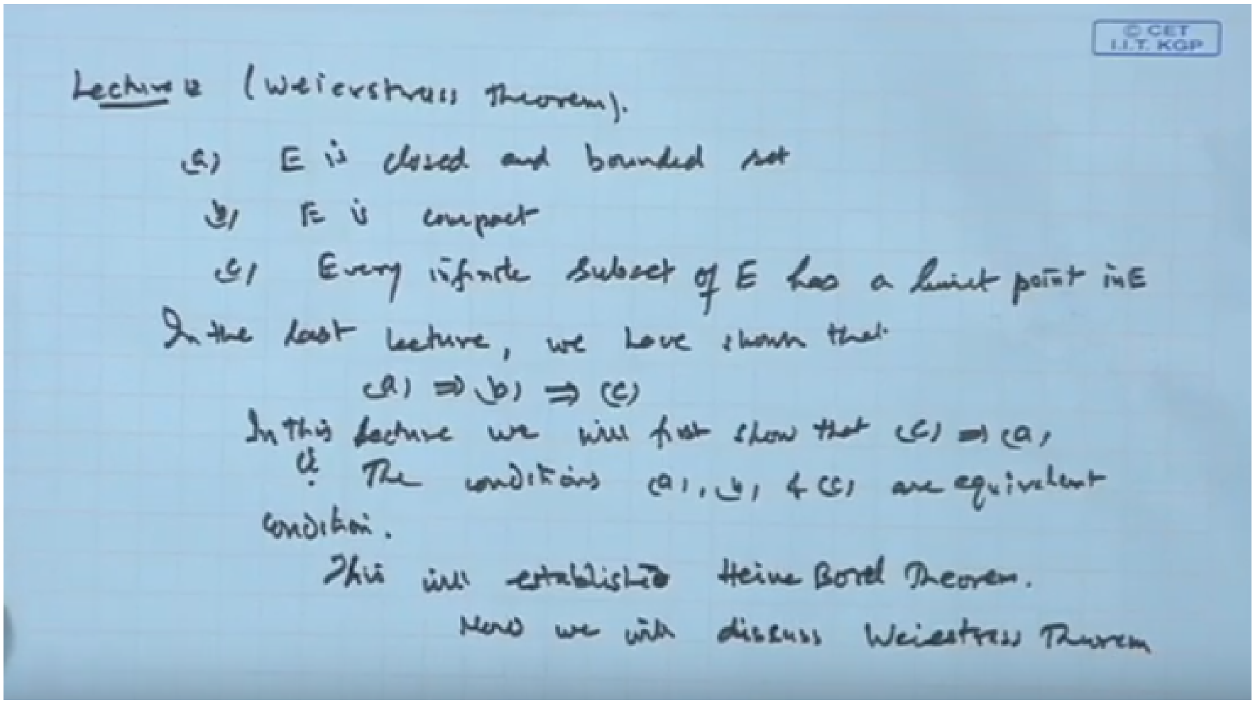

Okay, in this lecture, we will discuss the restores Weirstrass theorem. In fact in the previous lecture, we have established the three relations, ABC. The relations was, a, was, that E is compact, E is closed and bounded, sorry, e is closed and bounded set, second was, E is a compact and third is, every infinite subset of E, has a limit point in E. So these three conditions listed and we have proved in the last lecture. We have shown that, the first condition a, implies the second condition b, implies the third condition c. So first before we starting, for going for the Weirstrass theorem, we will first prove that c implies a. So, that all three conditions, ABC, are equivalent. It means, when e is closed and bounded set, then e will be compact and every infinite subset of e, has a limit point in e. So all three are equivalent conditions for that, Okay? So in this lecture, we will first show, that c implies a. That is the conditions, conditions, A, B and C are equivalent conditions, are equivalent condition. In fact, once we establish this condition, then automatically this will prove or this will establish, Heine Borel theorem, sorry Heine Borel theorem, Okay?

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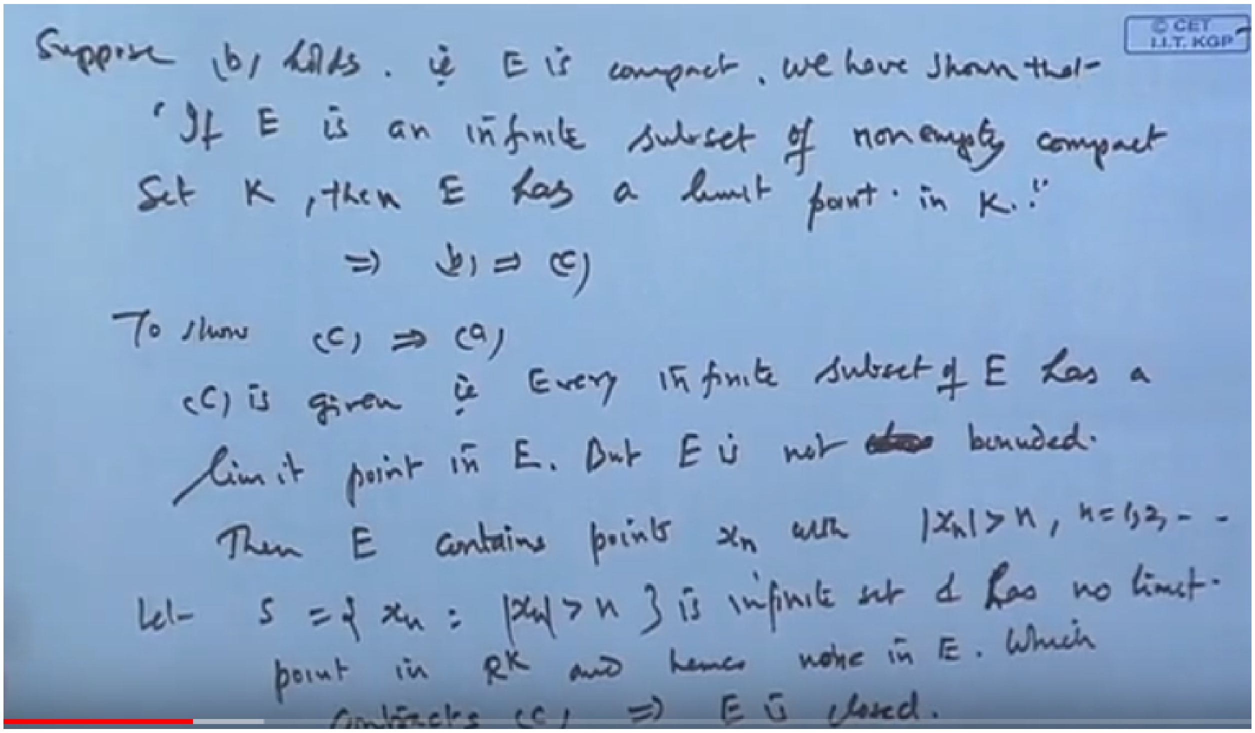

Now we will discuss Weirstrass Theorem, thank you. Now, to show, C implies a. This we will prove by contradiction. So given that, every infinite subset of e, has a limit point in E, Okay? So c is given. That is, every in finite subset of e, has a limit point in e, this is given, Okay? So we wanted to prove a is closed, e is closed and bounded. But suppose, but e is not closed, not bounded let it see first, not bounded. It means, if it is not bounded, it means there will be a sequence of the points in E, which can, whose bond will go to infinity. Means each x 1, x 2, xn, will be there, such that mode of xn will be greater, than any arbitrary number n. So we get, then E contains, E contains the points, xn, with the property, that mode of xn is greater than n, when n is 1 2 3 and so on. x1 will go greater than 1, X 2 there, so the limit of this xn will not exist. So let s, be the set of all such xn , such that xn and such that xn is greater, the mode of xn is greater than n. Then obviously this is an infinite set, is infinite set. Because if it is finite, then we cannot, we can get the bond for xn. So it is any finite set. And clearly there is no limit point and has no limit point, limit point in RK. This is the set we are choosing in our s consist. So once it is so it has no, hence has none in E and, and hence not in, none in E. Means, this sequence will not have any limit point in E also, Okay? That is one thing. Okay, thus is born. So, but what is this? This contradicts C, which contradicts C, because C said that if you take any infinite subset of e, it must have limit point E. S is an infinite subset of e, but it does not have a limit point in C, so contradicts. This implies that E is closed.

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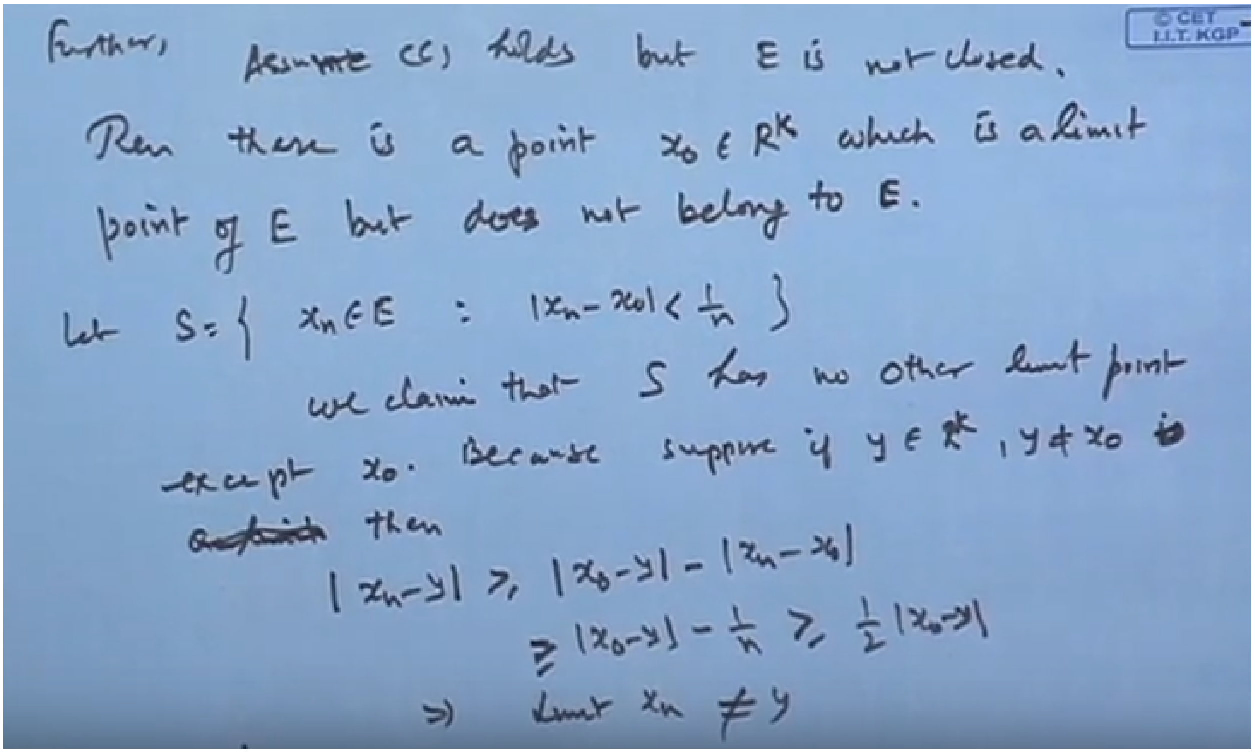

Again to show E is bounded, so support further. Assume C holds, but E is not closed. It means the limit point of the E, all the limits points is not an E. So thus we can get the limit point, then there is a point, then there is a point, say there is a point, suppose X naught, belongs to R K, which is a limit point of E, which is a limit point of E. But, which is limit, but not a point of, but does not belongs to, does not belong to E. Because all the limits point does not belongs to E, because E is not closed. So if it is not, then we can get this, sequence of the points in E, that is, so the set Xn, belongs to E, such that mode Xn, minus X naught, can we made less than say, 1 by n, this collection will be there, Is it not? So set of all points E which satisfy this. In some sequence will be obtained via this. Now let us find S with the set of those points. Let s, be the set of those point of a, each satisfies this condition, Okay? We claim, that s cannot catch x not as a limit, s has no other limit point, s has no other limit points, no other limit point, except X naught. Because, if suppose Y is another point, of this suppose, because suppose if Y belongs to Rk & y is different from X naught, is a limit point, is a limit point suppose, then we get , then the distance between Xn minus y, because you know if y then a limit point we will just clear.

The suppose y is another point, which is different from x, then we can say XN, Yn is greater than, equal to, x naught minus y, minus Xn, minus X naught, this way. But X naught minus y, because X naught and y naught, is different, so we can just put it as it is, x naught minus y. Now Xn minus X naught is less than, so minus of this is greater than y. Now n, we can choose in such a way, so that the whole thing and this is greater or equal to, the whole thing, is greater than equal to, half of this. Now, Y and X are different, so this is fixed point. It means, that this as n tends to infinity, X n does not go to Y. So this implies the limit of xn, is not y.

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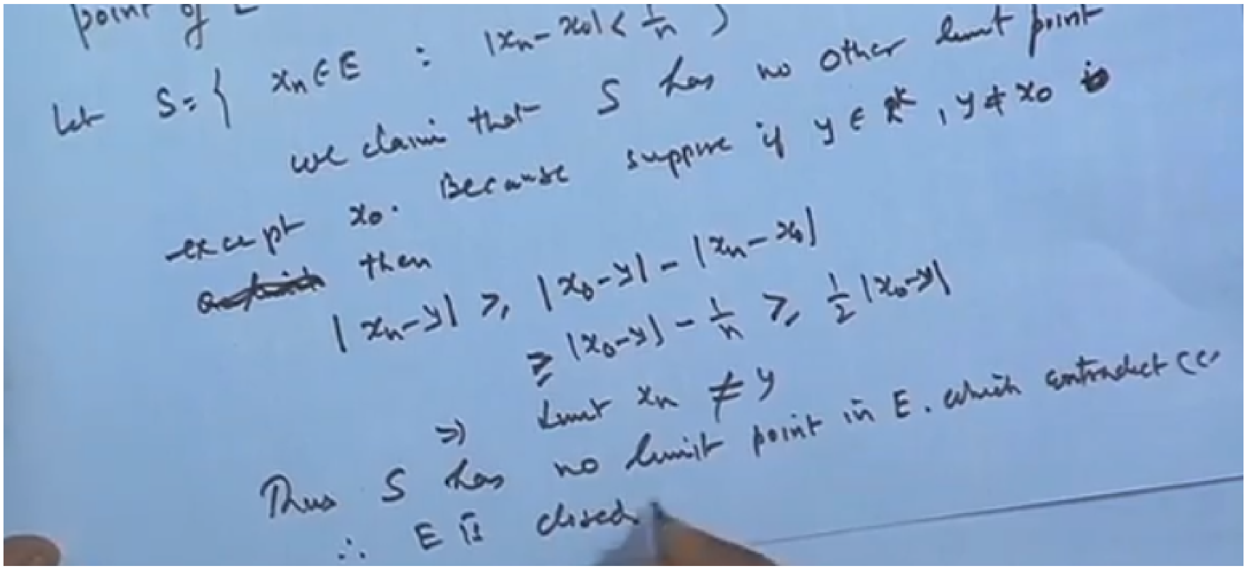

So it cannot have a point, other than X naught, as a limit point. But X naught is a point, which does not belongs to our set E. So this sort y, is not a limit point ,of any, the has a no limit point in E, thus S has no limit, because X naught is not in E. So X h as no limit point, no limit point in E. Hence contradicts our assumption three which contradicts C. Therefore our E is closed. So equivalence of one and two, will implies the Heine Borel Theorem. Okay? B will imply Heine Borel Theorem. That is the proof for the Heine Borel Theorem.

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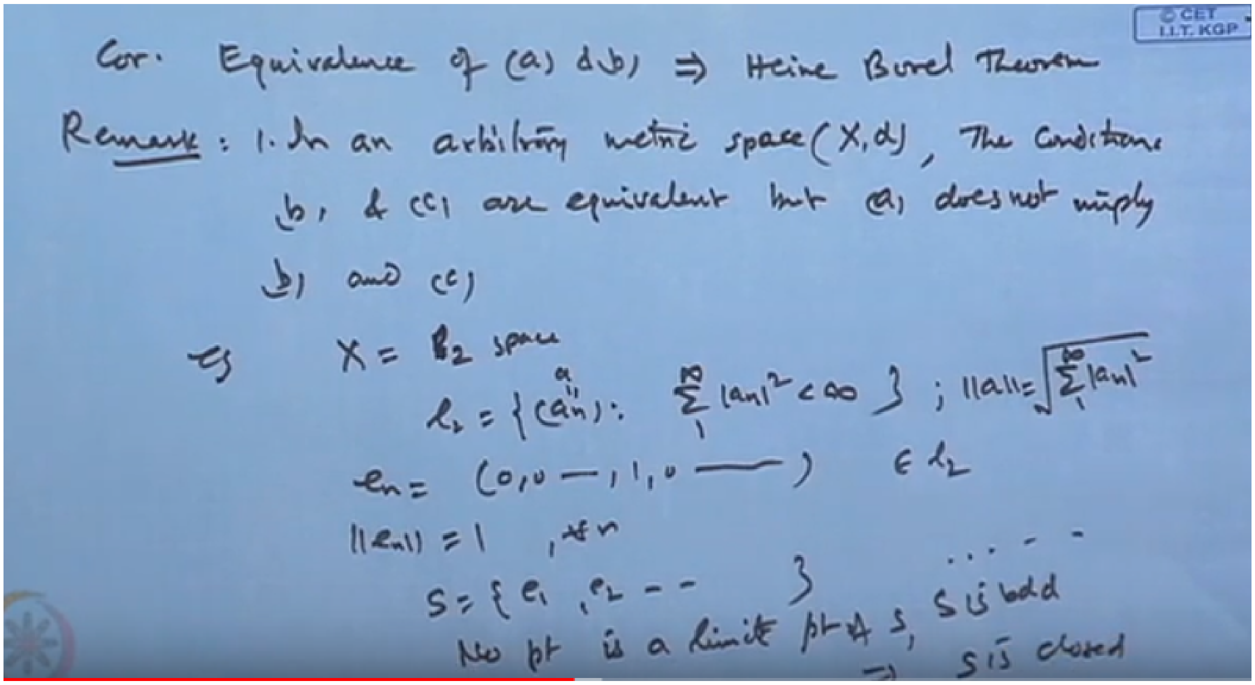

Okay? Now here we put some remark. The remarks say, that in an arbitrary metric space, in an arbitrary metric space, X d, the conditions B and C are equivalent, but, but A does not. It means, in general, B and C, it does not imply, but A it does not imply, B and C in general, it means, for arbitrary metric space, the compact set and the infinite subset of E, as a limit point, all equivalent. But if a set E, is bounded and closed, then you cannot say whether it remains compact or it will have a finite, infinite subset of e, will have a limit point in E. If not, may not be true. For example, if suppose I take x as a Hilbert is, l2 space. L2 space means, set of those sequences in such that sigma mode a n square, 1 to infinity, is finite, in l2 space. And if I take the sequence en h, 0 0 0 1, 0 0, this is the points belonging to l2 space. Now if we take the norm of en. Now here, norm of this, if suppose this, I denote by a, the norm of a and of course this is a case of functional analysis, but I will show, we are not much going in detail, but this is the norm. So norm of e n, is 1 for each n. Therefore e 1, e 2, e n this set is there, e 1, e 2, en and so on. Then each point, each point, having norm 1. Is it not? It is bounded. So s is bounded. And none of the point is a limit point. No point is a limit point. Because it is a point set only, limit point of s. Therefore we can say all the limits points of s, belong h. So this shows s is closed also.

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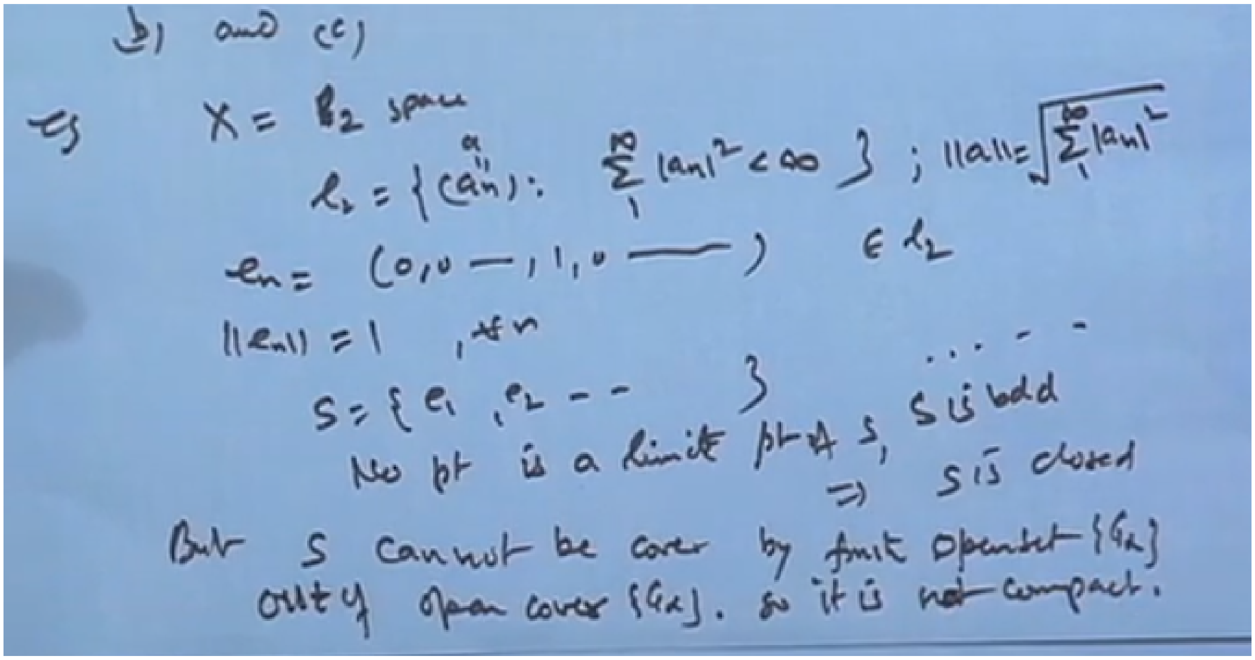

So it is a closed. But, it is an infinite set, so we cannot cover it, by means of a finite sub cover. But s, cannot be covered, by finite, open sets, say, G alpha, out of out of open cover G alpha. Means many open cover here, we cannot choose the finite curve which come, because it is infinite. So it is not compact, it is not compact. So this show contradicts our, Okay? So that is what we are not going detail for this, because it is part of the functional issue.

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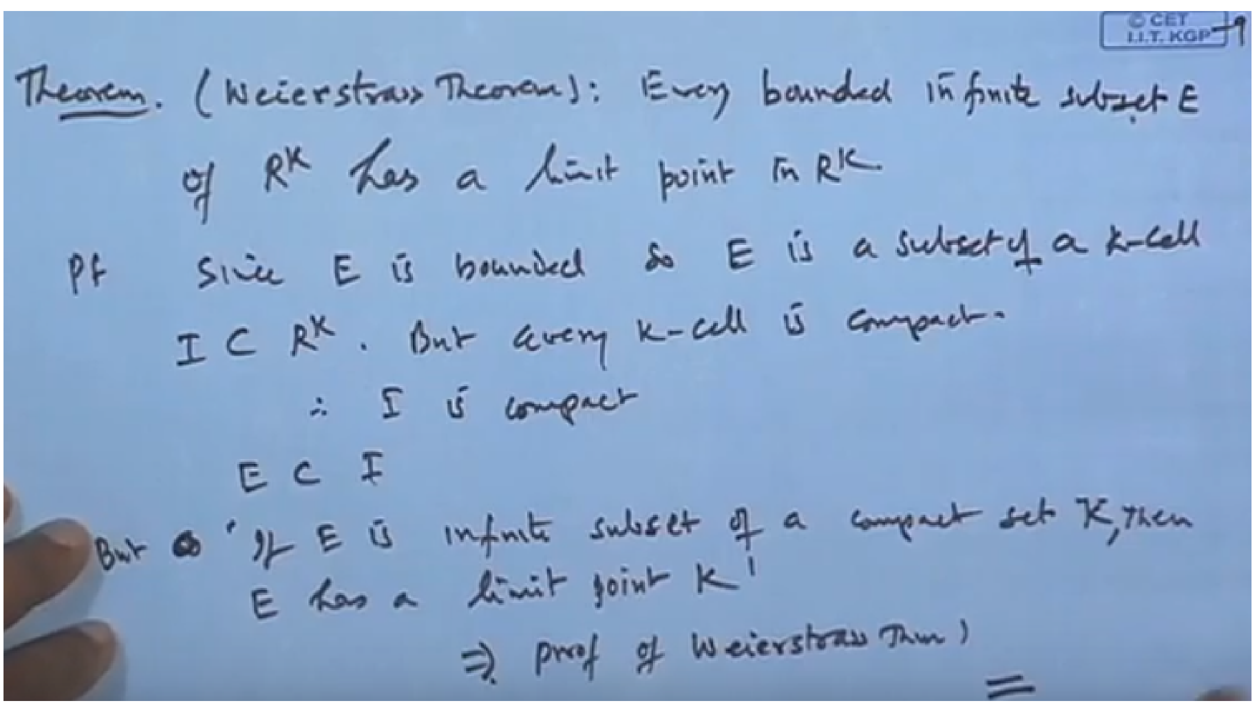

the next result which we say the Weistress theorem. The theorem says, every bounded, every bounded infinite subset, infinite subset of RK, every bounded in finite subset of RK has a limit point in Rk. The the proof is very easy. Since E, every bounded and infinite subsets, say E, of RK, since E is bounded, which is infinite also, set, so E a subset of a K cell, K cell I, which is contained in Rk, because it, bounded means, it will covered by a K cell, Okay? But, what is the earlier theorem says? That every k cell is compact, but every K cell is compact, therefore I is compact. So E which is contained in I, which is a compact set, Okay? And what this result says. One result that which we have already shown, that, that every, if we choose a infinite, then e has limit point in K, this we know. But we know this result. So using this implies, that Weistrass theorem, the proof of the the Weistrass theorem. Weistrass theorem, we wanted to prove wounded, infinite subset of E has limit point of RK. Does this complete the proof of this, Okay?

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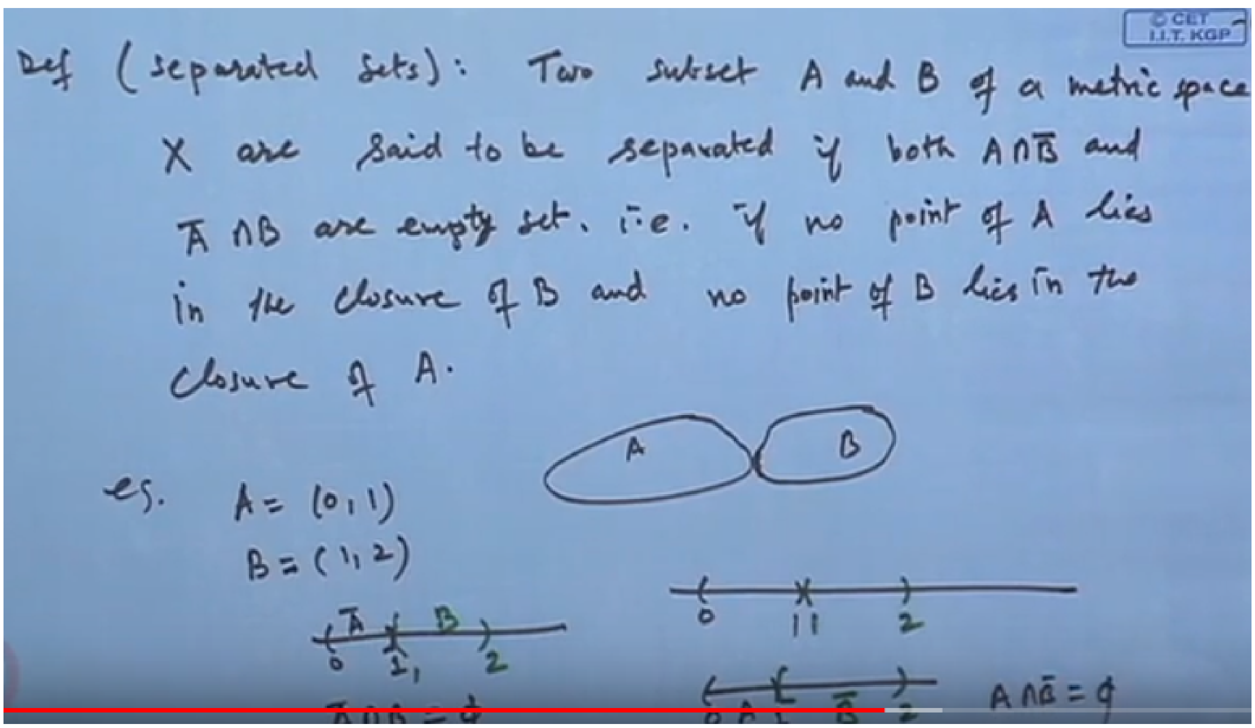

Now we have one more concept of said, which comes in the tail, is that, Connected Set. Okay, so let us define, connected set. So first we have a separate set, separated sets, Okay, two subsets, two subsets, A and B of a metric space, of a metric space, capital X, are said to be, are said to be separated, if both a intersection B closure and a closure intersection B, are empty set, are empty sets, are empty sets. That means that is, that is, if no point, no point of a, if no point of a, lies in, lies in the closure of, of B and no point of b, no point of b lies in the closure of a, closer of A, Okay? Then we say, these two sets are separate. It means this is the one set a and here is another set b, Okay? b, not this one. Now if I take the limits points of a, or the point of a, then it should not belongs to the closure of B. And if we take any point of B, all its limit point if it does not belongs to a, then we say a and B are separated. For example, if I take this set say, 0 1 and 0 1 suppose I take the set 01, b is the set, say 12, if i look this. So this is the 01 and here, it is 12, this is 12, this is. Now if we look, what is the closure of b? The closure of B is this. So this is our 01, is the set a. And then you take this, then it becomes b2, that is the closure of b bar, b bar, which imports, one and open at two. But one is a point, which belongs to only b bar, does not belong a. So I neither the point of b, nor its limit point belongs to a. Similarly if we take the others, say this is 0 and 1, this is the closure of this and the open sets, this one is 0 2, 12, 12 is our B. So A bar, intersection B, is empty and a bar intersection B, is empty. Then here A bar, A intersection b bar, Is also empty. So A and B, are separated set. Okay**?**

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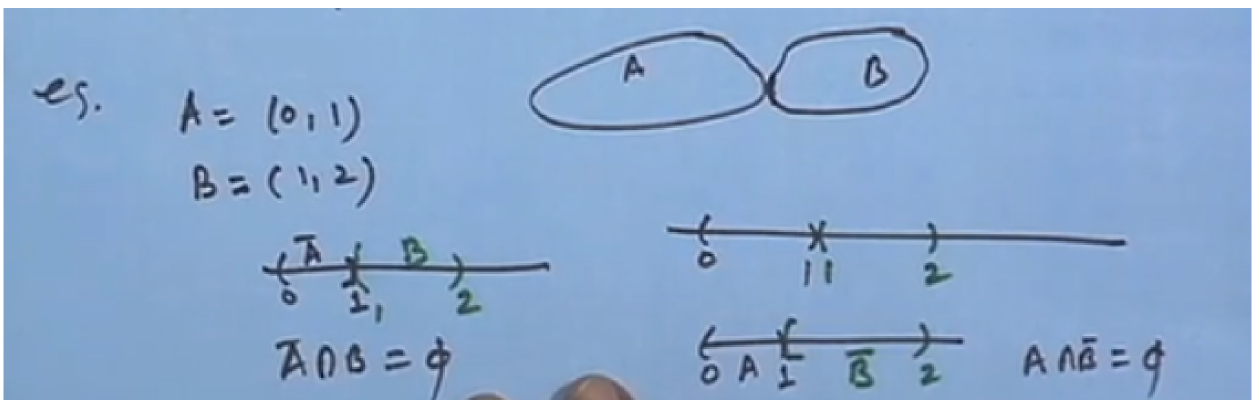

There is a difference between the disjoint set and separated. Separated set, of course are disjoint, but the disjoint sets, not be separated.

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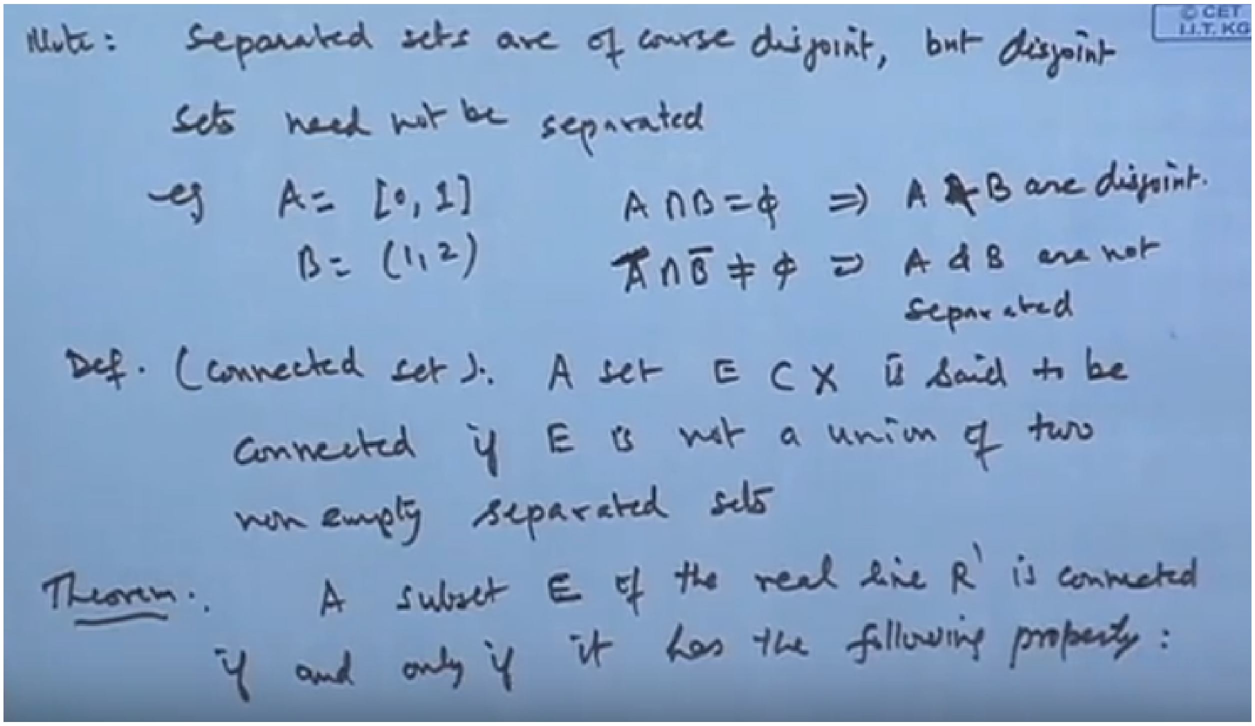

Separated sets, separated sets, are of course disjoint all of of course, disjoint set are of course disjoint. But disjoint sets disjoint sets, need not be, need not be separated. For example, if we look that interval 12, a, is the set. Suppose I take the interval, closed interval 0 1 and B is a set, which is an open interval, 12. Now a and B are disjoint. A intersection B is empty. So this implies, a and B, a and B, a and B are disjoint, disjoint. But a Bar, intersection B is not empty, a bar intersection B, because, no,no, a bar s intersection, B bar, is not empty. Because when you take the closure of this, one is the limit point, two is also limit. So intersection will include one. But the separate set said, it both these are empty sets. So el at least this is not empty. So this shows a and B, are not separated. Okay? Now, we defined the connected set now. Set T is said to be connected, if E is, e is, not a union of two, union of two, non-empty, union of the two, not in a union of two non-empty, separated set, separated sets, Okay? So this is. Now, one result we have and this connectedness over real Lin. What it says is, a subset e f the real line r 1, is connected, is connected, if and only if, if and only if, it has the following property, firm properties.

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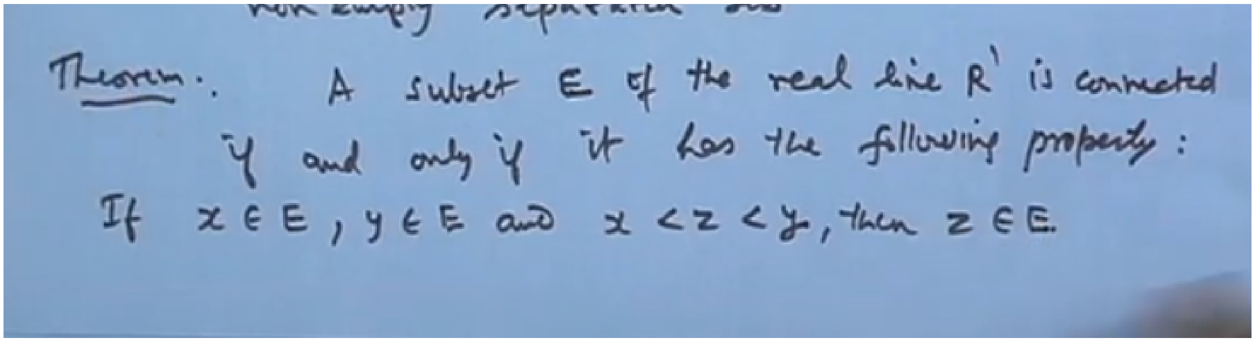

The property says if X, if X belongs to e, y belongs to e, and x is less than Z, less than y, then Z is also in E.

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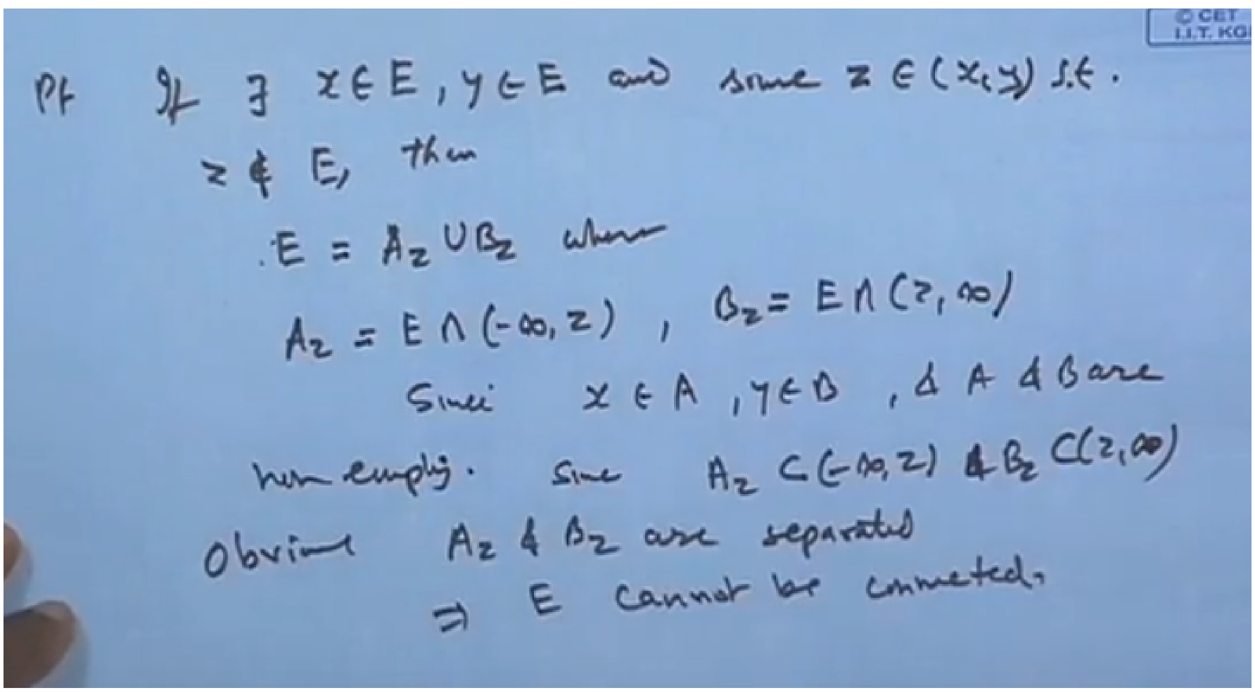

The proof is very simple. So you just, I will just give the outlines. If there exists, suppose if there exists, a point Z, if there exists X belongs to e, y belongs to e and some Z belonging to the interval X Y, such that Z is not in E, then we will reach a contradiction? Then, then e can be expressed as the union of this set, where a Z, is the set of e, intersection, minus infinity Z, bz, is the e intersection, Z infinity, Okay? So, since x and y are in way, since x belongs to a, because it is in e and y belongs to B and a and B, are non empty, and a and B are non empty, non empty. Since this is all non empty, then. Since A is the AZ, which is subset of minus infinity, Z and BZ which is subset of Z infinity. Therefore these are two separated set. Then obviously a Z and bz are separated, are separated. So once they are separated, then e cannot be connected, cannot be connected set. Because e is the union of these two sets, so it is not, a so contradiction. Therefore this result may not be true.

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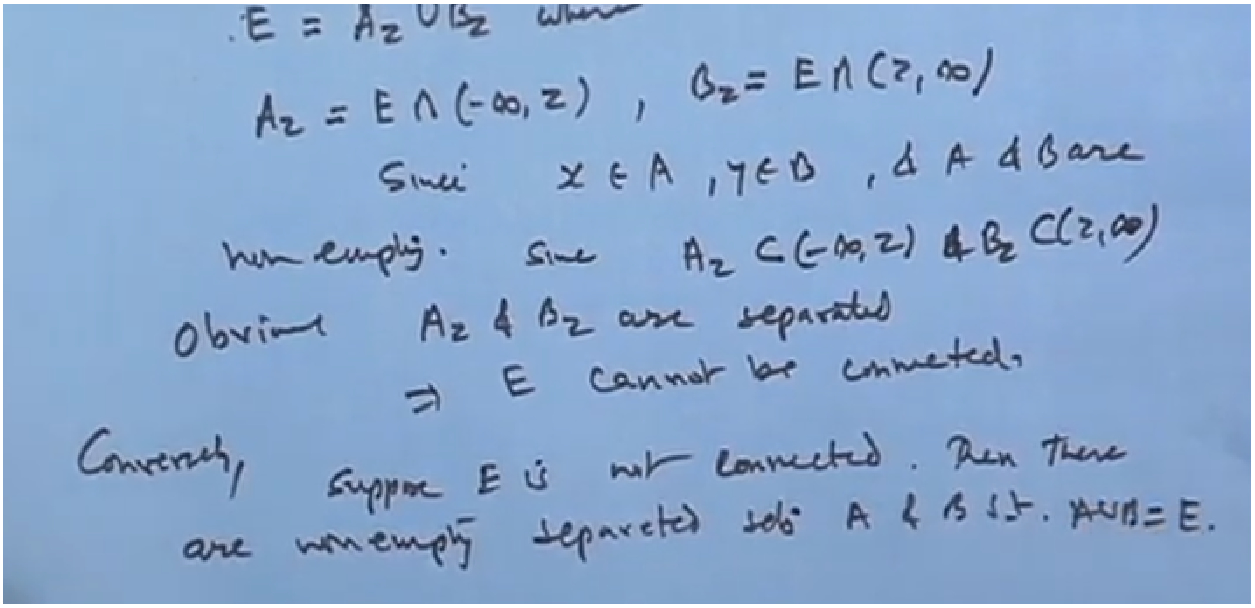

Conversely, if suppose e is not connected, e is not connected, then, there are non empty separated set, separated sets, a and B, such that, their union, a union B, is E.

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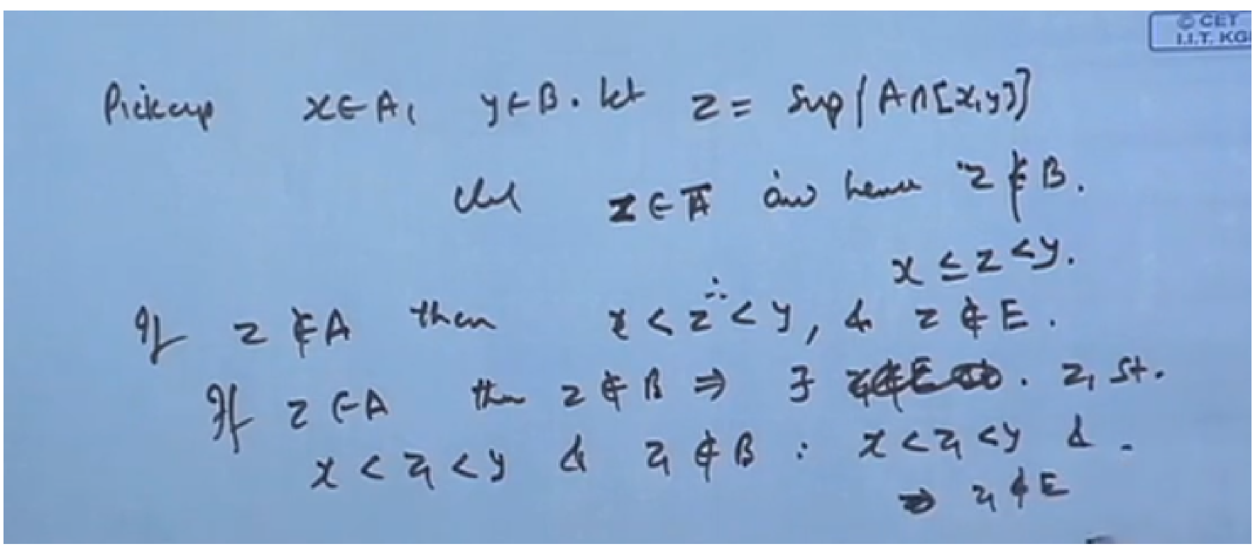

Now pick up x and y now pick, pick up x belongs to a, y belongs to be and let jet is the supremum value, of a intersection, this closed set x y. Now obviously, this set a, obviously, clearly x belongs to the a closure, x closure and hence, and hence this Z, sorry Z belongs to a, closure. And this Z cannot belong to b, Okay? So what we get? In fact, so therefore x may less than, equal to z and is strictly less that y. Now if Z does not belongs to a, then in case, then we have, X less than Z, less than t and z is not in E. If Z belongs to a, then we have, Z is not in B, so we get that exists as Z1, belongs to, does not belongs to e, but such that Z 1, there exists a z1, such that Z X less than Z 1, less than y and Z1 is not in b. Therefore X less than Z 1, less than y and this implies that Z1 is not in y and Z1 is not y. so this completes up to, Okay.

Thank you