Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture – 09
Solution of 3 Dimensional Parabolic Problem

Good morning everyone, welcome to this session. We are looking into the solution of multi dimensional parabolic partial differential equation that which we have well posed.

So, in the last class we have seen how an actual problem can be imposed problem and how it can be converted into well posed problem in order to have seperational valuable type of solution. Now, in the last class at the end we were starting about the about the 3 dimensional parabolic partial differential equation and solution by separation of valuable. Now, in today's class we will be finishing it over and see that how the solution will be evolved using seperational valuable. Then next after that we will be looking at to the problem which will be 4 dimensional problems.

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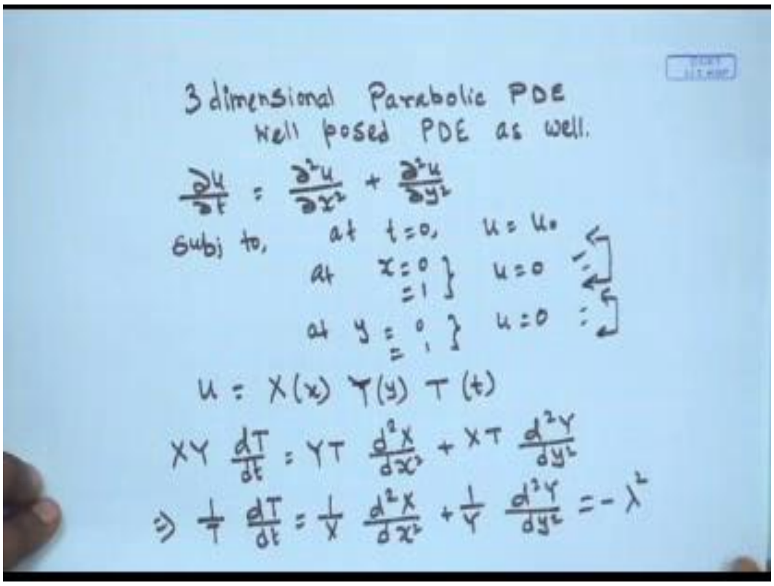

So, let us look into a 3 dimensional parabolic PDE, but which is a well posed PDE as well the governing equation will be del u del t is equal to del square u del x square plus del square u del y square subject to at t is equal to 0, we have u is equal to u naught at x is equal to 0 and 1 u is equal to 0 at y is equal to 0 and 1 u is equal to 0.

So, we have all the four boundary conditions to be homogeneous and 1 initial condition to be non-homogeneous and then will be looking for a seperational valuable type of solution. Therefore, the solution will be will be assuming that it will be a product of 3 function and each function will be a sole function of the independent valuable that means, u will be a product of a function which will be a solute function of type another function will be solute function of x and then it will be another function which will be solute of function of y.

Therefore, u can be represented as X of x, Y of y, T of t then what we are going to doing we are substituting the solution with the governing equation and let us see what we get x y d t d t is equal to y t d square x d x square plus x t d square y d y square then we divide both side by x y t and we will be getting 1 over t d t d t is equal to 1 over x d square x d x square plus 1 over y d square y d y square.

Now, the left hand side is function of t alone the right hand side is function of space it is a function of x and y. Therefore, they are equal and they will be equal to some constant and this constant has to be either 0 negative or positive and we have seen earlier that if it is 0 then will be; if it is 0 and positive will be getting a trivial. We will be landing into a trivial solution, but we are looking for the non trivial solution.

So, therefore, this equal to some constant and that will be a negative constant minus lambda square because we are going to formulate a standard and independent Eigen value problem in the x direction and independent Eigen value problem in the y direction, simply because the boundary conditions at these directions are homogeneous. So, if we have a homogeneous set of boundary conditions in a particular direction then will be formulating an independent standard Eigen value problem in the direction. So, let us look into the solution of the special valuation part.

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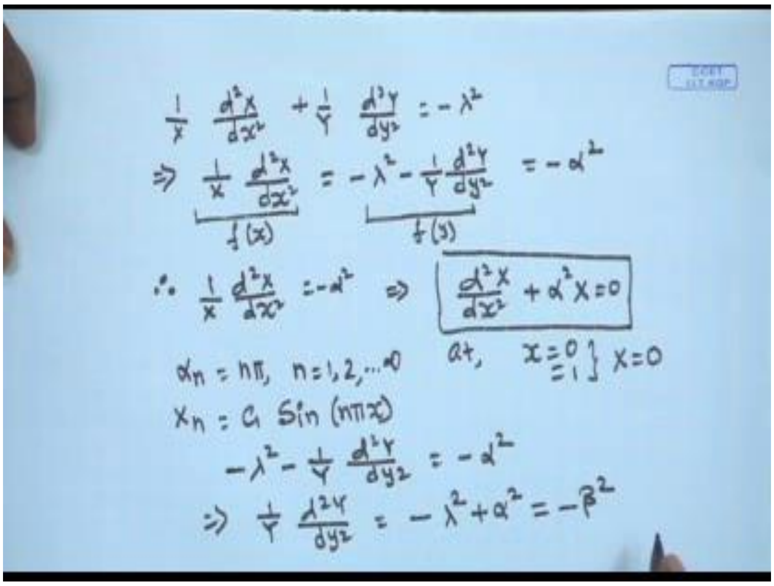

So, 1 over x d square x d x square plus 1 over y d square y d y square will be is equal to minus lambda square. So, again we can write 1 over x d square x d x square take the y wherein part on the other side minus lambda square minus 1 over y d square y d y square. Now again, this part is a function of x alone and this part is a function of y alone and they are equal they will be equal to some constant and let us say this constant is minus alpha square because if this constant is positive and 0 positive and 0 then we will getting a trivial solution and we are looking for the non trivial solution.

Therefore, we can formulate the standard Eigen value problem in the x direction as d square x d x square is equal to minus alpha square. So, you can get d square x d x square plus alpha square x is equal to 0 and the boundary condition of the original problem in the x direction should be satisfied by the x value part and the capital x. Since they were the boundary condition, we will be having a boundary condition here as well that is the x is equal to 0 and 1 capital x is equal to 0. So, we can formulate a standard Eigen value problem in the x direction because the boundary conditions are homogeneous in the direction and we know the solution of this problem, the Eigen values are n phi and the Eigen functions are sin n phi e x.

So, you write them, alpha n is equal to n phi where n index n runs from 1, 2 up to infinity and the Eigen functions are sin functions sin n phi x, this is the solution. Now, let us turn back our attention to the y varying part, let us see how the y varying part looks. So, minus lambda square minus 1 over y d square y d y square is equal to minus alpha square. So, therefore, 1 over y d square y d y square is equal to minus lambda square plus alpha square and again the combination of the 2 two constants will be a constant and again this constant has to be a negative constant, otherwise we will be landing up with a trivial solution which is not desirable and we are looking for a non trivial solution. So, let us look solve the y varying part now.

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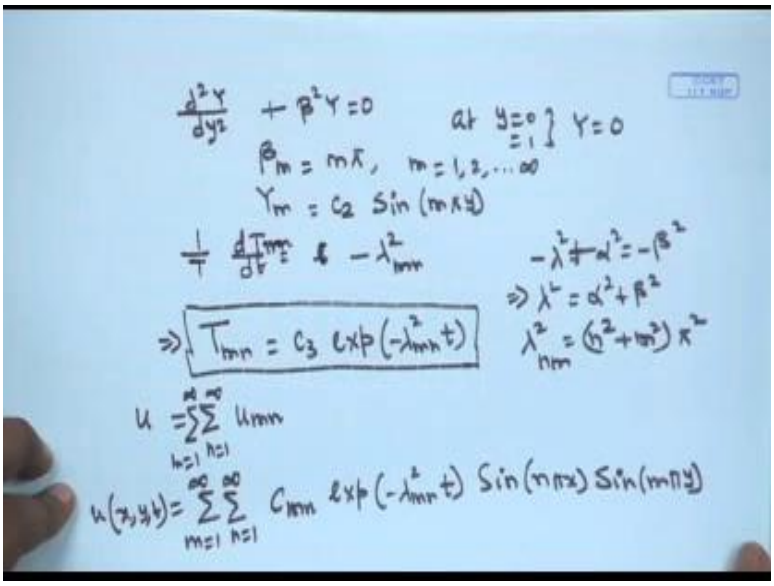

It becomes d square y d y square plus beta square y is equal to 0 and the boundary conditions of the original problem should be satisfied by the y varying part. Therefore, at y is equal to 0 and 1 our capital y is equal to 0. So, you know again the solution of this, but in this particular case if we see that both x varying part and y varying part they will constitute an independent Eigen value problem. So, I will be writing a different subscript to indicate the solution in the x varying part and the y varying part because these 2 are not related at all, they are not they are not depending on each other and they will be formulate an independent set of Eigen value problem.

So, the boundary condition the Eigen values will be beta m is equal to m phi where the index m runs from 1, 2, 3 up to infinity and y m will be C 2 sin m phi y the index m stands for Eigen value problem in the y direction and m stands for the Eigen value problem in the x direction. Now, we can look into the solution of time varying part if you remember time varying part was 1 over t d t, d t is equal to exponential it would have been minus lambda square and since, let us look into the definition of lambda, lambda square minus lambda square is equal to plus alpha square is equal to minus beta square.

So, lambda square will be nothing, but alpha square plus beta square and let us put the value of alpha and beta alpha is nothing, but n m phi. So, it will be n square beta is m phi. So, this will be m square phi square. So, since lambda is function of both m and n we write lambda n m is equal to n square plus m square phi square. So, I write lambda n m here and this will be t m n. So, therefore, the solution of t m n will be nothing, but another constant, let us say C 3 exponential minus lambda m n square t. So, this is the solution of the time varying part and where the constant lambda m n square is it can be written as n square plus m square into phi square.

Now, we are in a position to construct the complete solution by multiplying each of these component together and then adding them up by using the principle of linear super position if we really do that then the solution will be constituted by summation of u m n and there will be a double summation because we will be considering 2 Eigen value problems 1 is the x direction and 1 is in the y direction they are independent to each other therefore, there will be appearing 2 summation in the y in the solution indicating 1 summation over the x over m and over m and another summation over y over m.

So, therefore, n is equal to 1 to infinity m is equal to 1 to infinity and this will be 1 to infinity n is equal to 1 infinity and multiplied up to 3 constants; C 1, C 2, C 3 will be it will be giving you new constant C m n and exponential minus lambda m n a square t sin n phi x sin m phi y. Now, these constitutes the solution of u now only 1 step is left behind that is the evaluation of constant C m n and the constant C m n can be evaluated from the from the initial condition that has not been utilized till now that is at e is equal to 0 u is equal to q naught.

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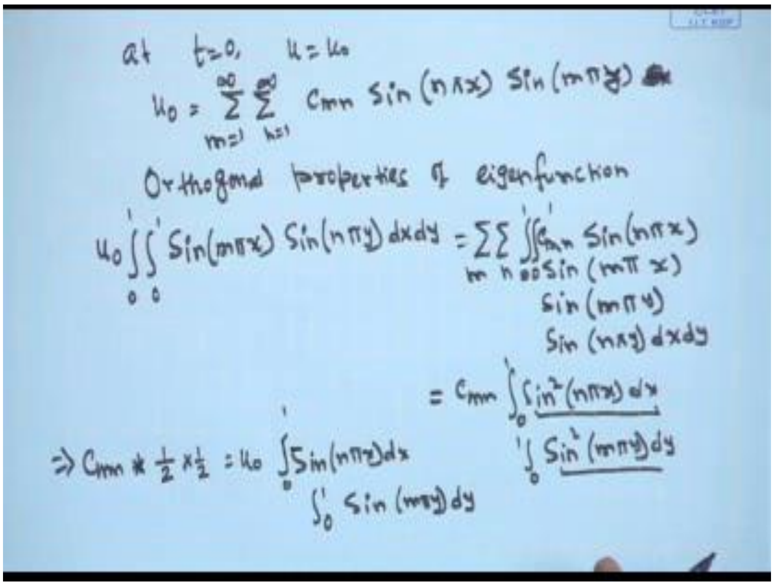

Now, at t is equal to 0 u is equal to u naught. So, if you write that let us see how it comes out u naught is equal to minus a is equal to summation m is equal to 1 to infinity n is equal to 1 to infinity C m n sin n phi x sin m phi x its not sin m phi x it an sin m pi y, now we can we can utilize the orthogonal properties of the Eigen function and can evaluate this constant C m n. So, what we are going to do next we are going to multiply both side by sin n phi x sin m phi y d x d y and integrate over determinate of x and y. So, u 0 double integral sin m phi x sin n phi y d x d y is equal to double summation integral C m n sin n phi x sin m phi x sin m phi y sin n phi y d x d y 1 integral.

So, this will be a double integral 0 to 1, 0 to 1. So, this integral over the domain of x this is from 0 to 1 over the domain of y that is from 0 to 1 if instead of a constant value of u at t is equal to 0, if we some function of some specified function of x or some specified function of y or some specified function of combined x and y that would have been included here and these could have been integrate by parts in that case, but since you know this constant in this case the left hand side the u naught can be taken out of the integral sign.

Now, if we utilize the orthogonal properties of the Eigen functions the and open up the summations series only 2 terms will exist only 1 term will exist that will be when m is equal to n in both the cases because when m we know that the ortho is a Eigen functions or orthogonal functions. So, integral of sin n phi x sin m phi x d x will be equal to 0 for m naught equal to n when they will be equal only that time we survive. So, we will be having C m n integral sin square n phi x d x 0 to 1 multiplied by 0 to 1 sin square m phi y d y. So, we can have C m n is equal multiplied by this integral as we have seen earlier this will be equal to half this will be again it will be half.

So, it will be 1 by 2 into 1 by 2 is equal to u naught integral of sin m phi x d x 0 to 1 will be giving you value sin. So, it will be just then then convert m into n science sin n phi x d x 0 to 1 integral 0 to 1 sin m phi y d y. So, what I am doing, I am converting the running index x and y respectively to their original function that means, sin index n was associated with the in the x direction and index m was associated in the y direction. So, I am just bringing them to that same direction because they are basically they are dummy variable. So, if you do that then let us see how the functional variation of u naught looks like.

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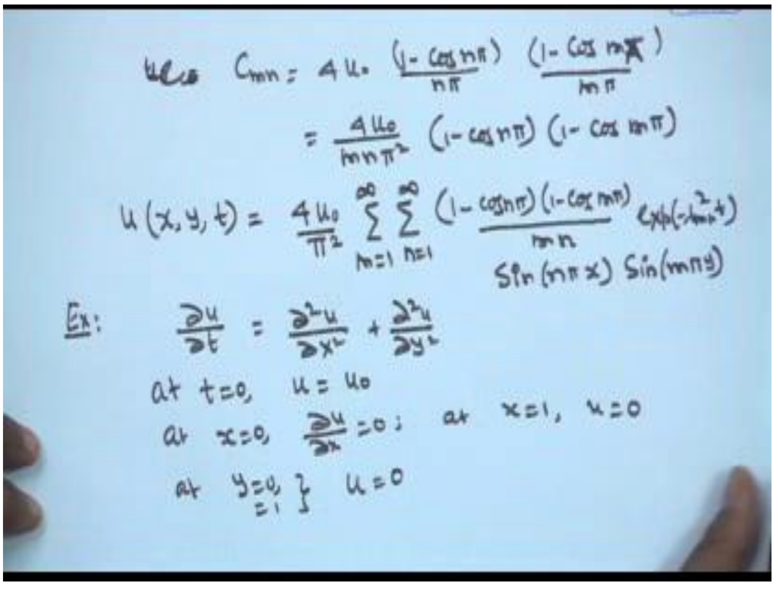

So, u naught will be nothing, but C m n will be nothing, but four times u naught 1 minus cosine n phi divided by n phi 1 minus cosine m phi divided by m phi and will be getting four u naught m n phi square 1 minus cos n phi 1 minus cos m phi. So, let us not construct the complete solution as u x y t is nothing, but four u naught over phi square double summation m is equal to 1 to infinity n is equal to 1 to infinity 1 minus cosine n phi 1 minus cosine m phi divided by m into n exponential minus lambda m n square t sin n phi x sin m phi y that gives the complete solution for a 3 dimensional parabolic partial differential equation.

Now, I will be solving 1 more problem if my 1 my boundary condition is in, in this particular case we are considered the boundary condition if 1 of the condition is that means, if it is the boundary at that particular location is insulated then how it looks like. So, the next example that I we will be solving is this is that we will be solving the exactly the same problem del u del t is equal to del square u del x square plus del square u del y square at t is equal to 0 u is equal to u naught at x is equal to 0 del u del x is equal to 0 and at x is equal to 1 u is equal to 0 at y is equal to 0 and y is equal to 1 both u is equal to 0. So, will we are having a Neumann boundary condition at x is equal to 0 and everywhere else they there it is a boundary condition and that is homogeneous. So, now let us look into the solution of this problem again, we will be attempting a separation of variable type solution.

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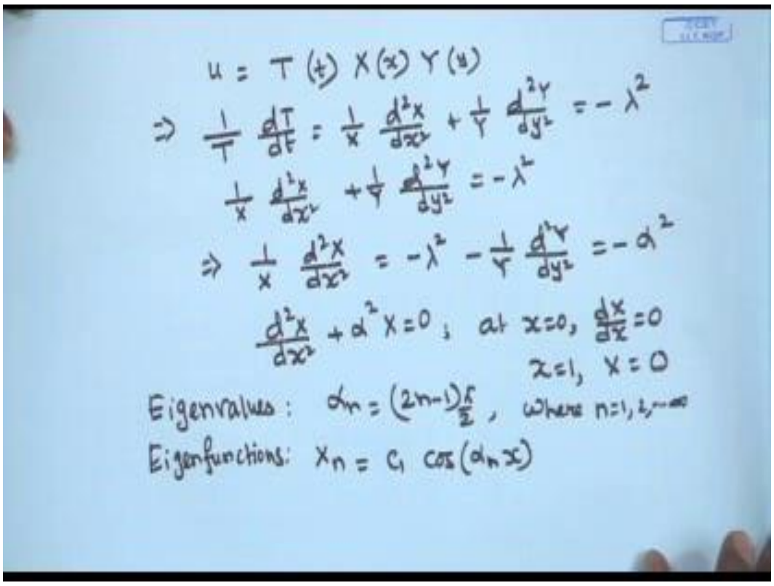

So, u is assumed to be constant of product of 3 function t of x of x y of y now if you substitute in the governing equation and after doing separation of variable 1 you will be getting 1 over t d t d t is equal to 1 over x d square x d x square plus 1 over y d square y d y square. So, again the left hand side is a function of type alone the right hand side is a function of space alone they will be equal to some constant, let us say minus lambda square and then we will be again re simplify the special varying part. So, 1 over x d square x d x square plus 1 over y d square y d y square is equal to minus lambda square.

Now, since in this problem we will be having the homogeneous boundary conditions both in x direction and y direction, then will be formulating a standard Eigen value problem was value problem in both directions independently. So, therefore, we write 1 over x d square x d x square is equal to minus lambda square minus 1 over y d square y d y square and this will be equal to minus alpha square in order to have a non trivial solution. So, let us formulate the standard Eigen value problem in the x direction plus alpha square x is equal to 0 at x is equal to 0 we have d x d x is equal to 0 because this must be satisfying the original with the boundary condition of the original problem in the mother problem we had the Neumann boundary condition prevailing at x is equal to 0. So, therefore, in the sub problem also the special varying part must be satisfied that boundary condition at x is equal to 0.

So, at x is equal to 0, so will be having a boundary condition here also now we have already seen the solution of these problem earlier that they if there is Neumann boundary condition then you know existing prevailing at x is equal to 0 and at x is equal to 1 then the Eigen functions where 2 n minus 1 phi by 2 and the Eigen values are 2 n minus 1 phi by 2 and the Eigen functions where the your cosine functions. So, let us write that down. So, Eigen values where alpha n is equal to 2 n minus 1 phi by 2 where n is equal to running from 1 to up to infinity and Eigen functions x n n corresponding to nth Eigen value. So, this will be C 1 cosine let us say alpha n x where alpha n is basically nothing, but 2 n minus 1 phi by 2.

So, these are this is the solution of the x varying part now let us look into the solution of the y varying part then we will be construct we will be getting the solution of time varying part then we will be constructing the complete solution.

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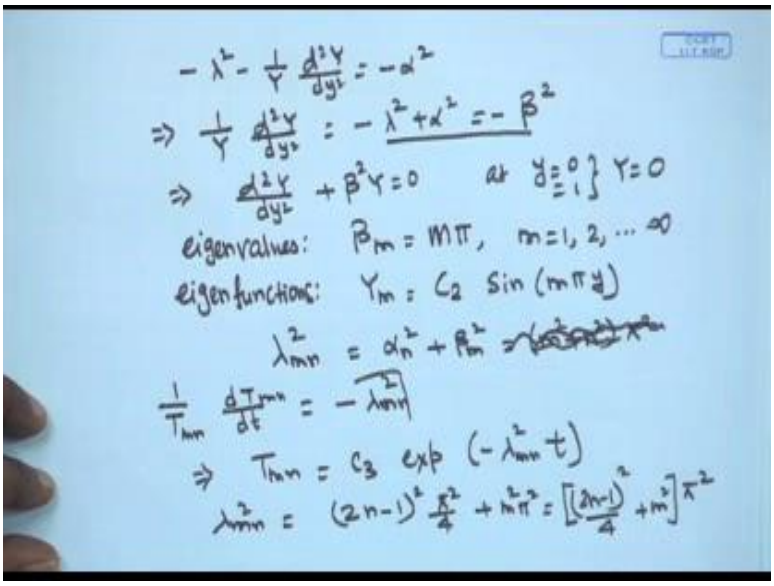

So, the y varying part minus lambda square minus 1 over y d square y d y square is equal to minus alpha square. So, 1 over y d square y d y square is equal to minus lambda square plus alpha square and again this will be a new constant minus beta square in order to have a trivial solution in y direction in order to have a have a non trivial solution in the y direction as well. So, you will be we will be constructing the standard Eigen value problem in the y direction as d square y d y square plus beta square y will be equal to 0 at y is equal to 0 and 1 we have both boundary condition because in the original problem we had boundary condition in the direction in the y direction.

So, we know the solution of this problem the solution is that the corresponding Eigen value are m phi and Eigen functions are sin m phi x m phi m phi m phi y. So, therefore, the Eigen values are beta m is equal to m phi where index m runs from 1 2 up to infinity and Eigen functions are y m is equal to C 2 sin m phi y and now let us look into how the lambda varies. So, this is the definition of lambda. So, lambda m n square is equal to nothing, but alpha n square plus beta m square.

So, this will be m square plus n square into phi square and we can get the time varying for now t m n d t m n d t will be is equal to minus lambda m n square and after integration, we will be getting t m n is equal to let us say C 3 exponential minus lambda m n square times t. So, this gives the solution of the time varying part now we are in a position to construct the complete solution of u. Now, if you really look into that then u as a function of x y t.

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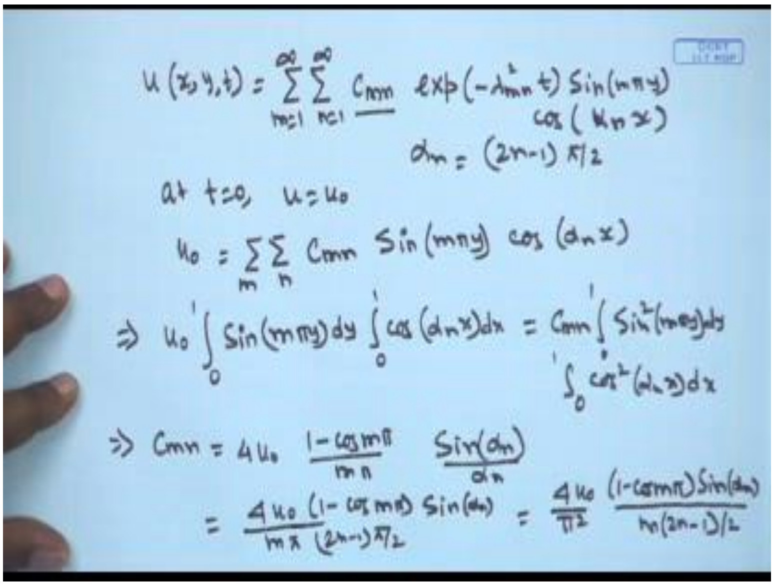

Now, becomes there will be 2 summation 1 over m 1 to infinity and over n 1 to infinity multiply C 1, C 2, C 3, 2 con 3 constants they will be multiplied. So, they will be giving raise to a new constant C m n exponential minus lambda m n square times t. Here, if we if we write down that, this is correct, but alpha alpha n is 2 n minus 1 phi by 2. Therefore, lambda m n square will be 2 n minus 1 square phi square by four plus m square phi square. So, if you take phi square common.

So, this becomes 2 n minus 1 square of that by 4 plus m square phi square in the previous case when we where doing that condition on on both x and y direction we had lambda m n square as n square plus n square into phi square, but in this case 1 of the Eigen value is 2 n minus n phi by 2 another Eigen value is m phi. So, therefore, the expression of lambda m n square will be 2 n minus 1 square by four plus m square into phi square.

So, C m n exponential minus lambda m n square t sin cosine sin m phi y and we had cosine alpha n x where alpha n is nothing, but 2 n minus 1 phi by two. So, again the unknown constant C m n will be now evaluated from the initial condition at t is equal to 0 u is equal to u naught. So, we had u naught is equal to double summation 1 over m another over n C m n sin m phi y cosine alpha n x. So, you will be utilizing the orthogonal property of the Eigen function will be multiplied both side by lambda by cosine alpha m x and sin n phi y and integrate over the domain of x and domain of y.

So, if you do that and utilizing the separation of variable then we will be getting u naught is equal to integral sin m phi y d y 0 to 1 cosine alpha n x d x from 0 to 1 and only 1 survive in the right hand side when m is equal to n and we will be getting C m n 0 to 1 sin square m phi y d y and 0 to 1 cos square alpha n x d x you have we have already seen that this integral is half and this integral is half. So, you will be getting C m n is equal to four u naught this will give 1 minus cosine m phi divided by m phi and this will be sin alpha n divided by alpha n. So, I will be getting four u naught by m phi and alpha n is 2 n minus 1 phi by 2 and on the numerator will be having cos m phi sin alpha n. So, this will be 4 u naught over phi square 1 minus cosine m phi sin alpha n divided by m 2 n minus 1 by 2.

So, we will be getting the complete solution the complete solution will be u x y t as summation of over m summation over n C m n exponential minus lambda m n square t sin m phi y cosine alpha n x where alpha n is 2 n minus 1 phi by 2 lambda m n square will be nothing, but 2 n minus 1 square divide by four plus m square multiplied by phi square and C m n just evaluated as four naught divided by phi square 1 minus cosine m phi and multiplied by sin alpha n divided by m 2 n minus 1 phi by 2.

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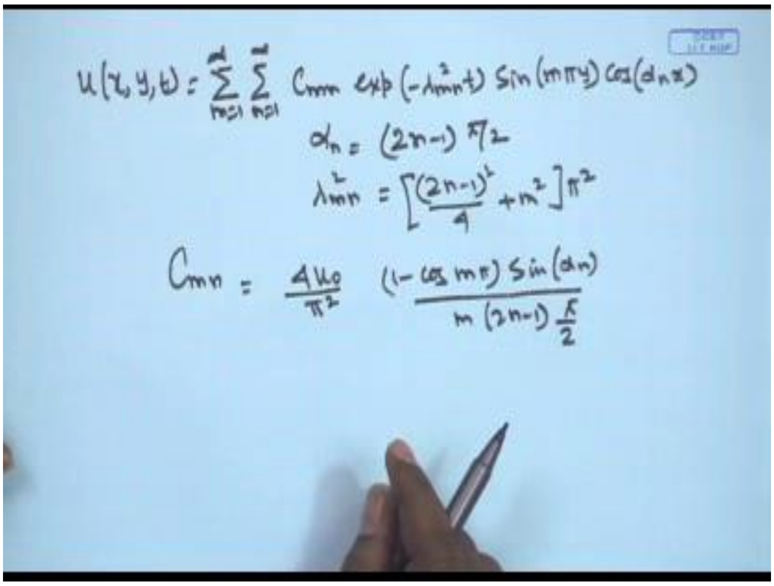

So, that gives the complete solution of a 3 dimensional problem parabolic partial differential equation with a boundary condition a Neumann boundary condition prevailing at x is equal to 0 and everywhere else all the other 3 boundaries is a homogeneous boundary condition with respect to y and with respect to x equal to 1. So, we will be getting, in the next class.

So, I will stop in this class. In the next class what I will be doing I will be taking 1 step ahead and I will be solving a well posed problem which is four dimensional in nature 1 dimensional in type and 3 dimensional in space.

Thank you very much.