Model 2

Lecture – 10

Compact Set

So today we will discuss compact sets in a general metric space. That requires the concept of open cover.

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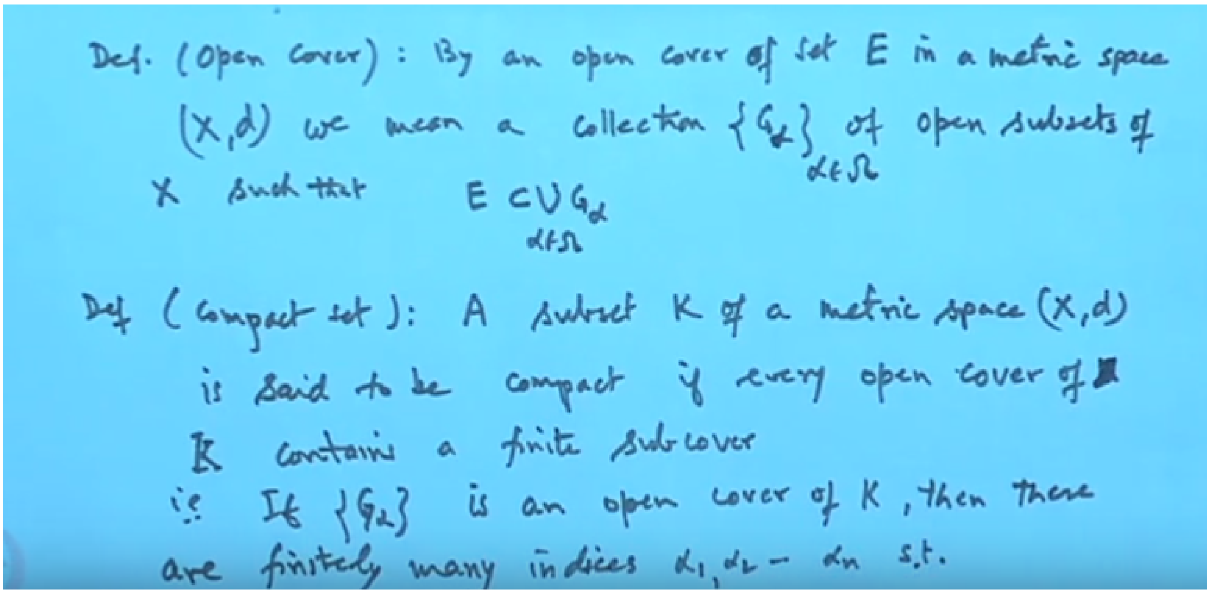

So first we see open cover. By an open cover of a set of a set E in a metric space X, D, we Mean a collection, a collection of open set collection G alpha we are alpha belongs to the index set. Alpha belongs to the index set Omega, of open subsets of X such that arbitary union of this open set G alpha, covers when alpha belongs to Omega. Okay so if an collection of the open set which goes unions are between and cover C then we said this collection of the open sets in the metric XD is called an open cover for E. so we define the compact set, a subset K of a metric space, of a metric space. XD is said to be compact, compact is said to be compact. If every open cover if every open cover of X of X every open cover of K sorry of K every open cover of K contains a finite sub cover, that is the meaning of this is that is meaning is if G alpha, if G alpha is an open cover of K, then there are finitely many indices there are finitely many indices indices alpha one say, alpha 2, alpha N,

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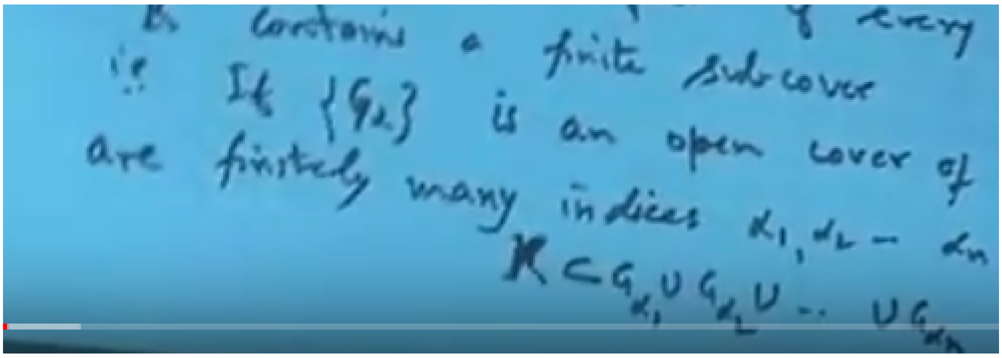

such that the finite union of these sets G alpha 1, G alpha 2, G alpha N finite union of these covers K okay covers K. So this set is said to be compact. So it just like this a compact set requires the finite number of the open sets which are responsible to cover the entire set. suppose we have a security system in our IIT’s and beyond is foolishness to apply to apply the security point wise, means infinite number of people you just put it on the security that's not a wise decision. So what we do we put of a poles, check post and then only finite number of check post is basically is sufficient to look after the security. So that way we said gate pass the security system in the gate pass. That forms a compact sets. Ok so it is just like an arbitrary a set E is said to be compact whenever any open cover of it has a finite circle. that is there are only finite number of open sets whose Union will cover the entire set X. obviously finite set is a compact set.

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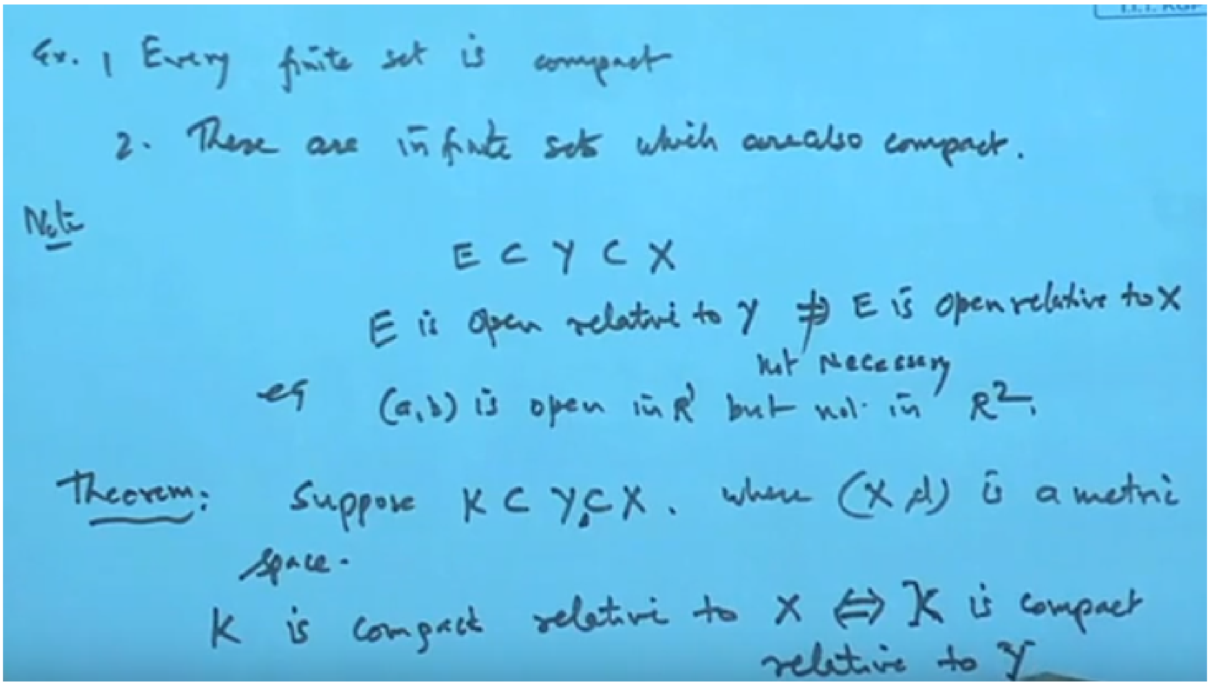

every finite set is compact. There are infinite sets also which are compact. So we will see that infinite complex sets in all case. We will see the they're all in finite sets which are also compact. So just I I take an example of a set of all the real numbers in the interval 0,1 it is infinite cuts X but it is compact because only we can divide say 2,3 intervals we can choose such that the union will cover the whole interval A, B. so that will be we will see the others example. For now one thing which we okay that suppose E is a subset of Y which is subset of X and if E is open, is open with respect to relative to Y relative to each open relative to Y, relative to Y, that does not imply that E is open relative to X, it not necessarily, not necessarily, not necessary that if a set is open in it or subset some metric space Y, because XT is a metric space, Y is a subset of X, so Y, D will also be a metric space. so if a set is open with respect to the metric space by D, then it may or may not remain open with respect to the metric X and examples we have seen the R1 and R2 an open interval a, b is open in R1 but not in R2. You know, so the openness all the closeness of a set depends under which the set is embedded. But this is not the case so far we considered the compactness. If a set is compact related to the Y, then it has to be compared related to X and vice versa. So this is an interesting result for the compact. We have this result as follows, suppose K is a subset of Y which is a subset of X, we are XD is a metric space, is a metric space. the result says, K is compact K is compact, relative to X if and only if K is compact, K is compact relative to Y.

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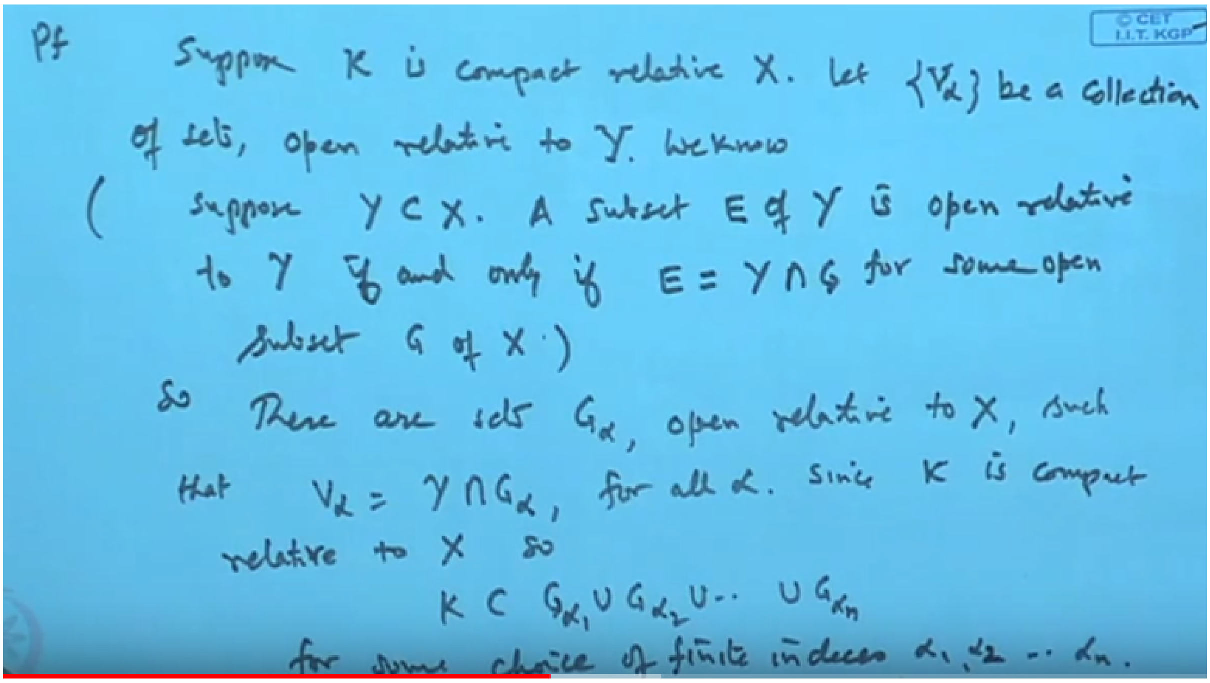

so let us see the proof of this. so what we want is that if K is compact with respect to Y, then it has to compare with respect to X and vice versa. So let us suppose K is compact, suppose K is compact relative to, with respect to or relative to X, it means if we choose a collection of the open set which are relative to X, those countable union covers K then there will be a finite subset, finite collection of those open set which can cover K and they say every open cover of K, in X will have a finite score. so this is no we want it to so K is compact relative to Y. so let us consider the open cover of K with respect to Y or relative to Y. so let V alpha be a collection, be a collection of sets open relative to Y, relative to Y. now prior to this we have one result, we know the result says suppose, Y is subset of a metric space X, a subset E of Y each open, relative to Y right through to Y, if and only if, if and only if there is a E can be can be written H, Y intersects and G, for some open subset G of X. this result we have already Shown. Okay. so using this result we can say if V alpha is giving to be an open relative to Y, so according to this result V alpha can be expressed edge, by intersection G alpha for some open subset G alpha of X. so we can say that they are all sets, so there are sets say G alpha, open relative to Y, relative to X, such that V alpha is nothing but the Y intersection G alpha for all form, mean V1 will be Y intersection G1, V2 will be a so correspondingly we can get the open sets G1, G2,...,GN, related to X. okay now what is given is K is compact relative to X. so G alpha is an open set relative to X. So this collection of this open cover will Have a finite circle. so since K is compact relative to relative to X, so there are, so K will be contained in the finite union of these open sets, G alpha 1 G alpha 2 G alpha n, okay? For some choice of finitely many indices for some choice is choice of finite indices and this is alpha 1, alpha 2, alpha N. okay like this.

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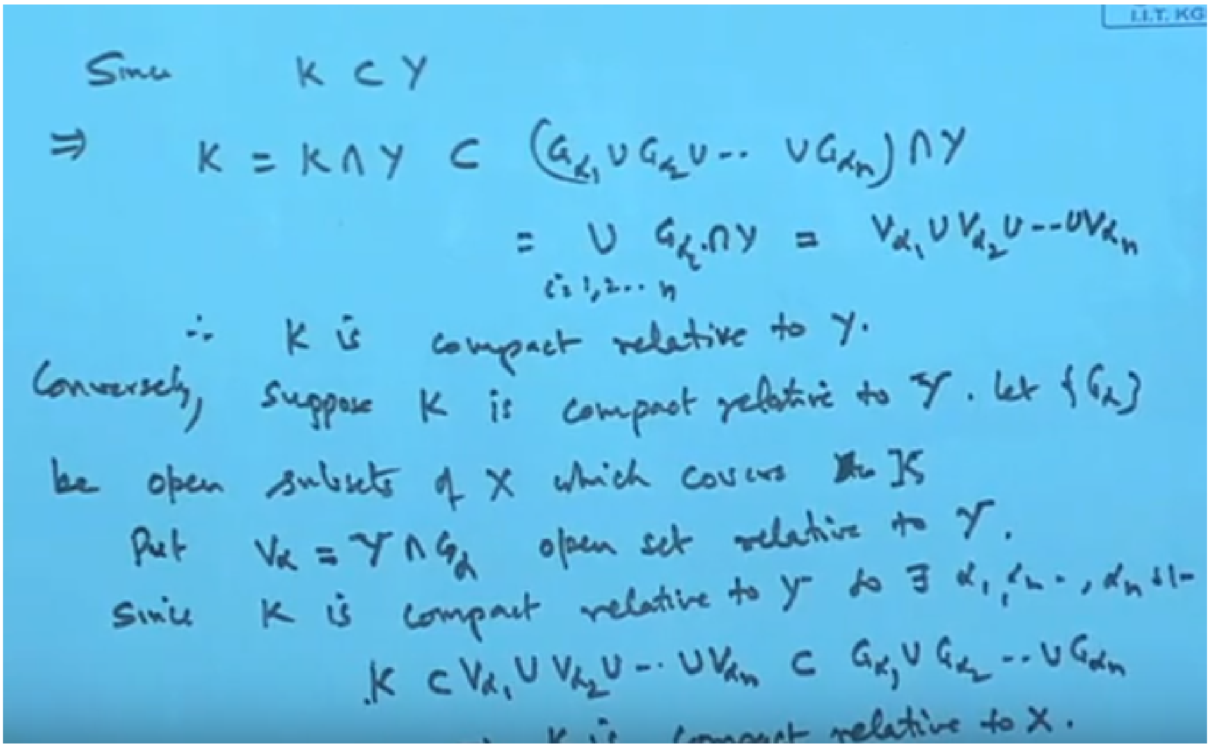

now if I take since K is contained in since K is a subset of X, K is contained in Y this is given this is given K is a subset of K is contained in by so K is contained by so this implies that K which is K intersection Y, is contained in, G alpha 1 Union, G alpha 2 Union, G alpha N, intersection Y and that is the same as this that union of union of G alpha I intersection Y, we are I is say 1,2 1, 2,.. n. ok but G alpha I intersection Y is nothing but V alpha, is it not so this is the same case the Union V alpha 1, Union V alpha 2, Union V alpha N. so case contained in this Union, that is they finite number of the open subsets relative to Y covers K. this shows K is compact, relative to Y case compact related to Y. now conversely suppose K is compact, relative to Y, relative to Y, we wanted to showcase compact relative to X, so let G alpha be a let G alpha be an open cover of X we open subsets we open subsets of X which covers which covers K, it's an open cover covers X okay miss covers K sorry because K is in compact it with respect to miss covers K okay. case compared with respect to Y what we are doing is we wanted to so case compared with respect to X, so let us find out an open cover G alpha of K. now if we prove that this open cover which is an open cover which are the open sets with respect to X and covers K, if it has a finite sub cover then obviously K will be compared with respect to X. okay so let us put the put V alpha, as the set Y intersection G alpha, okay by is a subset of X, G alpha is an open set, so this B alpha will be open set, relative to Y. relative to Y because of the previous result. okay now this given this is given case compact related to Y. so since K is compact relative to Y, to Y so this open cover of V alpha, has a finite sub cover, so there are these there exists alpha 1, alpha 2, alpha N, such that the union of these V alpha 1, V alpha 2, V alpha N, overs K, okay but V alpha 1, V alpha 2, V alpha N these are the subsets of what? subsets of G alpha 1, Union G alpha 2, Union G alpha, N. B alpha 1 is a subset of G alpha 1 B alpha and a subset of G alpha so K is also contained in the finite union of these open subsets, which are open with respect to X. therefore every open cover of X, K relative to X have a finite sub cover. So this implies K is compact relative to X and that's proved the result. **Ok.**

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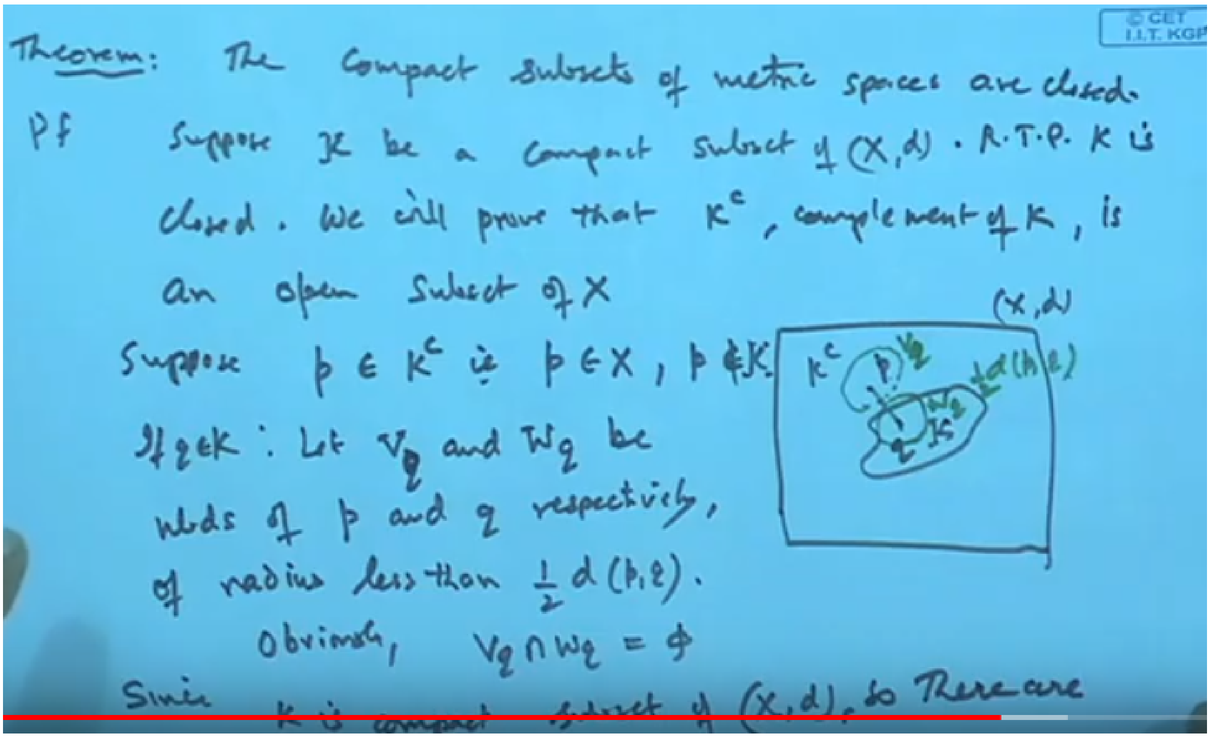

Another result shows that closed subset compact subsets. The compact subsets of a metric pace of metric spaces, spaces are closed. Every compact subset of metric space will be a closed set, whatever the metric maybe, is an arbitary metric space, every compare subset is a closed set. Let's see the proof for this we will prove by contradiction. Suppose K be a compact subset of X,d of a matric space. so in order to show the case is closed if I prove this complement is open subset of X then is ok so in order to prove required to prove is K is closed, closed so what we will do we will so we will prove that its complement KC complement of K, this is an open subset of X subset of X. so here we have this is our metric space this is a set K, what we want is the complement of this KC each open with respect to X. so it means if I take any point here say P which does not belongs to K and if we are able to so there exists a neighbourhood around the point P which is totally contained in KC then obviously this point P will be an interior point of KC but P is an arbiter so we can say every point along of KC we can draw the neighbourhood which is totally contained in casing so KC becomes open. So that's the idea of the proof. so let us take suppose P is a point belonging to a complement of K, that is KC, okay but compare that is an X, or you can say P that is P is in X, but but P is not in K, that is the meaning of this P is in X but it's not in K. now let us take a point Q. If Q belongs to K if Q belongs to K then there is a distance between P and Q, so this is the distance between PQ.   the distance of this is nothing but the D of P Q so if I take a radius less than half of the distance of PQ and draw the neighbourhood around these points, like this door the neighbourhood around this point like this then this neighbourhood will be disguised. so let this neighbourhood be denoted by Vq and this neighbourhood we denote by Wq. so let us take this, let Vp, Vq sorry and Wq be neighbourhoods of P, and q respectively, of radius less than half the distance between Pq. Okay, then obviously that Vq intersection Wq will be the joints that is two. Okay now let us take since K, is compact K is given to be compact is a compact K is compacts subsets of metric space X T so every open cover of K will have a finite sub cover so there are the points q1, q2,...qn such that the open ball drawn at q1, q2,...qn and finite number of pin bar will be sufficient to cover K so let us say there are finitely so there are finitely,

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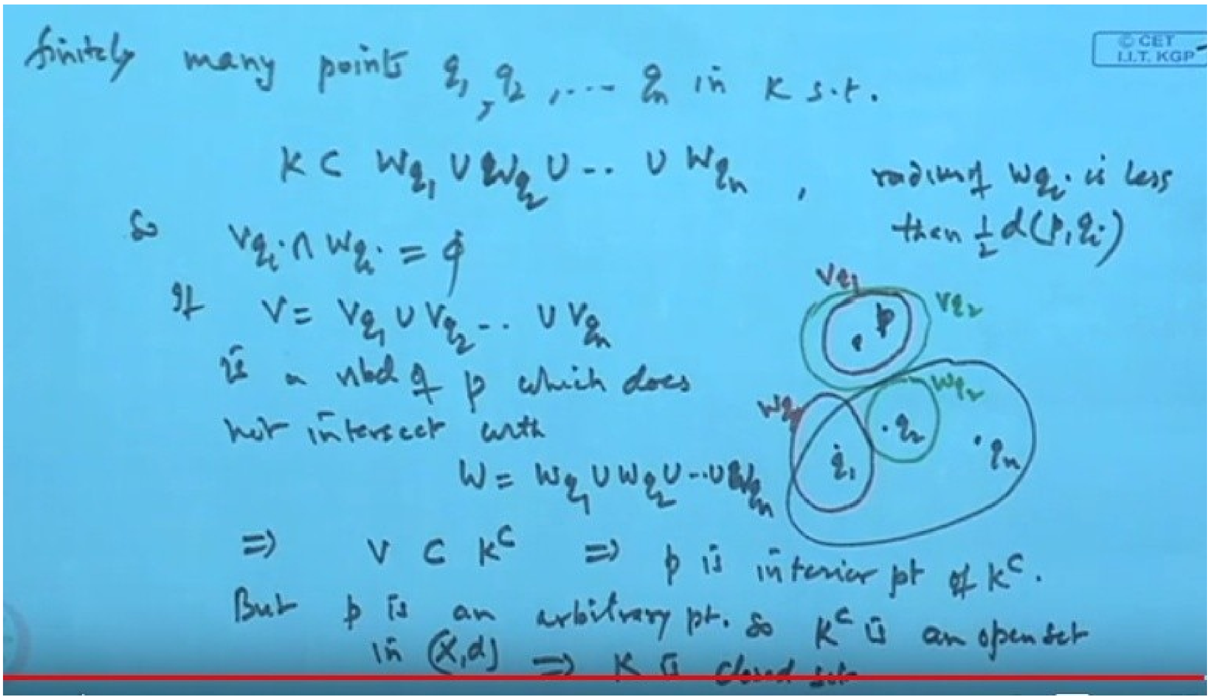

There are finitely many points say q1, q2,qn, in K such that such that the open balls or never were drawn at this point with the radius 1/2 of q1, P etc..., will remain. Such that the neighborhood drawn at these points with the radius I will say after so Wq2 Union, Wq2,.. Wqn, where the radius of this, radius of W say, qi is half of less than half of the distance from P to qi. okay now this finite number of these neighborhood will cover K because K is compact. okay now since these neighborhoods you are drawing this is our K, here is q1, q2 say qn, here is P, so what we are doing is we are drawing a ball say this ball okay and here it say this ball so here it is Wq1, this is nothing but Vq1, which are adjacent. Then we are taking say q2, so again P2 we are taking say this is our say Vq2 and here it is say Wq2, again they are disjoint because the distance does not match does not is less than half of the radius is less than half of P of q2. so obviously they this Vq1, Vq2, ...Vqn. will be designed with that okay so let if so vq 1 vqi intersection w qy is empty. Okay that is so if we take V as finite union of Vq1, Vq2, Vqn, suppose this finite union, then this will be a neighbourhood, is a neighborhood of P. is a neighborhood of P and this neighborhood does not intersect which does not intersect with W. with W which is W means Wq1 Union, Wq2 Union, this one say Wqn. this does not intersect with this because this is the smallest one which is de join which every man so we have it totally a neighbourhood. so this implies that this neighborhood V, is totally contained in the complement part of K. therefore P is an interior point of K complement, but P is an arbitrary, arbitrary point. so this shows that interior of this is an open set in x,d. so once it is open then K must be closed. so every compelled subset of a metric space is a closed set. And that's what.