Course

On

Introductory Course in Real Analysis

Properties of Compact Sets

Okay, so this is in continuation of my previous lecture, on the compact sets. In this particular lecture, we will discuss few properties of compact sets and also relation with some other set completeness and other continuity and so on another results. Okay?

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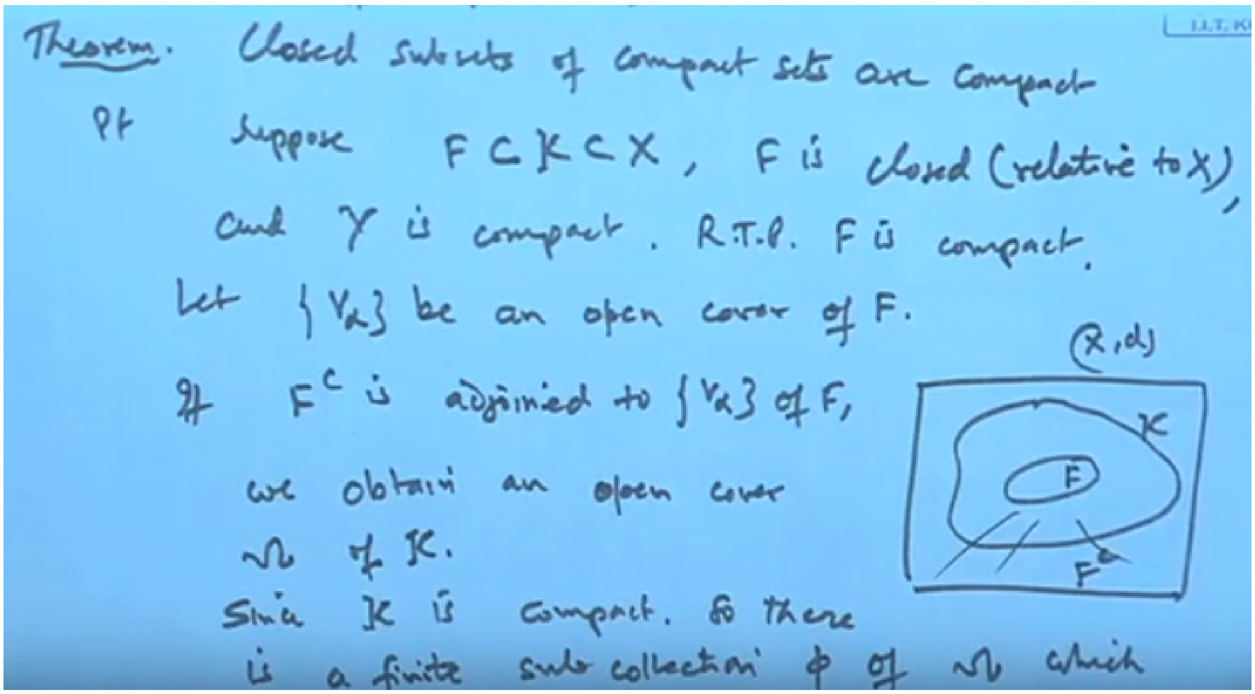

Close subsets of a compact sets, of compact sets or subsets of complex sets, are compact. So this we wanted to show proof.

So let us take an F, which is a subset of K, which is subset of X. Where f is closed, relative to X and Y, is a compact set. What we want, so this closed subset, that is F, which is a subset of a compact set, is compact. So every closed subset, of a compact set, is compared that is what we need show. So required to prove is, F is compact. F is compact means, that it will cover, every open cover of F, will have a finite sub cover. So let us take let V alpha, be an open cover of F, of in cover of F. Okay? Now this is our scenario. So here, this is our XD, here this is a set K, and here is somewhere F. Okay, now we are taking an open cover of F. What is FC FC? FC will be the complement of F. So here somewhere we have a FC.

This is our FC. Okay? Now if we take a open cover of F, then some of the open cover will intersects FC, also, because these are FC, if f is closed the FC will be complement will be an open set. So it will be adjoining to B alpha. So if F of C is adjoin, adjoin to the open set b alpha, open cover b alpha, of F, then we obtained n. We obtain an open cover, open cover, we obtain an open cover of omega of X, of K, omega of K. Okay? Omega okay this, this is the open cover of B alpha. Now some of them will definitely adjoin with this. Now we are taking the open cover of K, Okay?

Now this open cover Omega, may contains F of C, also somehow. Okay? So it is since K is, let us see, Since K is compact, so every open cover will have a finite cover. So there is a finite sub collection of a Omega. So there is a finite sub collection, finite sub collection, of Phi, of Omega, which covers, which covers K. By definition of the compact sets K, Now the possibility, Okay, Now once this Omega, which is your open cover of K and since K is compact, so it will have a finite sub cover.

So it means that Phi will cover F also. Okay? And hence, hence this Phi will cover, F also, finite sub collection will cover F also. Now in this Phi, if the Phi C, is FC is a also member, then we can drop that.

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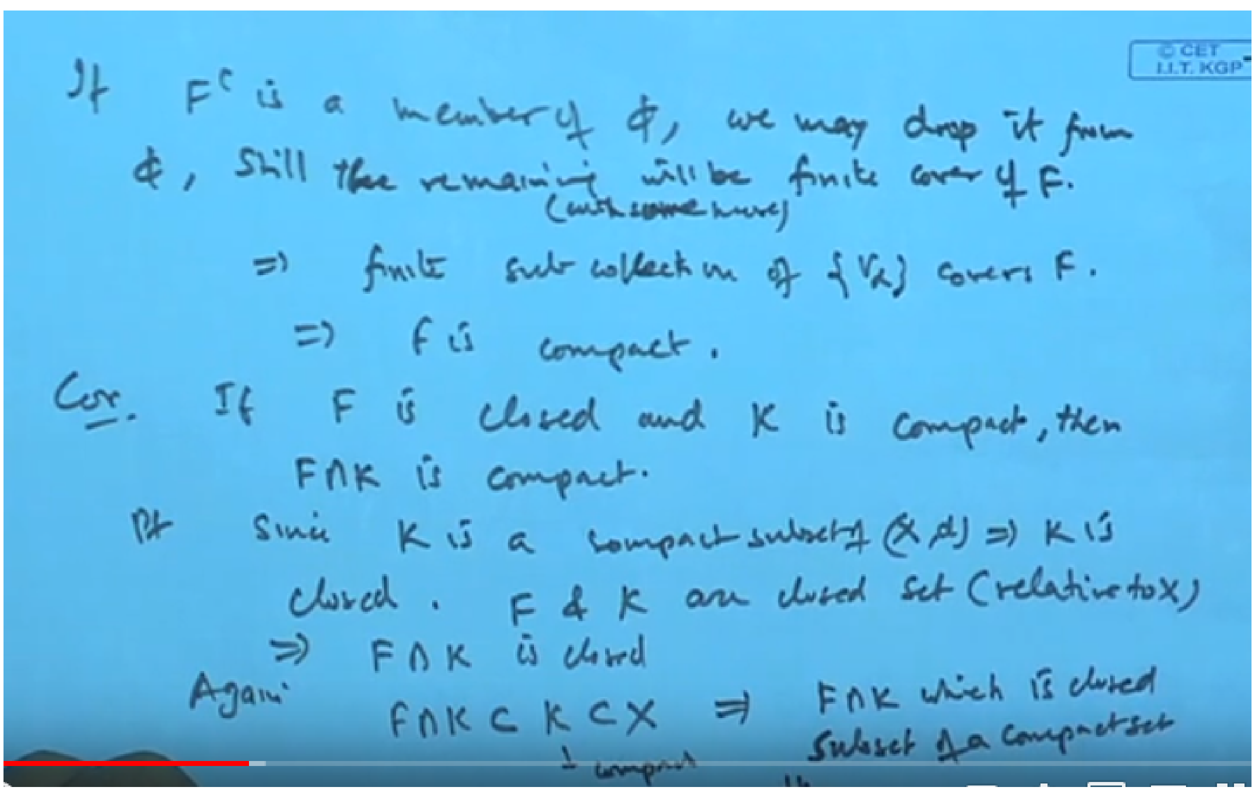

If, if, FC is a member of this, is a member of Phi, then even if we draw we can, then we may drop, we may drop it from 5. Still, still the remaining will be the finite cover of F. Finite cover of F, is still retain an open cover of F. Remaining with some with others, with some more, with some more, will be finite cover of A. So but this source, that this source, that this implies, that this sub collection of this, a finite sub collection of this open cover, V alpha, covers F. So this shows F is compact.

So that is the very intake. Okay? Now as a corollary to this a, if F is closed, F is closed, and K is compact, K is compact, then F intersection, K is compact. Okay? Now, proof photo, very easy, what is this? F is closed, K is compact subset of X, every compact subset of is closed? So since K s a compact subset, of say metric space XD, so it implies that K is closed every compare subset of this. Further F and K is all closed set, So relative to X, relative to X. So this implies, that intersection part of this intersection, of two closed set, is closed.

Again, this intersection, F intersections K, is totally contained in K, which is contained in X. K is compact, this is compact. So every closed subset of a compact set is compared. So this implies F intersection K, which is a closed Subset, of a compact set. Hence it is compact, hence it is compact. Okay? So that proves that. Okay? Hence it is compact. So this shows this. Okay? Next result in this, this is also testing result.

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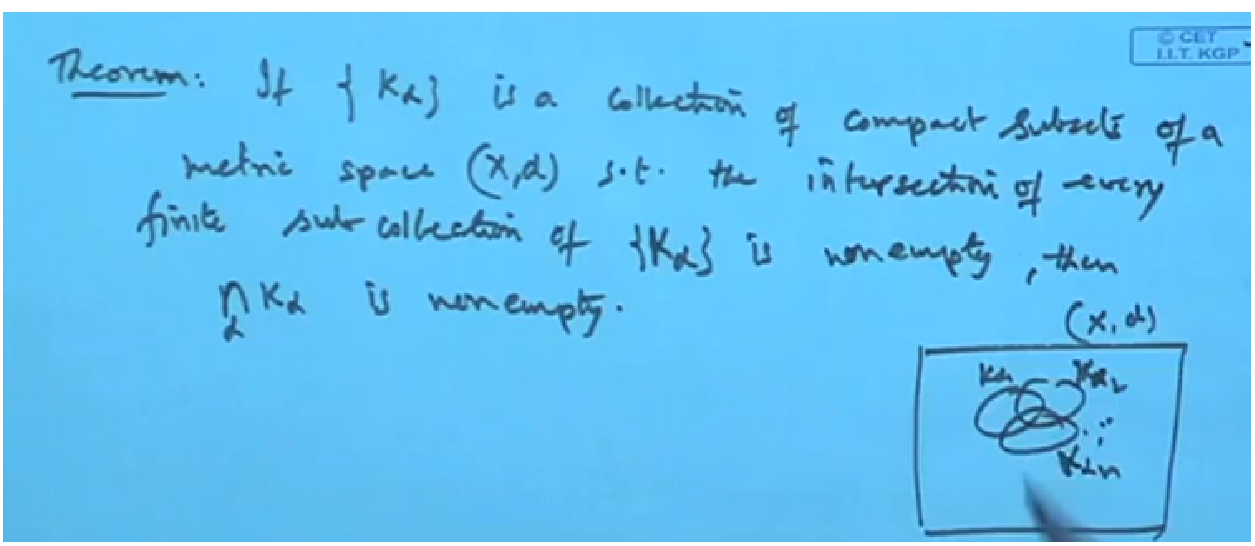

If K alpha is a collection of, if you can say K alpha, is a collection of compact subsets, compact subsets, of a metric space, of a metric space, say XD and such that, the such that, the intersection of, intersection of, every finite, every finite sub collection of K alpha, sub collection of these k alpha, sub collection of k alpha, is non empty. Then the arbitrary intersection of K alpha, K alpha is non empty. So this shows the finite intersection property basically. If K alpha is a collection of compact Subset, of a metric space XD, this is ourXD and K 1, K 2, K n, these are the compact subsets of this. K alpha 1, K alpha 2, K alpha 3, these are the compact subsets of this, XD. And if we take the finite intersection of these, finite intersection of K alpha 1, K alpha 2, say K alpha n, like this. If you take the final n, each finite intersection is non-empty set, and then Arbit intersection will be non-empty.

So this we will prove by contradiction. How will you say? Suppose, one of these sets, proof, what we will do is, we will pick up one of the X sets, out of K alpha, say K 1. Such that, that, no element of the K alpha, that set K 1, belongs to each K alpha. It means when you take the intersection of K 1 to K alpha, all intersection some. At least some of the points, of that, will be out of it, will remain out of it. Means, there are they come, not every point of K 1 is or none of the point of that set, belongs to each alpha, that each K alpha. That is what is. So fix a fix a member K 1, and then we will reach a contradiction. Okay?

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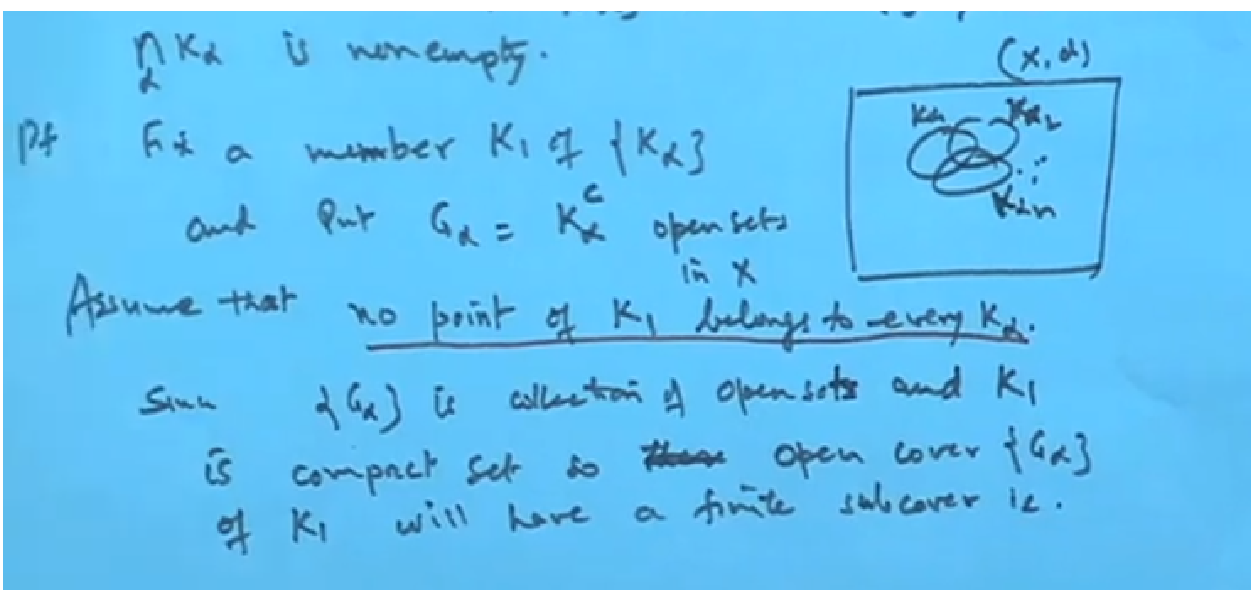

Then fix a member K 1, of this sequence of compact set K 1, from the sequence clear and put, and put say G alpha, is the complement part of K alpha, C. Now K Alpha is a compact set, set so it is a closed set, so G alpha will be an open set, in X. It is an open. Now, but what we assume, that assume that, this K1, assume that no points of k 1, no point, no point of K 1, e K 1, belongs to, belongs to every K alpha, every K alpha. So this is our assumption. This is very, no point of K 1, belongs to K Alpha. It means, when you find a say picked up the point 1, 1 point of K 1, then at least one of the Alpha, K alpha will be available, where that point does not belongs, to like that.

So no point of K 1 is, no point of K 1 belongs to every Alpha, K alpha. Since G alpha is an open sets and K alpha and since this collection G alpha, is a collection of open sets, collection of open sets, and k1 and k1, and k1 is a compact set. Because it is one of the set, which you are choosing out of K alpha, compact set. So by definition, so K1, says, so there are, so G alpha is a collection of insect and k1 is compressed, so the open cover, open cover, G alpha, of K 1, will have a finite sub cover, will have a finite, will have, then so open cover G alpha K 1 will have a finite sub cover.

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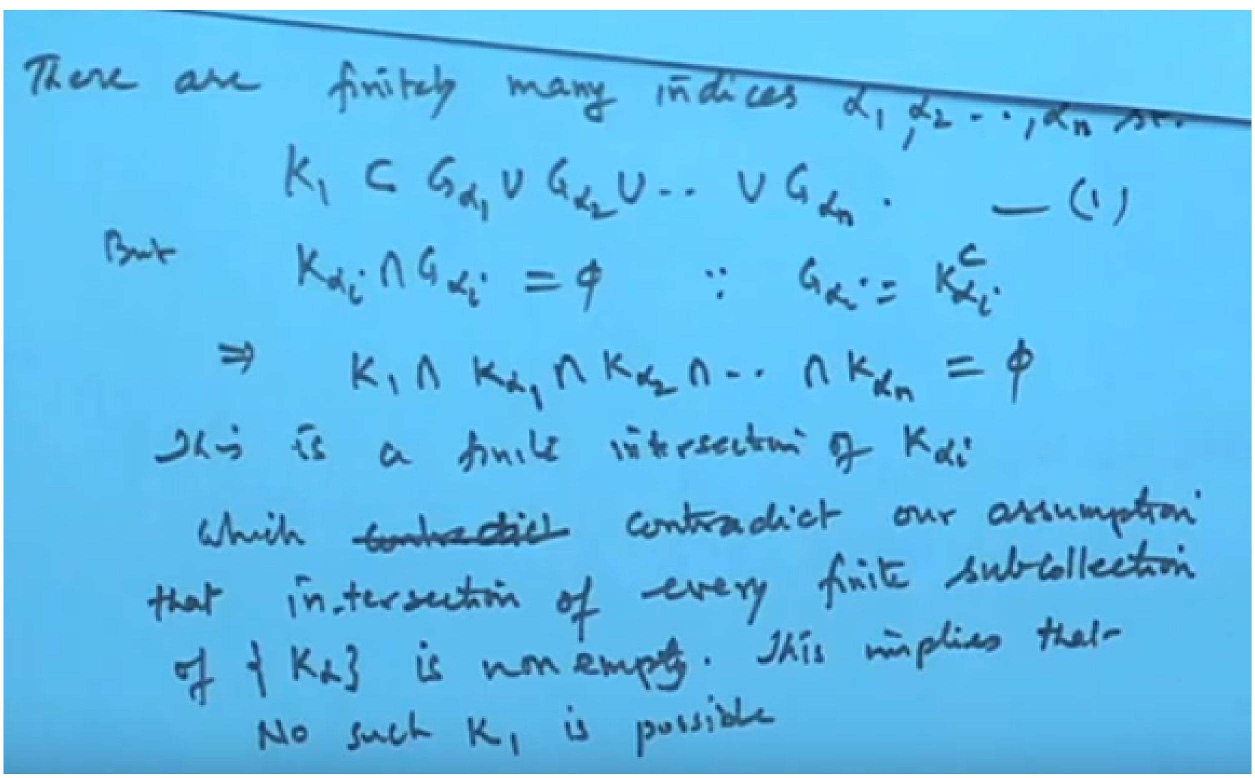

Finite sub cover means, that is, that is, there are all finitely many, there are finitely many indices, say alpha 1, alpha 2 ,alpha N, such that the K 1 is contained in, G alpha 1, G alpha 2, G alpha, Okay? This, but fine. Now, if we, this is say 1. But K alpha I, intersection G alpha I, is empty? Because, because, what? Because the G alpha I, is taking, is the complement of K Alpha I. So if I take the intersection, intersection this ,so this implies K 1 intersection, with K alpha 1, intersection with K alpha 2, intersection with K alpha n.

Then this will be empty set? But what is this? This is a finite intersection of K alpha? But this is, this so, this is a finite intersection, finite intersection, of K Alpha H. Is it not? That is if I picked up the finite collection of this set K alpha, then this are the finite. But what is the condition? If K Alpha is a collection of the complex of set of metric space, such that intersection of every finite sub collection, is non-empty. So which contradict, which contradict, Contradict, contradict, which contradict our assumption, that that intersection of, intersection of, every finite, every finite, sub collection of K Alpha is is non-empty. Now it is coming to be empty, so it is contradiction. And contradiction is because our wrong assumption, that K1 is one of the member out of K alpha, is there, which has a property. Then no point of K one belongs to every Alpha. So this needs, this implies, this Implies, that no debt, that no such k1, is Possible. It means, that if we picked up, it means, whenever you take the points, it is belongs to at least one of the point, at least it will belongs to all of that, alpha. That is what is saying. Okay? So this so, this is only possible.

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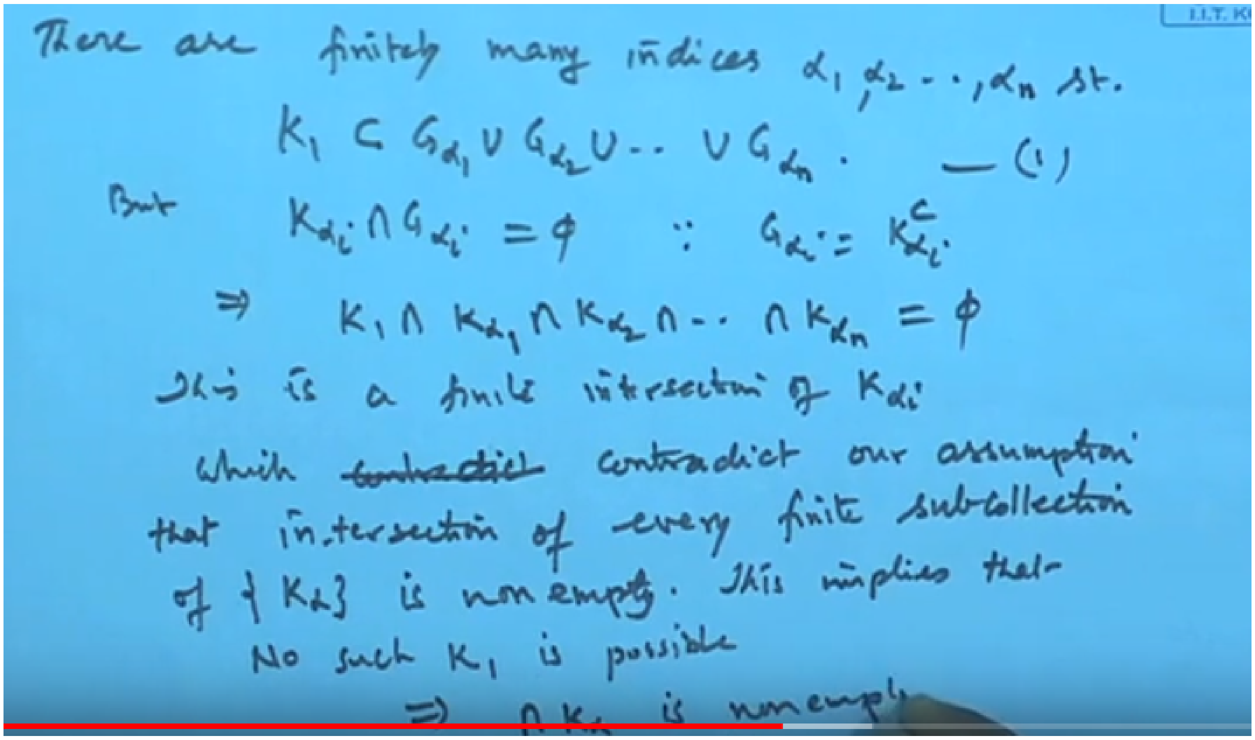

That this implies that the arbitrary intersection of K alpha, over alpha, is a non-empty set, because our assumption is wrong. Assumption is that there is a sum K 1, which has a property, assuming that no point of K 1 belongs to every alpha. So if you picked up one point, then it is one of the alphas, are there, where that that point does not belong. So this leads to a contradiction. It means, whatever the point is choose, it will belongs to every alpha, K alpha.

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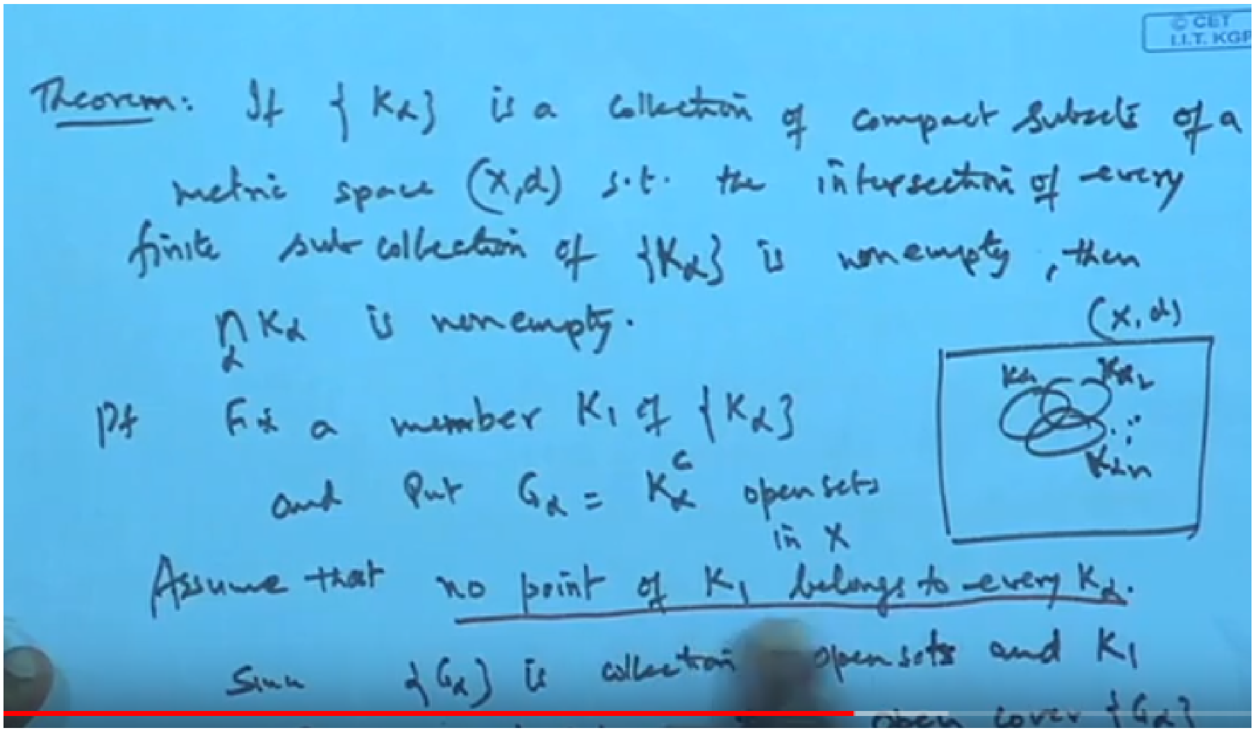

Hence the intersection will be non-empty, so that is proves that result. Okay?

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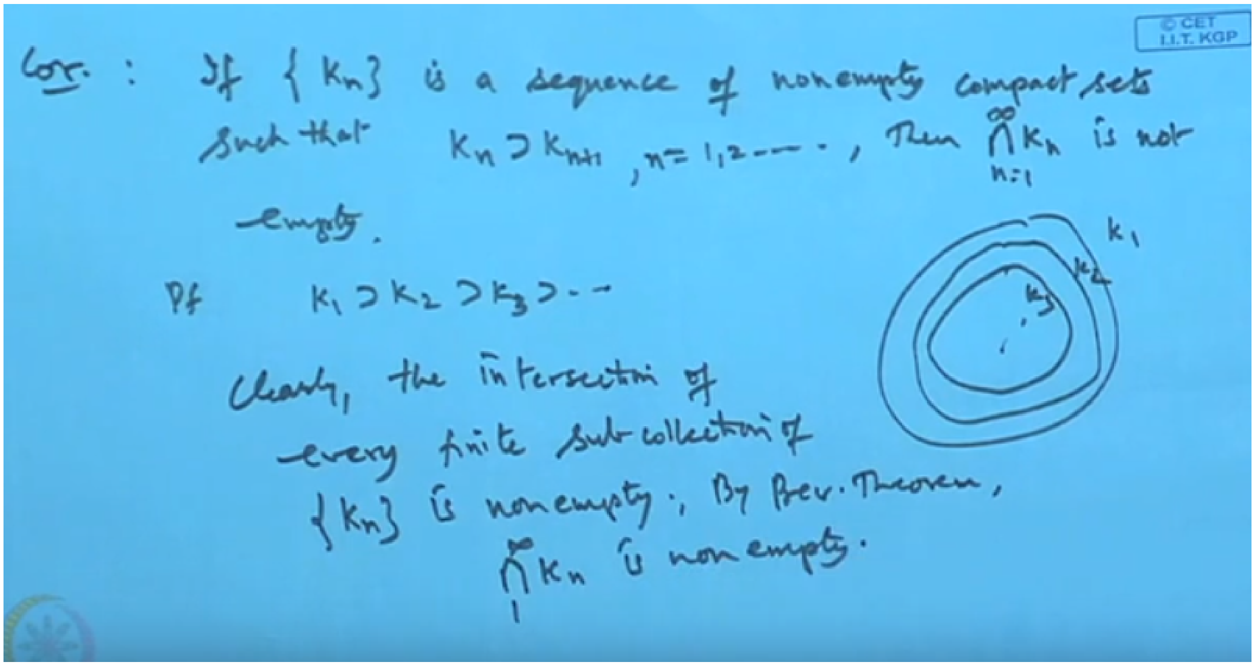

Now consequence of this, is the following results, as a corollary, the corollary is, what she says, if, if K N, is a sequence of, sequence of non-empty compact, non empty compact sets k, if a non-empty compact sets, such that, such that, KN, covers KN plus 1 and so on n plus 1. Where n is 1 2 3 and so on. Then the arbitrary, then the countable intersection of KN, 1 to infinity is not empty. The proof is follows from there, because. What is given? K 1 contains K 2, contains K 3, contains and so on. So this is our K 1, here is K 2, here is K 3, and like this. So if I take any finite collection of KN's, then their intersection is non-empty. So clearly, clearly the intersection, intersection of every finite, every finite sub collection, of Kn's, of sub collection is non-empty. And Kn's are the sequence of compact sets. So from the, so by previous theorem, the arbitrary into, the countable intersection of this, 1 to infinity, is non empty.

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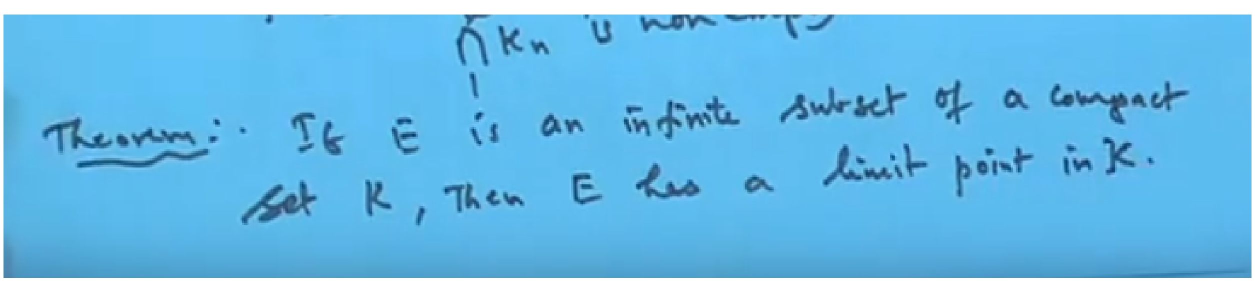

Okay, now we have another result. This result will also be used. If e is, e an infinite subset, infinite subsets of a compact set, of a compact set K, of a compact set K, then E has a limit point in K, limit point in K. So let us suppose, the contradiction, Okay?

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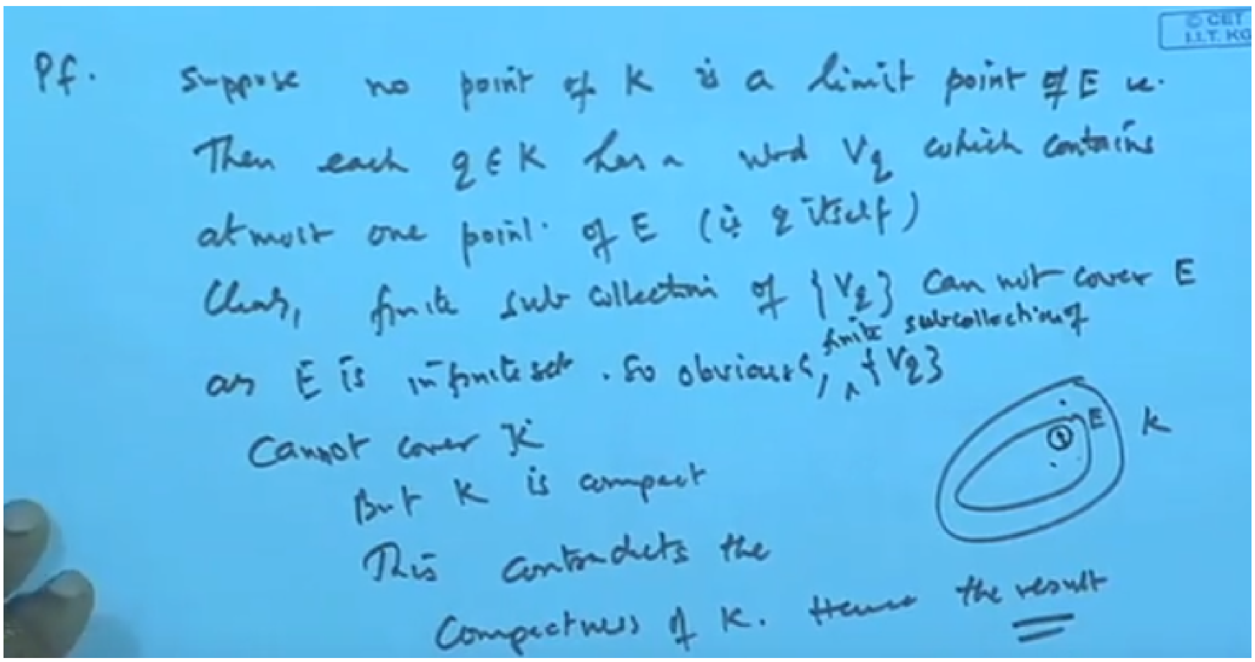

We prove again by contraction. So let us say, suppose no point of K, is a limit point of e. So suppose, no point of K, is a limit point, is a limit point of e. It means what? If no point of K, is a limit point of e, it means, if we take each K, that is ,then, then each, at each ,each Q, each point Q, belongs to K, have a neighborhood, has a neighborhood, has a neighborhood, say we Q, which contains, which contains, at most one point, at most one point. That is the Q itself, one point of e. That is Q itself. Centre, because if suppose, it contains infinitely many point, then Q becomes the limit point of it. So that gives the counter reason.

So suppose, it contains at most one point. Now if I take the finite sub collection of this, then clearly the finite sub collection, finite sub collection, of these neighborhoods, VQ, will not cover, cannot cover, cannot cover, say E. Cannot cover E. Because this is our, this is our K and E is this. Okay, now what we are assuming is, no point of K, is a limit point of this, suppose. Suppose, so Suppose, I take a Q, here, then this neighborhood Q, this neighborhood of Q will contain a neighborhood, which does not include the other points of the Q, it is Okay, because Q is not in a limit point. So at the most it will contain only the point.

But since e has a infinite collection of the points, so a finite number of the d this open, cannot cover e, because this will only have a finite number of points, available in this, cannot cover e. H is infinite set. So if this neighborhood cannot cover finite, it can also not cover K. So obviously, this neighborhood, obviously this neighborhood cannot cover finite, finite sub collection, finite sub collection, of this V2, cannot, cannot cover K. But what is K? But K is compact, but K is compact. So every open cover must have a finite sub cover. Now BQ, we are taking a open cover for this, so it cannot cover B Q, so if finite sub cover must cover Q, which is contradiction because it does not cover Q. So this contradiction, this contradicts the compactness of K, hence the result. Hence follows the result. Okay, now another 2 Theorems are there, of course just one more theorem and then proof is immediate.

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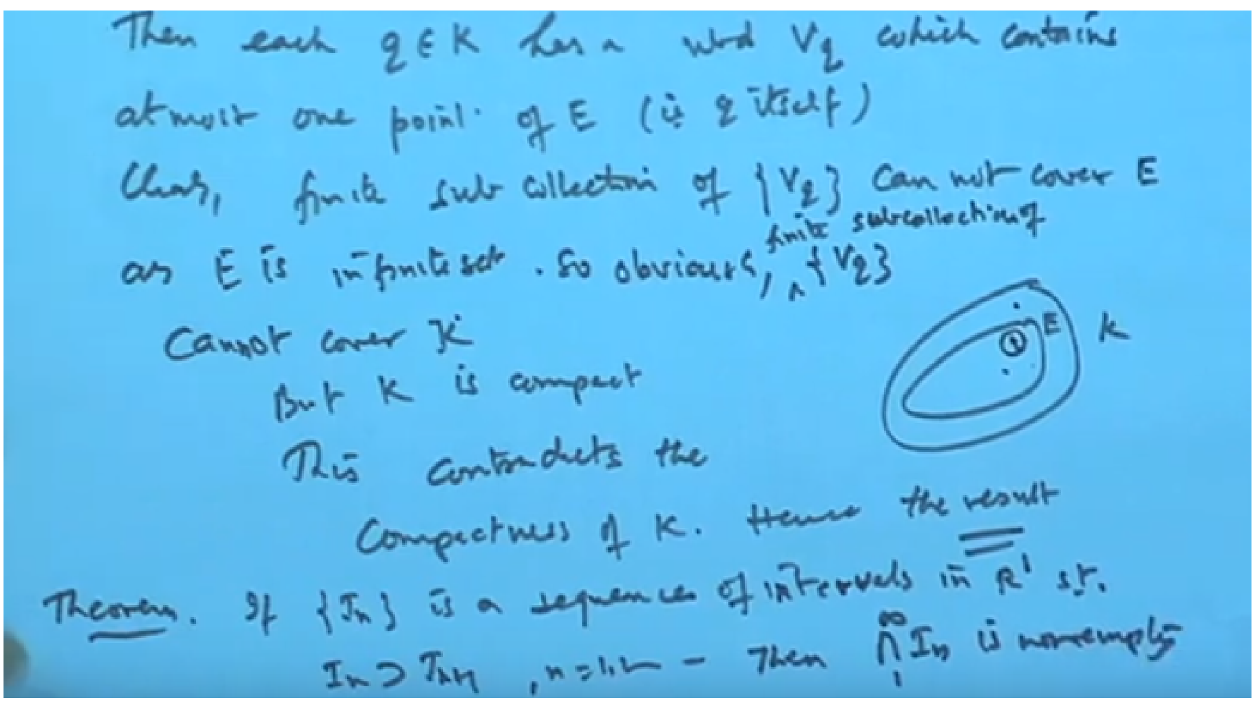

If I n is, if I N is a sequence of intervals, intervals in R 1, R 1 such tha,t the I ins cove,r I n plus 1, for n is 1 2 3, then the intersection of I N, 1 to infinity, is non empty. The proof is very simple.

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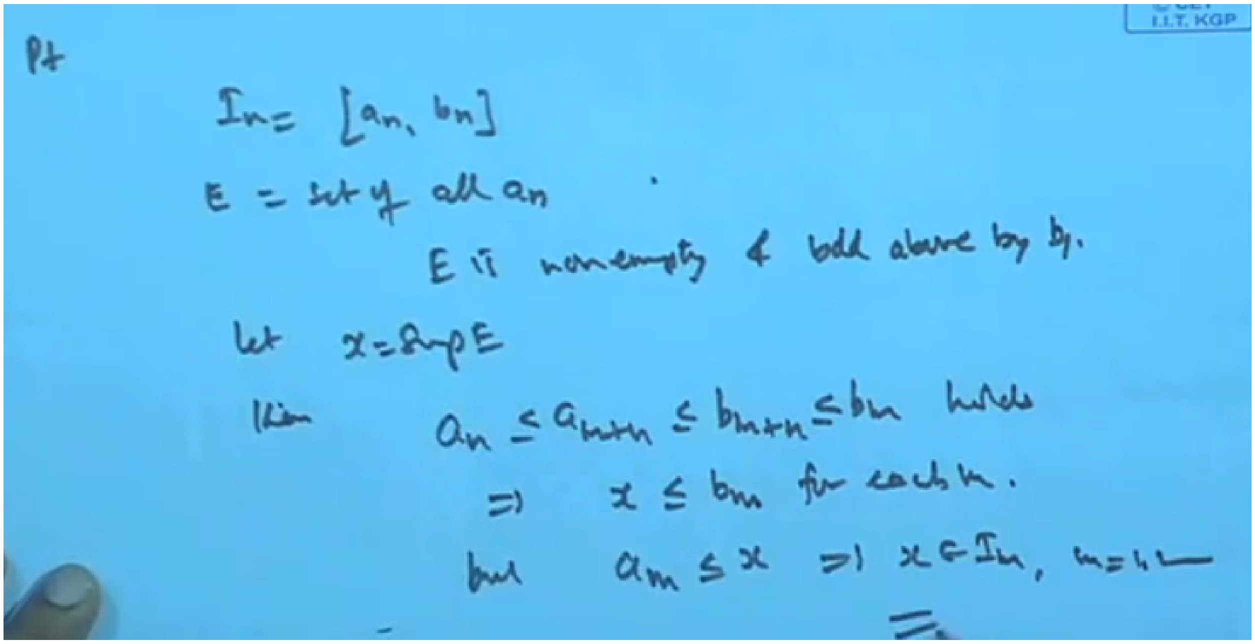

Suppose I n, I take aN, Bn at a closed Intervals, and let E be the set of all Ends. Then obviously is, obviously is non empty and bounded above by b1 and so let supremum of e, is suppose X, then clearly, then clearly aN, which is less than equal to, M plus, one which is less than equal to V M plus n, which is less than equal to V M, horse. Therefore when you take the supremum of this, we get X is less than equal to V M, for each M, for each M. But a n is already less, get less than equal to X, so what does it show? This shows that X belongs to IM, where M is 1 2 3. So we get the finite intersection of these, non-empty X, for any sub collection of IM intersection of this, is non empty, therefore the arbitrary intersection will also, so this proves that.

Thank you