Model 3

Lecture 13

Heine Borel Theorem

Okay, so before going for this, this Theorem,

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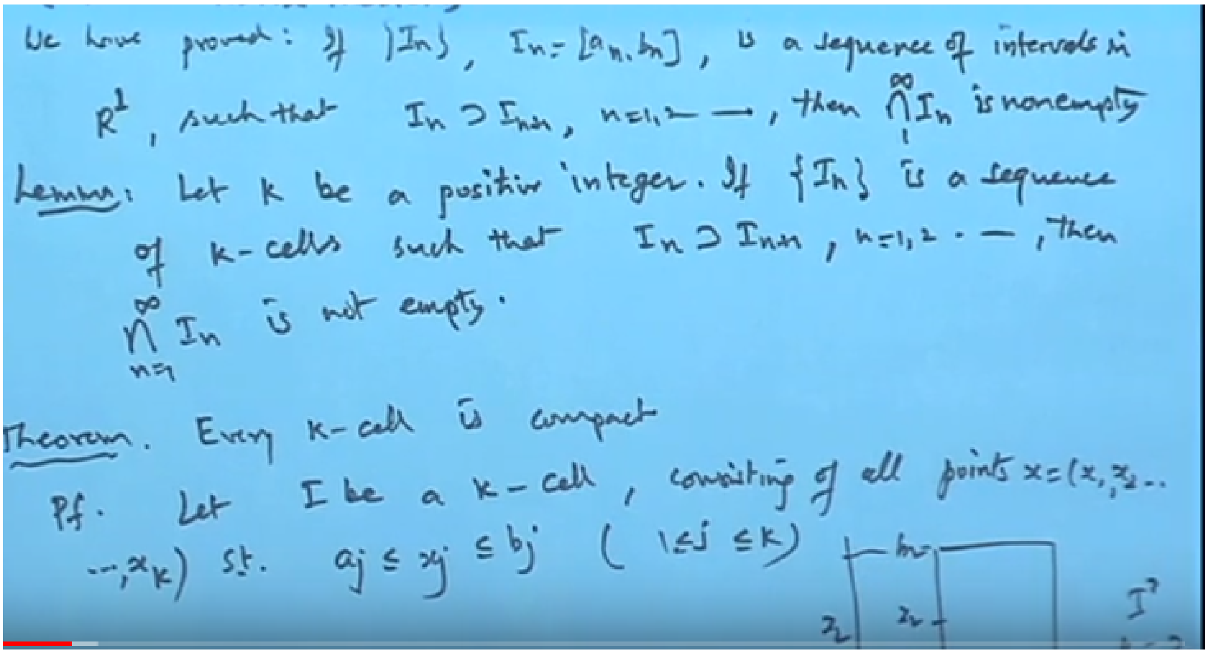

Heine Borel Theorem, we require one or two lemmas, which we need it in the proof of these results. We have seen already proved the following Theorem, that if In where the In is set closed interval an, bn, is a sequence of is a sequence of intervals in R1, that is real line interval of the real line closed interval real lines, such that such that, they are of decreasing nature such that In covers In plus 1 and so on, for n is 1, 2, 3 and so on. Then the intersection of In, when n is 1 to infinity of intersection of IN is non-empty, this result we have seen. Now extending this results we have we can extend this result to our K space. Let K be the positive integer, positive integer if In sequence In, B is a sequence of is a sequence of K cells, such that In covers In plus 1 for n is say, 1, 2, 3 and so on, then the countable intersection of these cells 1 to infinity, In is not empty. So the proof runs on the several line, as we have done it what we will do is we will fix up J each in j we can take it the four fixtures and apply this one, ok this proof we will get better so I am just dropping. What we are interested which is important part is that each or every K cell is compact. Every K cell is compact. So one cell means closed interval, 2 sell is a rectangle close rectangle, of course and third cell, fourth cell, like this. So every K cell is compact.

So let us see the proof of this first, suppose I be be a K cell, consisting of all points, all points, of the type say x1, x2,...xK such that the xJ lies between the bond aJ, less than equal to bJ, j is 1 to K. let this be a K cell. So here this is our set K is 2, so it is in I2 and this, is in 2 cell means it is in K is equal to 2. So we are having the shell like this, where the x1, vary from a1 to b1, where the X2, vary from a2 to b2, a 2 to b2. Say this is our here, so here, this is X 1, this is X 1, Okay? So this will vary from a1 to b1 and here is x2, which vary from a2 to b2, this is our, Okay? This is a one, so. This is a case, a cell in R2 space, in the R2 space; this is in R2 space, like this. Now what you say is, if I generally if I take a case L, then I will suppose case L, In that case it is, of the, it will contain all the points of the type x 1, x 2, xk, where the each X j, will lie between this, x1 will lie between a1 and b1, X2 will lie between a2, b2, xk will lie between AK, BK, so this one. Okay?

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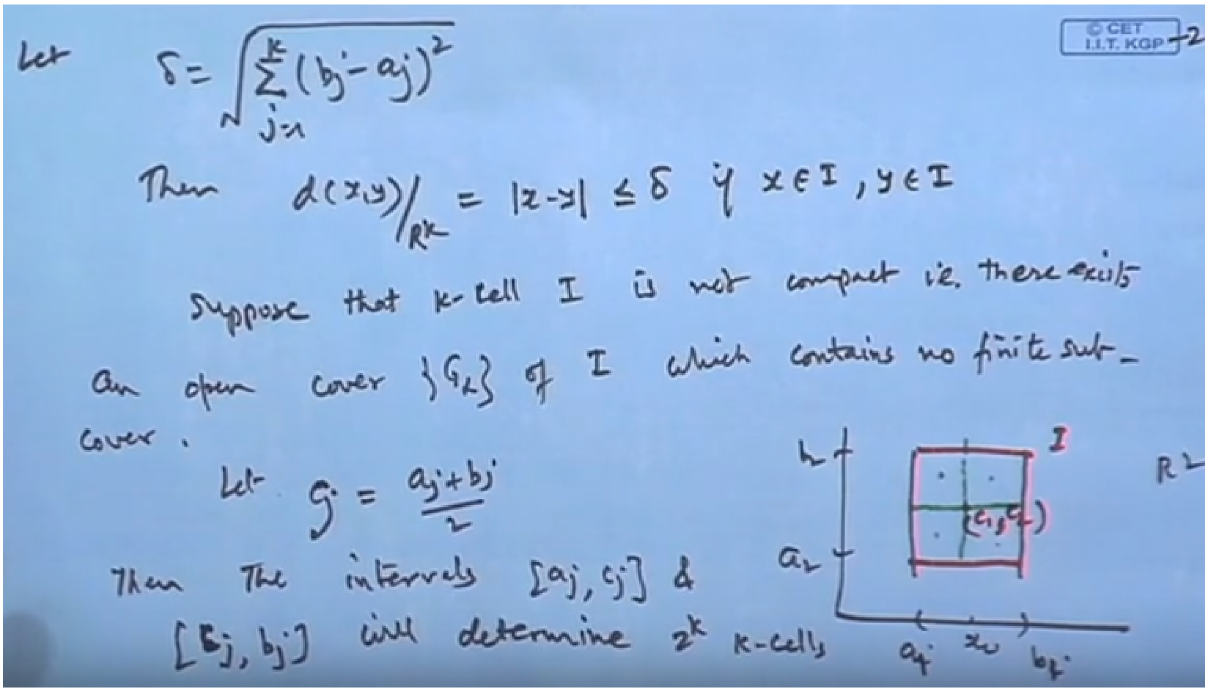

Now let us suppose, the distance between there is Delta. Let Delta B Sigma. Bj minus AJ, whole square j is 1 to K, under root. This is our, so if you look this one, this is the same as what? so b1 minus a1 square, plus b2 minus a2, whole square, Okay? So this will be the same as if I picked up the two point, x and y in this set, x and y, x is, say x1, x,2, y is say, y1, y2, If I picked up two points, then the distance between X and y, in this r2 space topology, will remain less than this number. So obviously, then distance between Xy, in our rk space, that is we are denoting by MS mode, of x minus y, will be less than equal to delta, when if x belongs to I, y belongs to I. So X is X1, X 2, xn, Y is y1, y2, y n, Okay? We wanted to show, this I is compact. So assume that. let us suppose, I is not compact. Suppose, that support, that I, the case L, I is not compact.

It means that out of open cover of this, any open cover if I choose, then it cannot have a finite circle. So that is, that is, there exists, there exists an open cover, say G alpha, of open sets, open cover of I, which contains, which contains no finite sub cover, no finite sub cover. That is the meaning of this, if it is not the compact, so finite subs cover. So let us suppose, this is our, say here is ai, this is bi and here is somewhere X I, this is our say, another point. So when you take choose, let us take cover, let, let us split up CJ, as a point, CJ, which is aJ, plus BJ, by two, Okay? It means C1, I am taking as a1 plus B1, by 2, C2, I am taking as a2 plus b2. So if suppose this is in R2 space, here this is a1 this is b1, here is a2, this is b2 and this basically, we have this, this is our rect I, interval. So what we are doing, we are taking a point here and which is the point c1, c2. C1 is the middle point of c2. So once you take the point, it will divide this whole I, in r2, in four parts. This will be a one cell, another cell, like this, Okay?

So by choosing this way, by choosing a CL, we can get then, then the interval, the intervals, intervals, AJ, CJ & D. Then the interval S is CJ and the interval CJ, CJ , bj. These intervals will determine the cell, will determine will determine, 2k, k cells.

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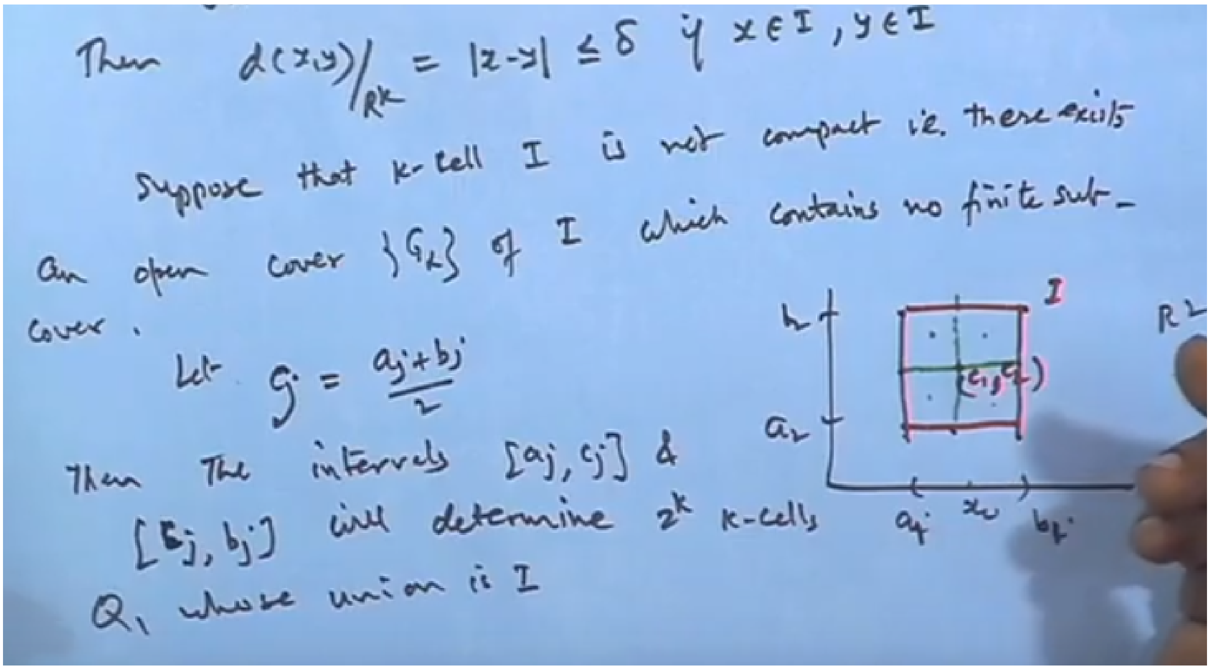

Suppose Q1, K cells Q1, whose Union is I, this is what, yeah, Okay? Just like in case of R2, I have picked up, That, if we take these intervals, c1, c2, o, AI, a1 c1, then c1, b1. Similarly here we get c1, B2, C2, B2 and like this. So we get the full interval 2 to the power 2, that is 4k cells, we get it here.

Now if I, is not compared, then at least one of the cell, will not be covered by a finite, will not be compact, that is open cover of one of the cell, will not have a finite cover, there exists an open cover, of one of these cells, Okay? so, at least one of them union. So what it says, I is not compact.

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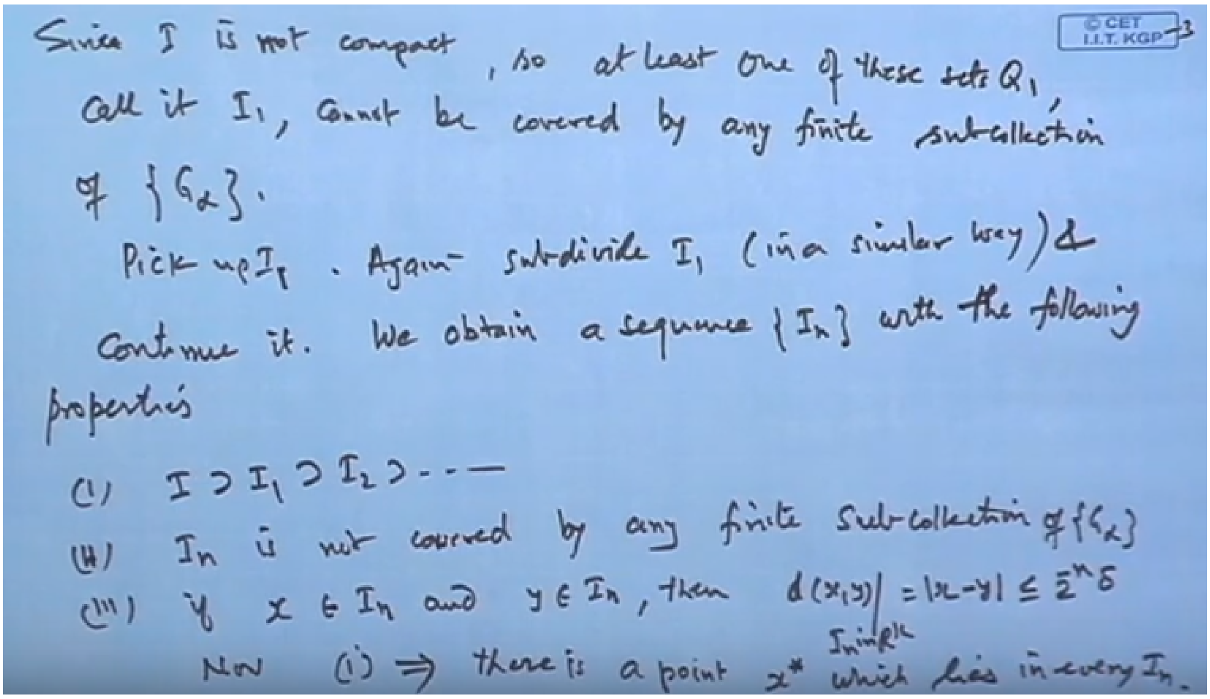

Since I is not compact, is not compact, so, so at least, so at least, one of these sets, say q1, say collect I1, cannot, cannot be covered by any finite sub collection, any finite sub collection, of the open cover G alpha, G alpha, otherwise if it is so, then I one will be compact and contradicts to I. Okay? So that will be next step. Now pick up, now I1, now pick up I1 now, and again divide, again subdivide, this I1, in a similar way and continue this process. So when you divide again, you are getting I1, 1 sector I2, is one of the cell, which is not covered, by any finite sub collection. So again I2, you again subdivide it and like this, so continue this process. So when you continue this process, then we obtain, continue, then we obtained in this process, we obtain a sequence. Say I, with the following property, with the following properties. The first is, this I, which you are getting sale, will be satisfy this inequality, I is covers I 1, covers I 2 and so on.

There are big reaching nature then second way is In is not covered, In is not covered, In by any, In is not covered by any finite sub collection, sub collection, of the cover G alpha. And third is, that if X is a point in I N and Y is also in IN, then the distance of this x y in Rk, that is mod of x minus y, over IN, this is over IN, of course, in IN, which is in RK. This distance will be less than equal to, 2 to the power, n minus n into Delta. Because here, if this length is divided, then what you are getting, this is the one for basically ,when n is 2, you are getting the half, 1 by delta by 2, delta by 2 and so on and it is 3 and like this. So this will be divided by this, I1, I3, and so on, Okay? Now what is 1? 1 says, so obviously 1 will here, no 1 will imply because they are of decreasing nature. And any finite sub collection of this will be non-empty. So the intersection of this will be non-empty. So by thus, there is, there is a point, X star, which lies, which lies, in every In, in every In. This follows from the results, which you have already proved, that if I1, covers, I2 covers I n, and if any finite collection, of this, that is, intersection of this, will be non-empty. Just now we have seen that, In. Okay?
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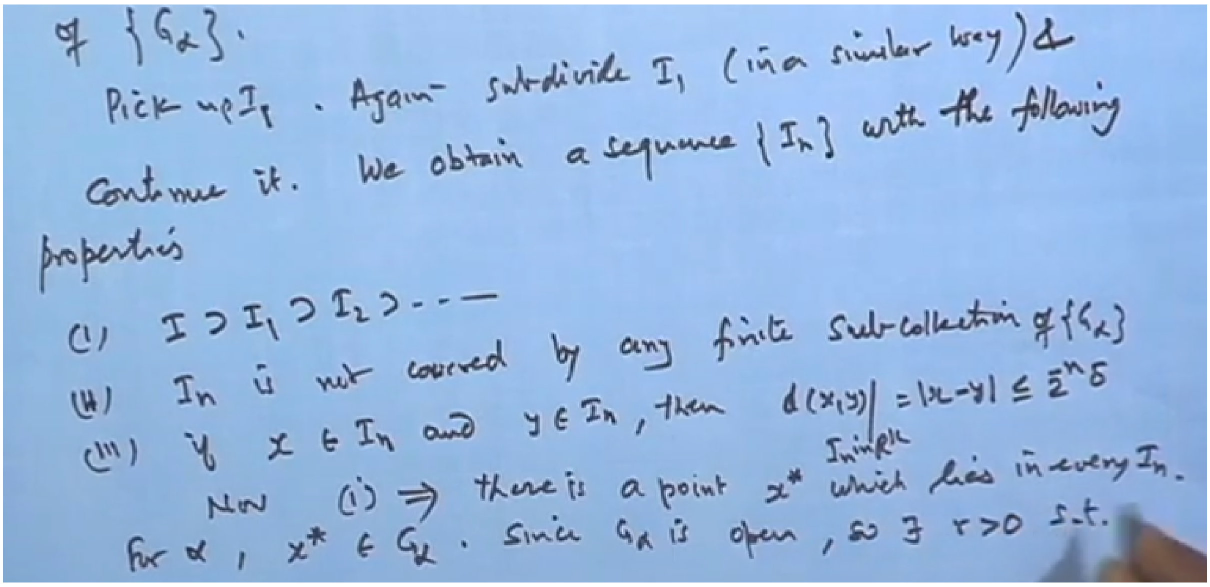

Second one is, so if I, X star belongs to some I n, so I n, is a substation in case L in part of the case L, so we can find some alpha. So for some alpha, alpha, x star, this will belongs to, the one of the element, for some element of the open cover set, G alpha. Because we have taken G alpha as a open cover of the I. So this H star will belong to, some of the G alpha, for some alpha, Okay? Welcome. But G alpha is open, so this star will be have an integer point, so we can find out a neighbourhood around the point, which is totally contained in this. So there exists an R, greater than 0, such that, the distance,

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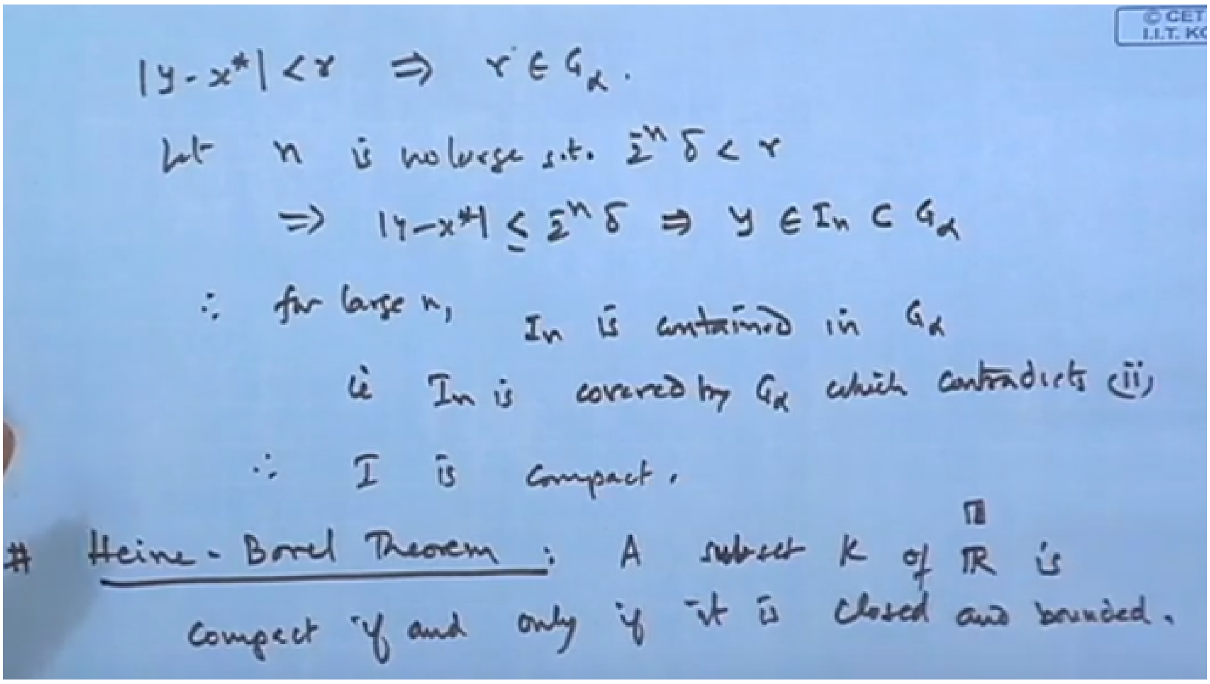

y minus x star, is less than R, will imply R is in G alpha. That is an integer point that is everything. Now R is our own choice, so, let us picked up n. Let n is so large, such that this 2 to the power, minus n Delta, is less than R. That is possible because all is one seen fixed. You

can take n such a large, so that this r is remaining less than. So if it is so, then what happened? This implies that, bi which you are choosing, their basically is in our In, also Because, I n, is satisfied this condition. That if X belongs to Yn, then this is 2 and this is less than R, so any element of I n, will be the element of R. So this implies, that distance y minus x sta,r is less than or equal to, 2 to the power minus delta, which implies the element Y, which is in G alpha, belongs to I n, which is subset of G alpha.
So, every element of Y n, I n, will be in G alpha, for some large n, for some large n. Which therefore, for large n, so for Particular n, I n is contained in, the set G alpha. So it contains G alpha, then I n is covered by G alpha, that is, I n, is covered by G alpha, which contradicts the second part of it. What is the second part says? I n is not covered by any finite sub collection, but it is covered by the G alpha, that is a finite sub collection, G alpha. So Contradiction, this contradiction is, because of a wrong assumption, that I is not compact. So this so, therefore, I is compact. So this proves, sorry, that means every case L, is a compact set. Particular every closing cover, is a, in that, is a compact set because that is also. Now this gives you a very interesting result, which we call it as a Heine – Borel Theorem, Heine – Borel Theorem, Okay. But basically this forum says, that the statement of the theorem let it be, the theorem is, Heine – Borel Theorem is, if we, a subset k, a subset k, of R, subset k of R is compact,if and only, if if and only, if it is closed and bounded, it closed and bounded. So this is a particular, when you are choosing R, especially. In general, Rk space, we will prove some result. So in the particular case, we can say drive the handle alone. So the result is first, that is important.

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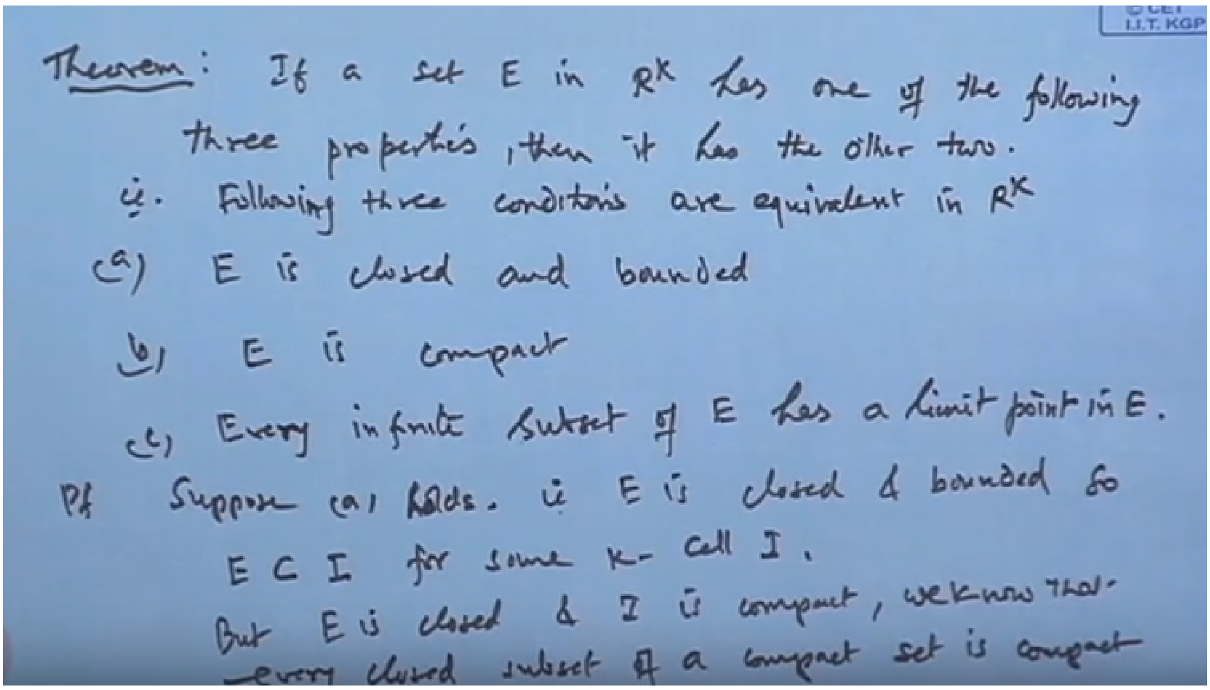

the proof of this. The proof follows, from the theorem, from the next theorem. What is this theorem is? The theorem says, that if a set E, in RK, in RK, has one of the following, one of the following three properties, has one of the following three properties, properties, then, then

it has the other two. That is, all three are equivalent. That is the following three conditions are equivalent, in the space RK. So in particular in case one, it is also true, in case of over the real time. What is the condition is, the condition says, E closed, is closed and bounded. Second condition shows, that E is compact. Third condition says, every infinite, every infinite subset, of e has a limit point, has a limit point in e, so this thing, the proof of this. So obviously, if it is true, then in case of R1, if e is compact, then this closed and bounded and vice versa. So the result follows immediately. Heine – Borel, Heine – Borel Theorem follows immediately. So let us see the proof. Suppose A holds, A holds, it means given each closed and bounded, what we want to is compact, Okay? And that is e is closed and bounded. So once it is bounded, it means, it will be enclosed by some cell. So let, so E is contained in I, for some K cell, K cell, I. Okay? But what is e? e is closed, but e is closed and I is compact, that we already shown. So every closed subset, of a compact set, is compact. So and we know, that every closed, every closed, this result we will, every case compact, and every closed and compact sets, are compact. So this is what we have proved earlier.
Closed subset of a compact set, are compact, we know. That every closed subset,

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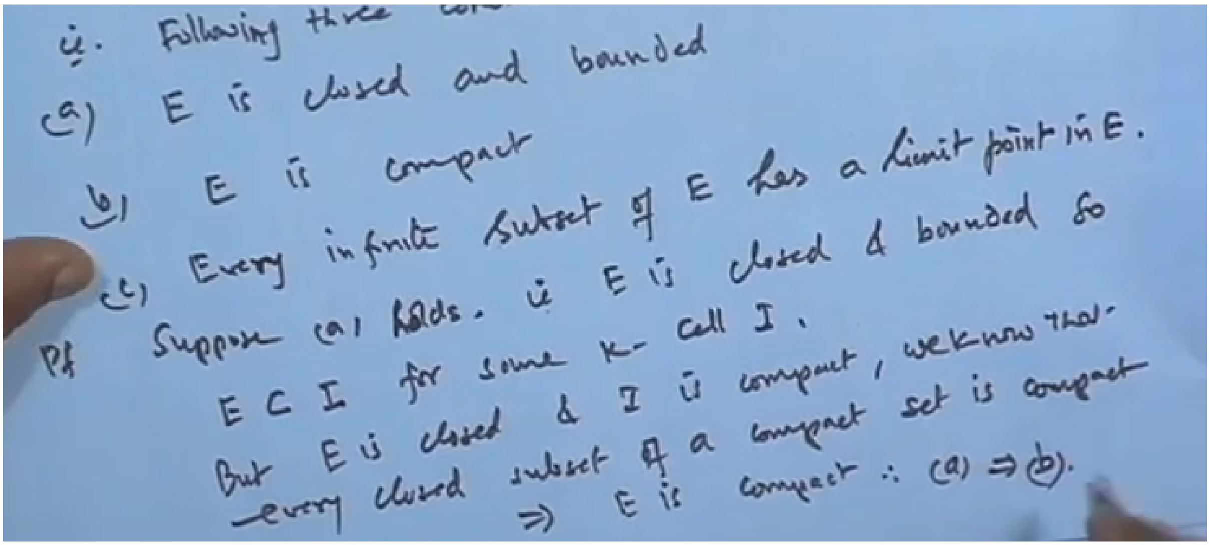

of a compact set, is compact. This we have already proved. So here, e is closed, I is compact. So e is a subset, of a compact set, therefore this implies, e is compact, so once it implies means, therefore a implies B, holds, Okay?

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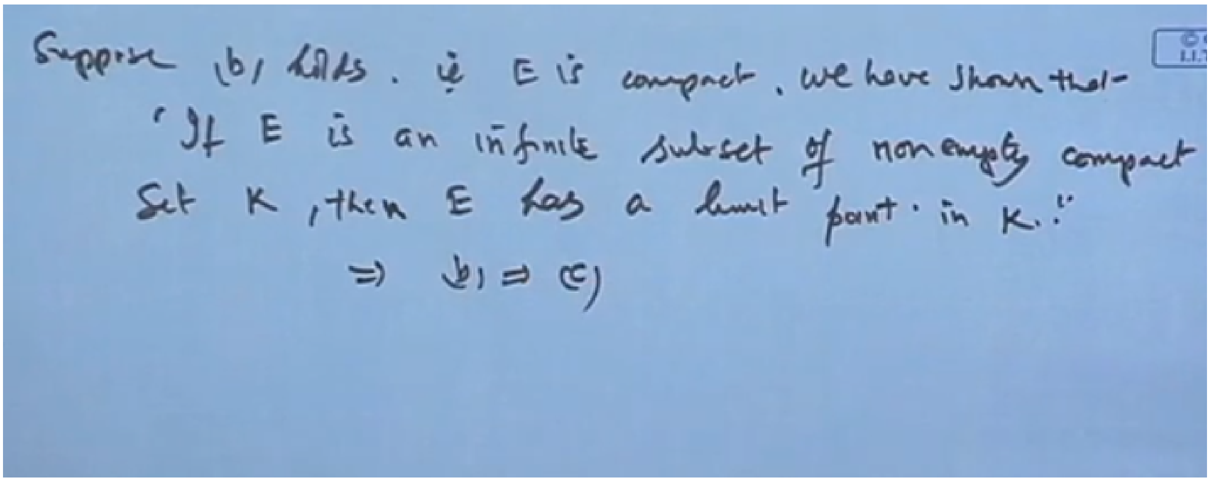

Now suppose B is true, suppose B is true, then given, suppose B holds, b holds, then automatically C implies, why? Because B means, E is compact, that is, E is compact. So take a sequence, any sequence, of the subsets of e. This we result already proved. That if e and is a finite subset of a compact set, then e has a limit point and we have shown earlier, that if e, e is an infinite, infinite subset, of a non-empty compact, of non empty compact sets k. Then e has, then e has a limit point in k, limit point, limit point in k. This we have shown. So here, e is compact already given, we wanted that every infinite subset of E, has a limit point in E. So it, what it shows it, if this infinite subset of e, if I choose and infinite subset of e means, we can have it in fine sub sets. Non-empty compact subsets of k. Then this will have a limit point in e. So obviously, b implies C. So that is nothing to prove in this.