Model 1

Lecture - 1

Course

On Intodructory Course in Real Analysis

Okay, so this Introductory Real Analysis. This course is a basic course in the analysis which is a big one, of many courses, in pure and applied mathematics and statistics. This is very useful course, for any branch of Science and Engineering students. The present course has been designed to introduce the subjects, to UG undergraduate, as well as, for postgraduate restraints in Science and Engineering. The course contents a good introduction on each topic and an advanced treatment of the theory, at a fairly under less, suitable, understandable level to the students, at this stage. Now before starting the lecture, let us see the real number system, how the real number system is developed. In fact, dead kinds and **Cantors** give a systematic way of its development, by using the curse and the sequences respectively. Now this we will not discuss the **Kent Ithorian** and dedicatory. We will start the real number system, concepts of the sets, continuity, differentiability, Riemann integral and so on. But this says, it also an important topic, dead kinds and **canta story.** So the interested students may go through the book, which is available, these topics are available any real analysis book for that.

Okay, now we know the natural number system, so we assume, the students are familiar with the natural number system, 1 2 3 and so on. When we include 0 in this system, then you get the whole number, then we get the integers, rational number, real number, and so on. So, this is, these sets are developed slowly, as we recall the things.

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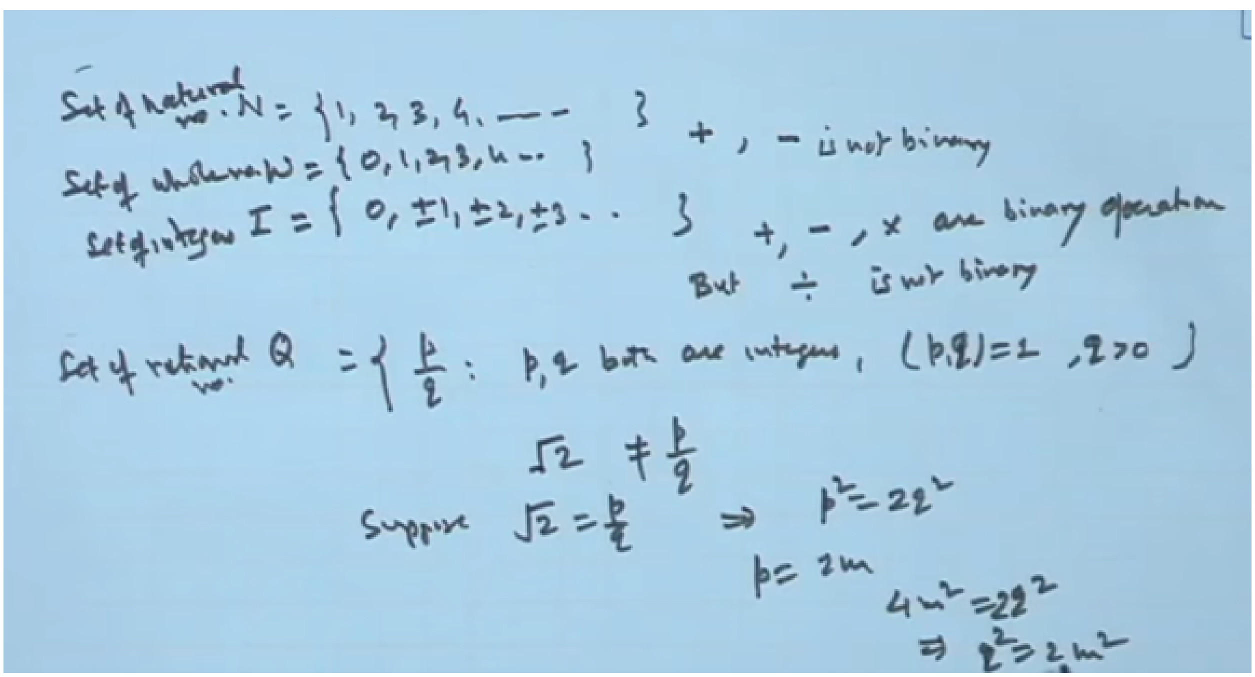

In fact when we say the set of natural number, it is basically one, two, three, the collection of these numbers, one two three four and so on. Now over this set the addition is a binary operation that is if I add any two natural number, we always get a natural number. But the subtraction is not binary. It means, that if I picked up the two elements of this, then and subtract them, then it may or may not belongs to n. For example if I take, 4 & 3, then 4 minus 3 is available one, but 3 minus 4 is not available 1. So this shows the system N or the class n of natural number is not at all complete still. So in order to get the subtraction, also as a binary operation on this collection we have to take all these numbers 0 plus, minus 1 plus, minus 2 plus minus 3 and so on and this correction, we say, it is the set of all integers.

There is one more extension of N. When we add 0, in the set of natural number, then this new is known as the set of whole numbers. So this is the set of natural number, this is the set of whole numbers and this is the set of integers. Now over the integer I, we see plus minus and multiplication are binary operations. However, the division is not binary. That is, if I picked up any two integer and divide, provided the denominator is nonzero, then we get a fractional, not a integer. So this again shows, there is a possibility of further extension of I and that leads to the concept of Q, the set of rational point, number. So the set of rational number is of the form P by Q, where P and Q both are integers, the greatest common divisor of PQ is 1 and Q is positive. So this collection, gives you the, all possible number, which is of the form P by Q, integers can also be a part of it, when you take Q to be one, set of all integer will come and like this. So natural number, whole number and integer, they are the subset of the set of lesser number. But still this set is not complete, is not complete. Because if I looked at root 2, then Root 2 cannot be expressed in the form of P by Q and the reason is very simple. Suppose it can be expressed in the form of P by Q, suppose, then this implies that P square becomes 2 Q square. So when we divide by 2, it means P must be a something, divisible by 2. So we can write P as 2 times of M.

So P can be written as 2 times M and once you write 2 times M, then we get this, 4 M square is 2 Q Square, this implies Q square is also is 2m square. So again there is a factor, available in Q as 2. It means there is a common factor in P and Q so it cannot be expressed in the form of P by Q, where the common divisor is 1, highest sefe is 1. Okay? Therefore it is not at all a irrational number. So, where we should put? So such a number which are not rational number and radical science and so on, these are called, 'Irrational Numbers'.

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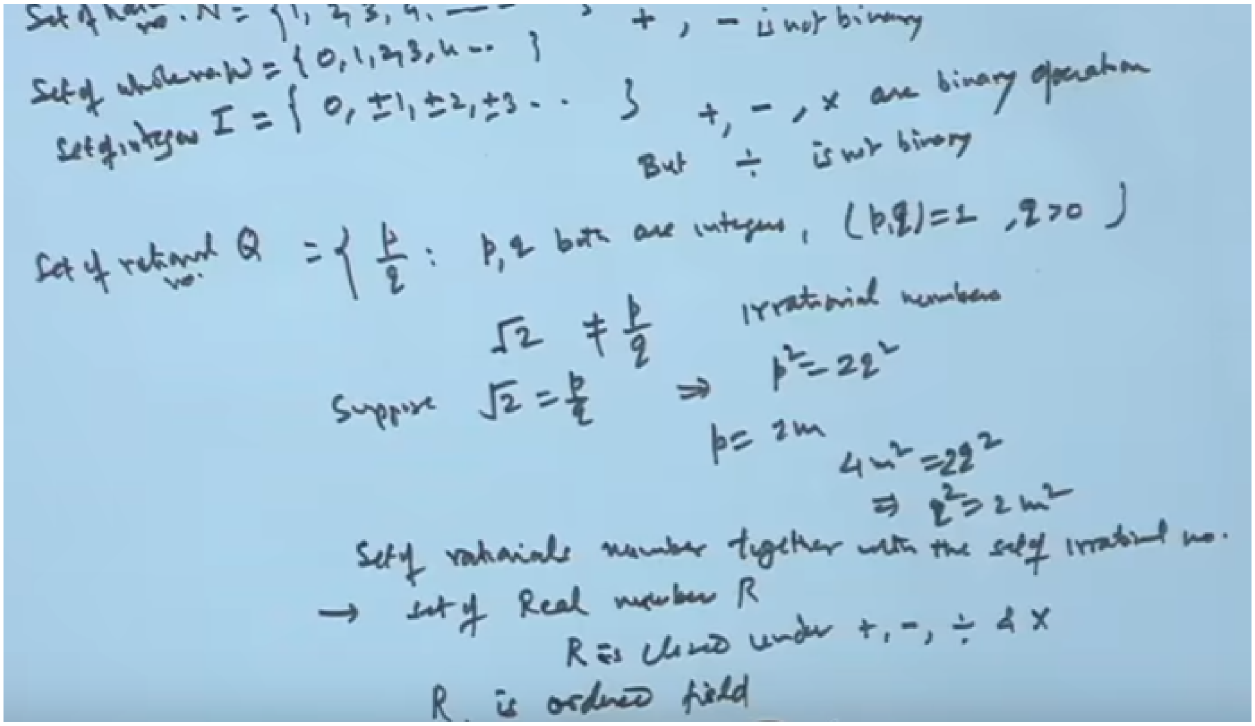

So set of a rational number and set of rational number, together with the set of irrational number together with the set of irrational number this gives you the set of real numbers. It means, our set of real number R, contains all the rational points, all the rational point, and in this set of real number R is closed, under addition, subtraction, division and multiplicatio**n.** Okay? And not only this. This R is an ordered set, ordered field. R is an ordered field, ordered field. It means, we can introduce the some partial relation on it or if we picked up any two real number, one can always decide, or one can always find out, whether one is strictly less than other or is strictly greater than other or both are equal. So this shows the come structure is an ordered structure and the field of course when we discuss a you have gone through the real come algebra part filled, a vector space, together with the addition and multiplication and all other conditions are satisfied and form the field.

So this is a topic, which we may discuss, you may go through, in the lineage of apart field, but it is an ordered field for Union. Now throughout this lecture, we will use three books. The books which we will use is,

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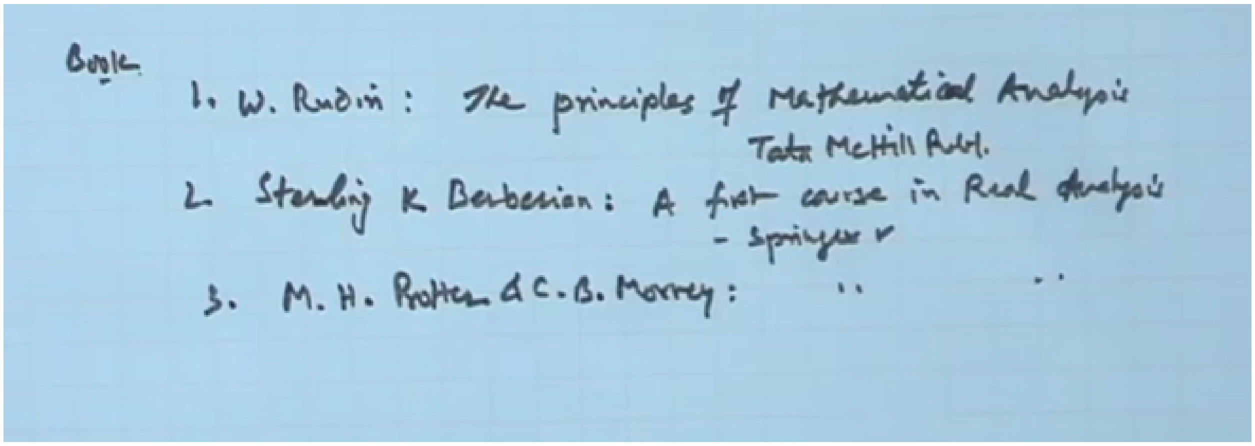

the first book is our Water Rudin, this is a book, the principle of principles of Mathematical Analysis, analysis. This is basically a Tata Mchill publication. Second book we will use Sterling K Barbarian, a first course, in real analysis. This is an Springer for log, book publication and third is MH Protter and CB Morrey. The same title will spin develop. So we, okay so today we will discuss the basic topology,

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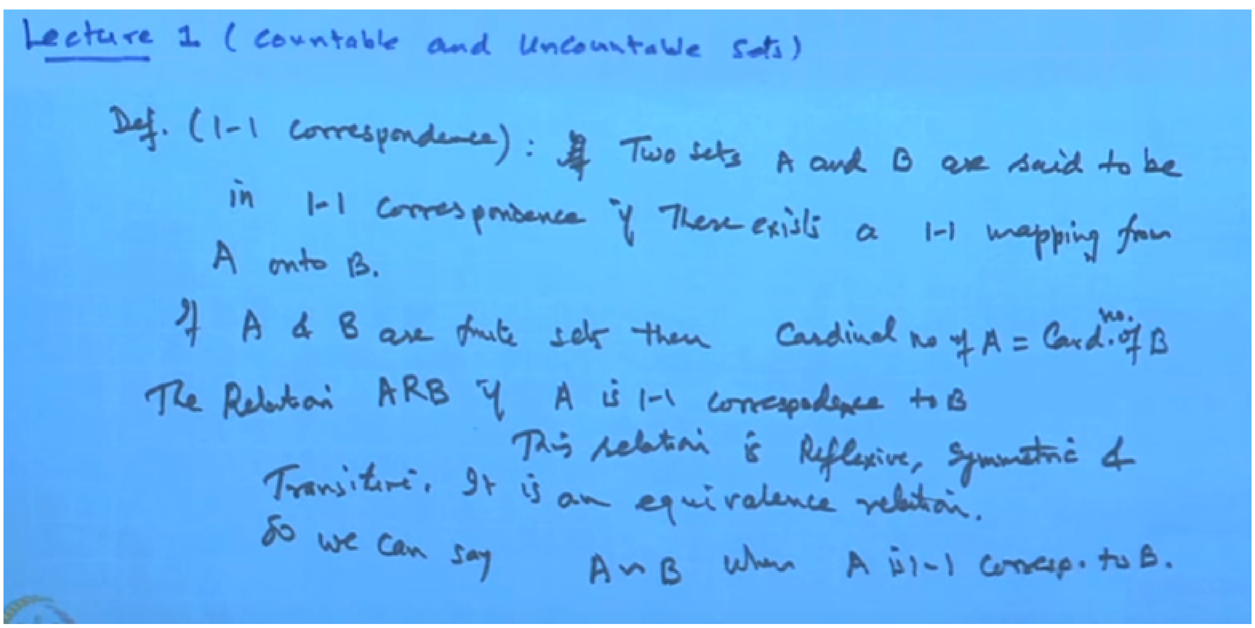

On the set of real numbers and first we will discuss what is one one correspondent, then we will go for the countable and uncountable concept of the countable and uncountable sets. The two sets A and B, are said to have one-two-one corres, are said to be in one-to-one correspondence, in one-to-one correspondence. If there exists if there exists, a 1:1 mapping, 1:1 mapping, from A on to B. And then, if A and B are finite, then we say the cardinality of A and cardinality B are the same if A and B are finite sets, then the cardinality then the cardinality or cardinal number of A is the same as the cardinal number of B, cardinal number of B. But if A is infinite, then there is no sense of talking the number of element in this set, both. So in that case when A is, we are infinite set, then instead of saying the cardinality is the same, we say,' they have a one-to-one correspondence', that is a more meaningful, than saying,' the numbers are same'.

So this is and the relation which we have, if we put the relation, suppose A is related to B, if A is one-to-one correspondence is one-to-one correspondence to B, then this relation, the relation, this relation is this relation is obviously, is reflexive, symmetric and transitive and transitive. So it is a reflection it is a equivalence relation. so it is an equal so it is an equivalence relation. So we also say, so we can say, we can say that A is equivalent to B, when A is one to one correspondence to B. So that is the way we define.

Now using the concept of the one-to-one correspondence, we can now define the finite set, infinite set, countable and uncountable set.

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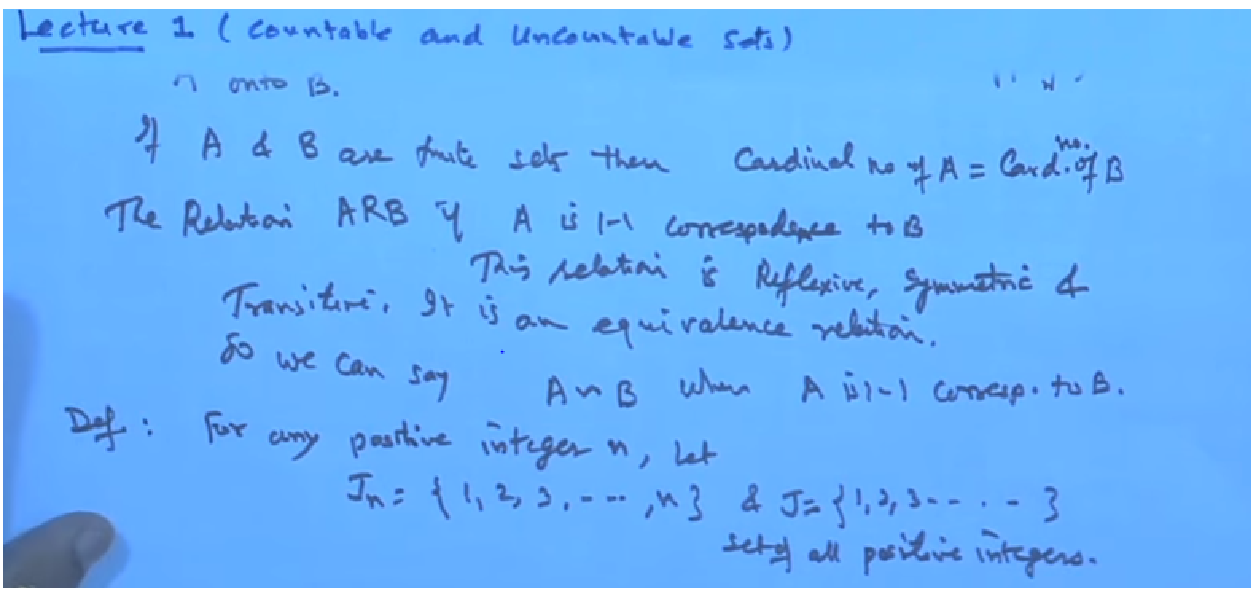

The definitions for any positive integer, for any positive integer, positive integer, say n Let J n represent the sets having the element 1 2 3 say up to n. The first n in natural number, of positive integers and let J is the set of all positive integers 1 2 3 and so on. This is the set of all positive integer, positive integers, okay?

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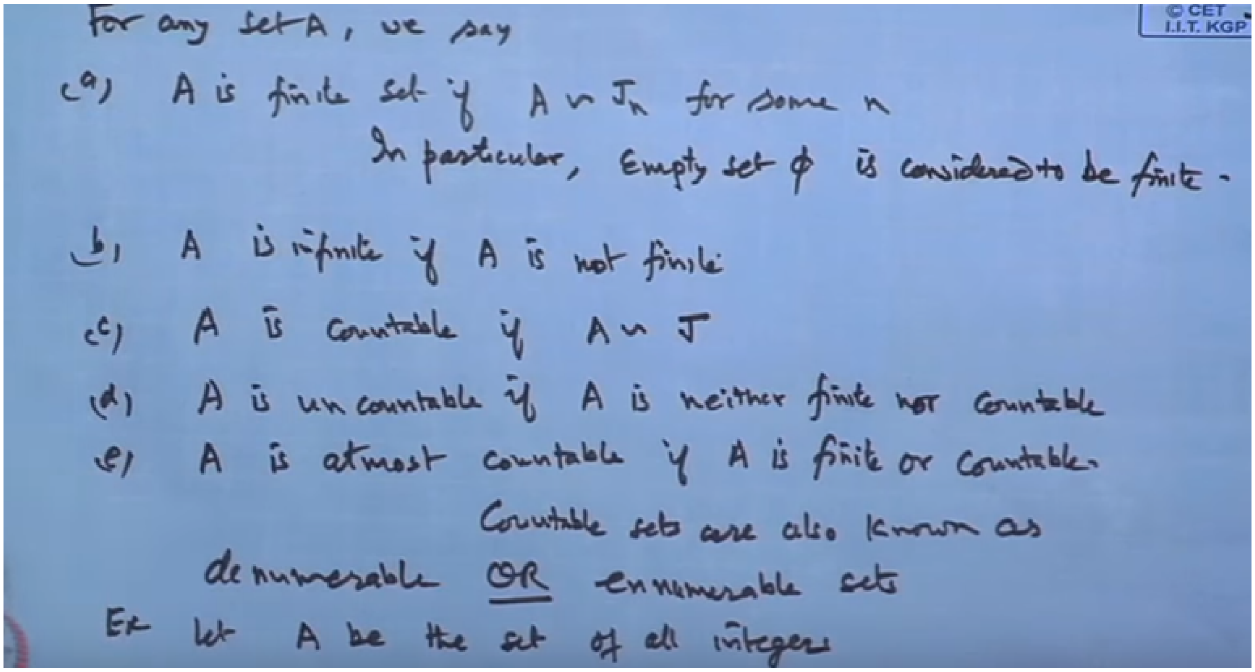

Then for any set A we define we say A is finite is a finite set, is finite set if A is equivalent to j n, for some n, for some n, obviously, once it is equal to some n, then n is fixed, so J n is finite, the number of the terms is n only, so a will also be finite one-to-one correspondence and the set will be a finite set. Empty set in particular considered to be a finite set, so in particular empty set Phi is considered, to be considered to be finite. Then a set B set A, it is set to be infinite, if is not finite. In fact this definition, we can further modify it and we get a better way of defining the infinite set in in the next top and next part, when we discuss about countability, okay? So A is finite means if it is not infinite, A is infinite means if it is not finite. And C, A is countable, if A is equivalent to J, that is there is a one-to-one correspondence, between the elements of A and the J. Or we can define a mapping, from set of positive integer to A which is 1 1, then such a set A, is said to be a countable set, okay? So this J we have though is started from 1 to N, 1 to infinity, we can also take J from 0 1 to n we starting at the point X 0 0 corresponding the point X 0 X 1 corresponding to point X 1 and so on. So we can also consider then positive or non-negative integers, positive integers, including 0. OK, the A uncountable, is said to be uncountable if A is neither, neither finite, nor countable, nor countable, and E we say A at most countable, is at most countable if A is finite or countable.

Finite set is also countable, set, but if the set is finite, advantage of and also I'll count means finite, that set will be considered countable, countable set it countable. So we say is almost countable means, either a is finite or may be a countable set, that's what countable. Countable sets are also known as, countable sets are also known as, 'denumerable set', denumerable or enumerable sets, Okay? Let us take some examples. We are this we say let A be the set of set of all integers, may be the set of all integers, okay? Then, set of all integers, we claim that this set of integers, A is countable.

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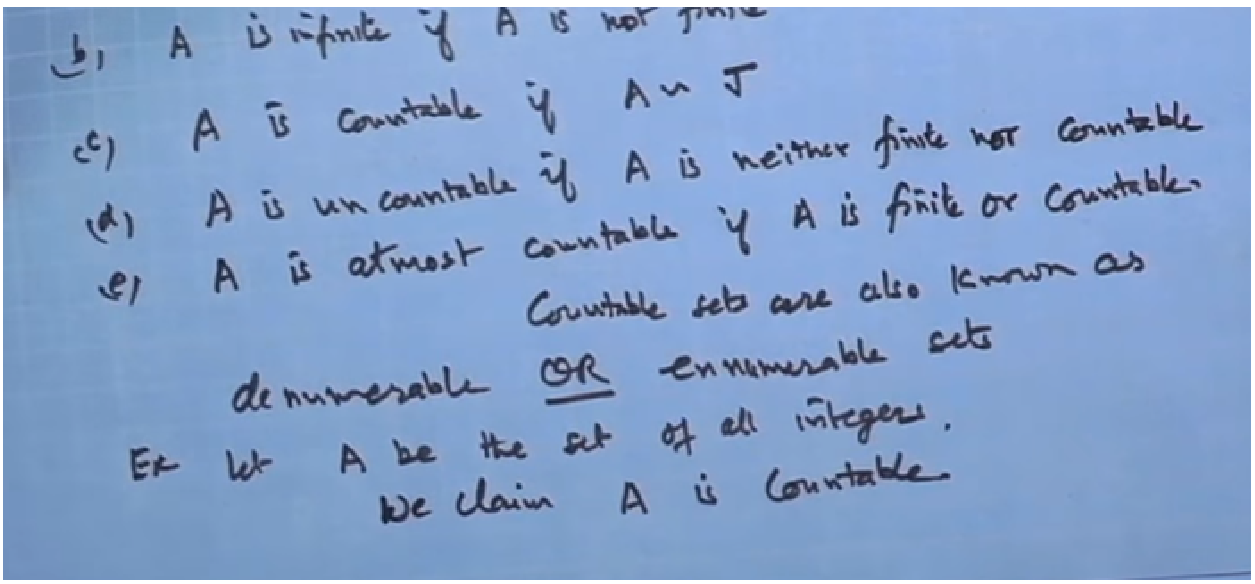

We claim A is countable. It is countable. It means we are able to define a one-to-one correspondence between the sets, a positive integer and the set of the elements A.

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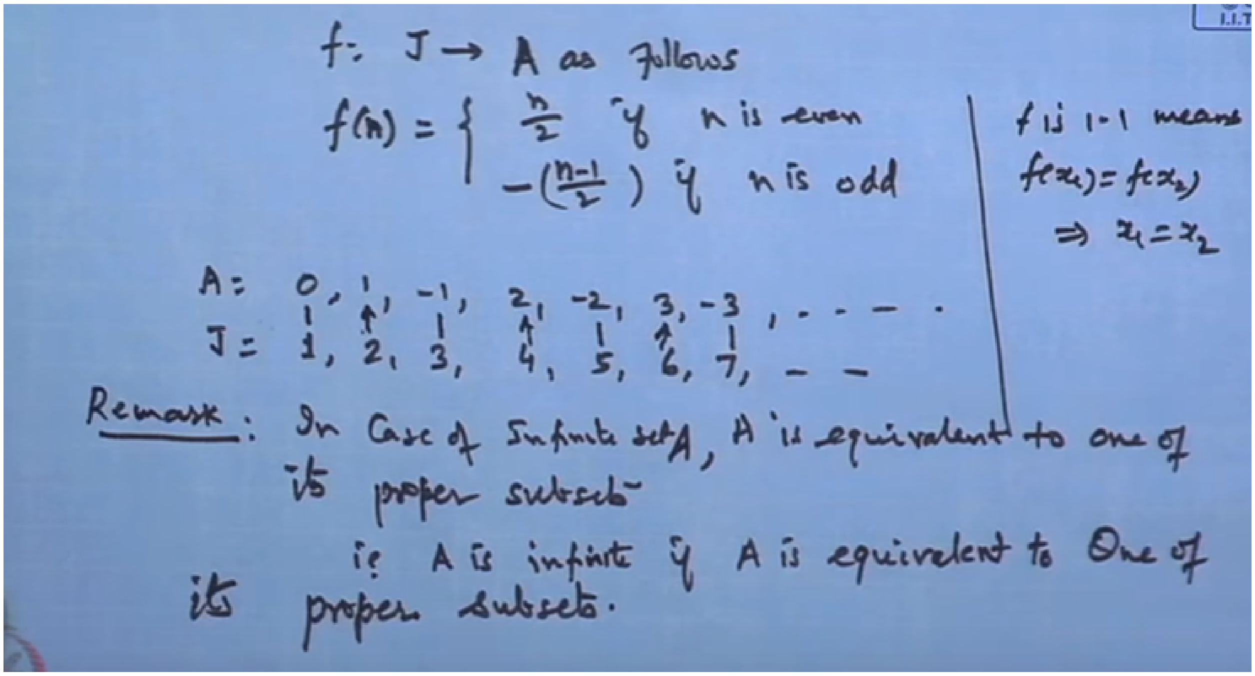

So let us define a mapping F, from the set of positive integer J to A as follows. if I take the image of any n under F, is say and by 2, if n is even integer, even positive integer and otherwise when we say n minus 1 by 2, if n is odd, positive, integer. So what we see here is, that if we take A which is the set like 0 1 minus 1 2 minus 2 3 minus 3 and so on and J, which is the set of positive integer, 1 2 3 4 5 6 7 and so on. So what it says is, as soon as n is even it will give n by 2 so this is related to here, okay? Then 4 will go to here, 6 will go like this, and when n is odd, when n is odd, then you are getting this thing, this thing, this thing, this thing, and of course 1 will go to 0. So there is a one-to-one correspondence between the elements of the set a and set J so obviously this and this mapping is 1:1 mapping one-one mapping we can just check it, because F is 1 1. Means f of x1 equal to F of x2, so it implies x1 equal to x2 so here if it say n is even then, obviously when you take F N 1 equal to F n 2. obviously n 1 comes out to n 2 similarly here when n is odd, we are getting same so obviously F is 1 1 here. So that is why this step set of positive integer of integers is a countable set. One more thing which we can see here, is a remark. What we have seen is that this is a set of integer, but J is a set of positive integer, J is a proper subset of A. So, but they are having a one-to-one correspondence.

So what we can say what we conclude or we observe here , in case of the infinite set, we can say A infinite, when A equivalent to one of its proper subsets, in case of infinite set, a proper subset, may be equivalent to the set itself. So in case of in finite set, a proper subset, A may be, is equivalent to, A is equivalent to, equal one of its proper subsets. And this is also a way to define an infinite set. We say, a set is infinite, if A is equal to one of its proper subsets, so we say and that is that is A is infinite, A is infinite set, if infinite set, if A equivalent to equal to one of its proper subsets. That's how, okay?

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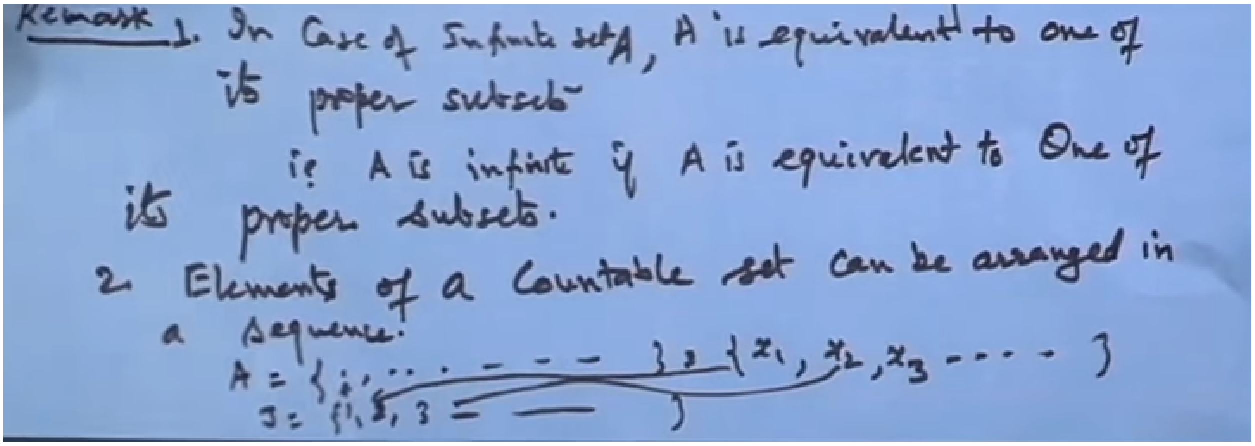

Another remark we can put it here that every elements of any countable set can be arranged in the form of the sequence. Elements of a countable set, of a countable set, elements of a countable set, may be arranged, can be arranged arranged in a sequence. Because basically what we we have a one to one correspondence with the set of positive integer so corresponding to one we are getting X 1, corresponding to 2 we are getting X 2, corresponding to 3 we are getting so this form basically sequence, because it's like this if is and he said having the elements see here these are the elements for this set a ok then J is what J is 1 2 3 and so on, it has a one-to-one correspondence so corresponding to 1 you are getting X 1, corresponding to 2 you are getting X 2, corresponding to three you are getting X 3 and so on. So this has a one-to-one correspondence with this is it not like this so we get the one-to-one correspondence between the set of net positive integer and the elements of the set. So, element of a countable set can be arranged in the form of the sequence.

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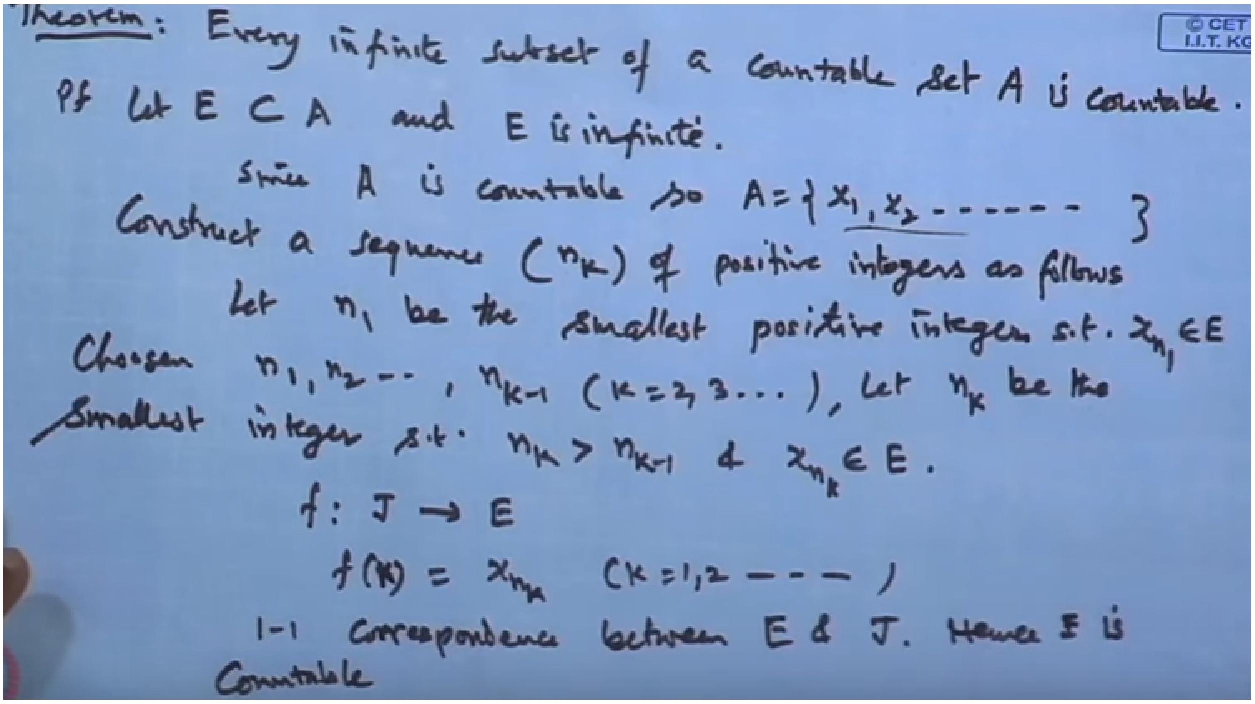

So this is also remark which we can't use we will use it now this is a interesting result and result says, every infinite subset, subset of a countable set, of a countable set, a each countable. So proof is, let E be subset of A, and E is infinite, is infinite. Now since A, since A is countable, so we can arrange the element of A in the form of sequence. So A will have the sequence like X 1 X 2 and so on. All the elements of the set can be arranged in the form of the sequence, okay? Now let us construct a sequence, construct a sequence NK of positive integers, as follows, as follows suppose n 1 with the smallest positive integer, be the smallest positive integer, such that, xnK, Xn1 is an element of e. Means, from this and 1 2 3 and so on, suppose I am taking the n 1, n 1 is the smallest integer, so that the first x + 1 corresponding to this X n 1 isn't it means out of these the first element which you are getting is excellent one belongs to E. Then assume that n 1 chosen n 1 and to say and K minus 1, where K is 2 3 & 4 so on. These are the, after choosing in such a way, when n 2 is greater than n 1 and such that Xn 2 belongs to E and so on. Let us take n K now. Be the smallest integer, with the smallest integer, with the smallest integer, such that, be the smallest integer, such that n K is greater than n K minus 1 and the corresponding term of the sequence xnk, belongs to e, okay?

So now let us introduce the function f from J to e. So if we take F of n, as X of n K F F of K let us take f of K is X of n K, where K is 1 2 3 and so on. Then what we see here there is a one-to-one correspondence between J and E because for K 1 X 1 X and one is any case 2 xn 2 is any xn

K is any and like this so this is a one-to-one correspondence, between E and J, hence E is countable, okay? So this shows that every infinite subset of a countable set, is countable, okay? Clear? Is it okay? Now here, as we have seen, that if a is countable, we can put it in the form of a sequence X 1 X 2 X and all the elements, we can arrange in the form of the sequence 1 X 2 X n. If I generalize it say why because 1 to N is basically the set of positive integer, so instead of this we can take the collection family of the sets also.

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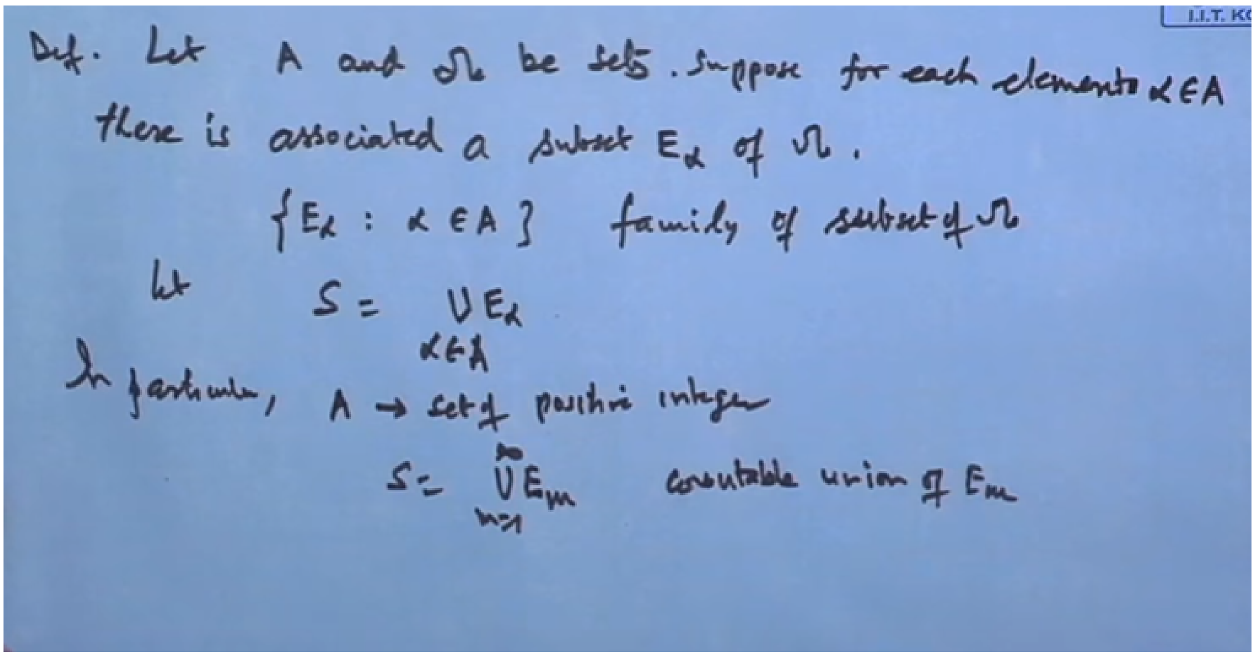

So we define like this let a and Omega be sets and suppose with each element alpha, of a, there is associated a subset of Omega, which is known as suppose, for each for each element alpha, belongs to a alpha, a there is, there is associated a subset e alpha of Omega, subsets are denoted by E alpha of Omega. Okay? Then the collection of all these alpha then this collection a alpha we are alpha belongs to a this collection alpha is the collection of these sets is the family of these sets family of sets or subsets of two subsets of Omega, family of subsets of Omega. Now if we take s, as the union of e alpha, when alpha belongs to a, then this for at least one we used, then any element belongs to this means, it will be in one of the alpha, e alpha like this. so for at least one of our used okay and in particular in particular when a is an positive integer, is the set of positive integer, then s becomes Union, e m, M is 1 to infinity and this we say it is a countable union, of EMS, like this. So even similarly for the intersection, also we can interface so this will we need it we can just.