Partial Differential Equations (PDE) for Engineers: Solution by Separation of Variables

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Lecture - 08
Separation of Variables: Rectangular Coordinate Systems

Welcome to this class. So, in the last we have seen how to solve a 2 dimensional parabolic partial differential equation. Which is highly well posed, and we have solved we are look into the complete solution of 3 problems. In each problem we have considered 3 different types of boundary conditions. In the first problem we encounter the dirichlet boundary condition, in the second problem we encounter the Neumann boundary condition, in the third problem we have taken care of the robin mixed boundary condition. And we have seen how the eigenvalues and Eigen functions will be defined in these cases. And who the complete solution will be obtain by evaluating the final constant c n, in the governing equation in the in the solution, by using the principle using the orthogonal property of the Eigen functions.

Now, in todays, in this class what will be looking into will be, looking into an exactly an actual problem and real valued problem, which will be ill posed problem, then will be converting this problem into an well posed problem, and we will see how the solution will emerge.

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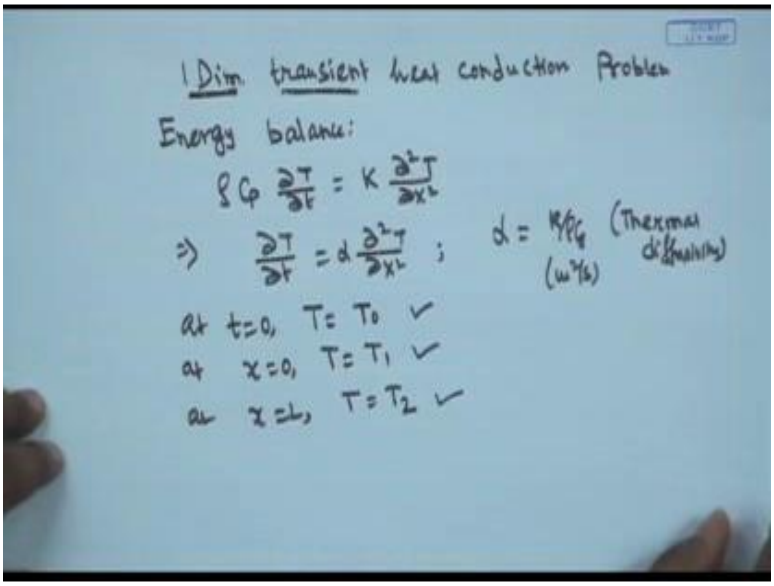

Let us solve a 1 dimensional transient heat conduction problem. So, since it is 1 dimensional. So, it is 1 dimensional in space and it is transient. So, 1 dimensional is time. So, it is a 2 dimensional problem. If you write down the energy balance equation, this becomes row c p, del t, del t is equal to k del square t, del x square. So, we can divide these we can convert, this as del t del t, is equal to alpha, del square, t del x square where alpha is called as the thermal diffusivity, k by rho c p. So, it is known as the thermal diffusivity. And it will be having a unit of meter square per second ah.

Now, we will see that, how these equation, let us look into the boundary condition at t, is equal to 0, we have capital t is equal to t nought at x is equal to 0 we are maintaining a temperature of t 1 and at x equal to l, we are maintaining a temperature of t 2. So, there are x is equal to 0, we are maintaining a temperature. Another boundary at located at x equal to l, we are maintaining another temperature. So, therefore, this is a complete problem, with a dirichlet boundary condition. And if you look into this problem, there are 3 sources of non homogeneity, are present in this problem. These are 1 source of non homogeneities, is in the initial condition, another is the boundary condition located at x equal to 0, another is the boundary condition located at x is equal to l. So, now, what will do next will be looking into will be first will be making it non dimensional, and during the non dimensional process will be trying to minimize the number of non homogeneities in the system. So, that you know our rigority of solution will be decreased.

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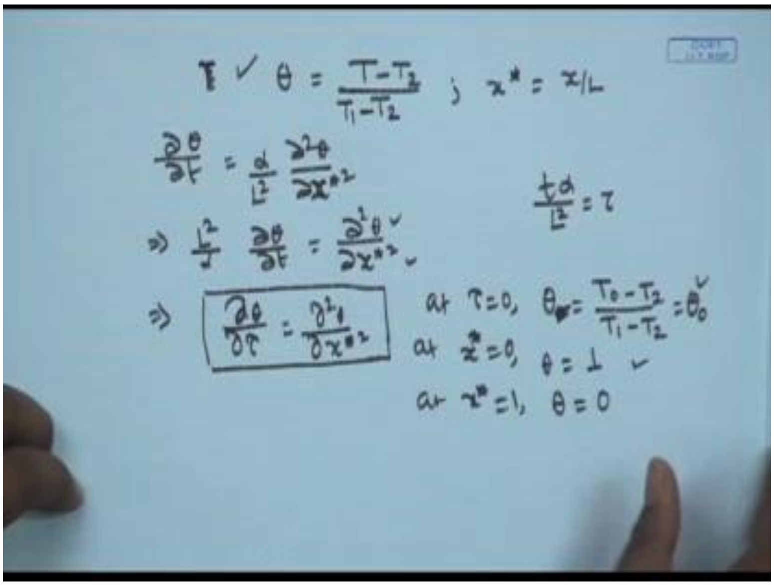

So, let us define a non dimensional temperature first. So, we define a non dimensional temperature t, theta, let us say theta, is equal to t minus t 2 divided by t 2, we can define anything, theta is equal to t minus t, 1 divided by t 1 theta is equal to t minus t 2 divided by t nought, but the non dimensionalization should be avoided in terms of initial temperature t, minus t nought divided by t 2 nought, t 1 because if we make it non dimensional with respect to by locating a difference, with respect to initial temperature at t minus t nought, in the denominator in the numerator, then that will make our initial condition to be 0, that will make my problem ill posed once again.

So, it is better to keep the, to define the non dimensional temperature in terms of in terms of temperature, that is appearing in the boundary. So, we define it as t minus t 2, divided by t 1 minus t 2. So, that will make my system 1 temperature at the location at x, is equal to 0, to be 1, and in other location it will be become 0. So, in terms of this, let us say x star is equal to x by l. So, now, let us make the system let us see del theta del t is equal to alpha, del square theta, l square del x star square. So, this becomes l square, by alpha del theta del t, is equal to del square theta del x star square. So, I get the, so, the right hand side is completely non dimensional the both the numerator, and denominator are non dimensional, on the left hand side numerator is non dimensional, but the denominator is not therefore, t alpha by l square is my non dimensional time.

So, I write this as del theta del, tau is equal to del, square theta del x star square. So, now, I expressed my boundary conditions, in terms of non dimensional, at you know variables. So, at t is equal to 0; that means, at tau is equal to 0. At t is equal to 0, means a tau equal to 0 t is equal to t nought. So, therefore, we call it theta nought as t nought, minus t 2 divided by t 1 minus t 2, at x star x is equal to 0 means x star equal to 0. At x star is equal to 0 theta is equal to. So, this is theta, theta, is equal to this thing, I will call you as theta 0, and x star equal to 0, theta is equal to t is equal to t 1. So, t 1 minus t 2 divided by t 1 minus t 2. So, it will be 1 and at x star is equal to 1, we have theta is equal to t 2. So, t is equal to t 2. So, it will be equal to 0.

So, this is now my governing equation. Now if you can look into this governing equation and the boundary conditions. Now instead of by the way we have defined theta, instead of having 3 non homogeneity in the equation. Now we have we are able to reduce 1 non homogeneity, and now we are dealing with 2 non homogeneities, located at time t is equal to 0, and at x star equal to 0, the theta equal to 1. So, these 2 nonhomogeneous present, but the non homogeneity at x star is equal to 1, is already vanished. So, now, we are dealing with 2 non homogeneities, in this system and will be define the problems in to 2 sub problems, considering 1 non homogeneity at a time.

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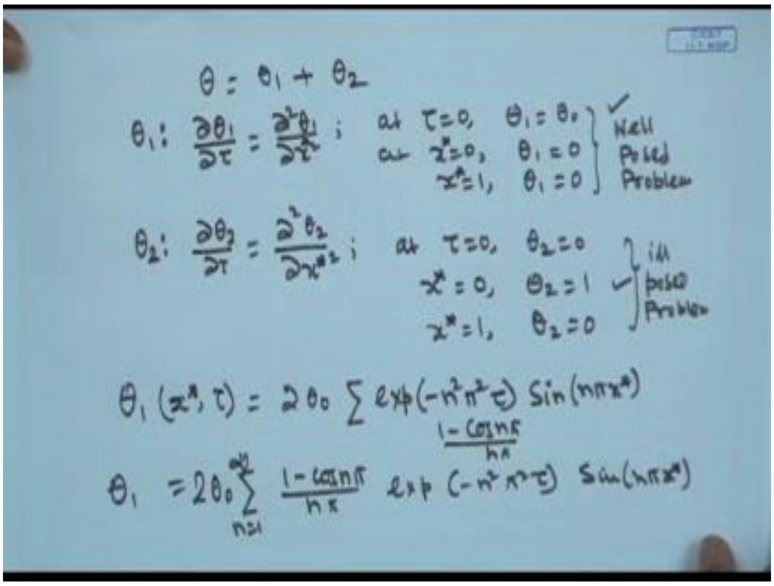

Now, let us do that. So, I define I break down this problem into 2 sub problems like theta is equal to theta 1 plus, theta 2 where my theta 1, is the sub problem del theta 1 del tau is equal, to del square theta, 1 del x square and at these at tau, is equal to 0, I considered theta 1 is equal to theta nought, and at x is equal to 0, theta 1 is equal to 0, and x is equal to 1 theta 1 equal to 0. So, I force the non homogeneity present at their location, at the boundary x equal to 0 to be 0, considering 1 non homogeneity at a time.

Similarly, I define the sub problem theta 2, as del theta 2, del t del tau is equal to del square theta 2, there will be x star, here del x star square, at tau is equal to 0. I force this non homogeneity to vanish. So, theta 2 equal to 0 I keep the non homogeneity at x is equal to 0 at x star is equal to 0, this will be theta 2 will be equal to will be 1. This non homogeneity intact and at x star is equal to 1, it was already 0. So, theta 2 was 0. So, I break down this, 2 this, problem into 2 sub problems considering, 1 non homogeneity in a time at a time. So, I keep the non homogeneity in the initial condition, and I keep the non homogeneity in the boundary condition, in this 2 sub problems respectively. Now if you look into this sub problem, you can identify that, these already a well posed problem. So, this is a well posed problem.

On the other hand, this is an ill posed problem. And we have already solved this equation in the earlier class, whenever we are talking about a parabolic partial differential equation, which is well posed part of the first kind, where we are talking about both boundary conditions to be homogeneous and dirichlet. So, these will be falling into that category, and we can directly write down the solution. So, this will be 2 x, star tau, will be 2 theta nought, summation of exponential minus n, square pi, square tau, and sin n pi x star multiplied, by 1, minus cosine n pi, by n pi. So, I write down the solution, quite neatly, 2 theta nought n, is equal to 1, to infinity 1 minus cosine n pi divided by n pi, exponential minus n square pi square tau, sin n pi x nought. This solution is known to us. So, we write down this solution.

Now, next will be looking into the solution of the, ill posed problem, how to convert the ill posed problem into a well posed problem. Let us see how to do that.

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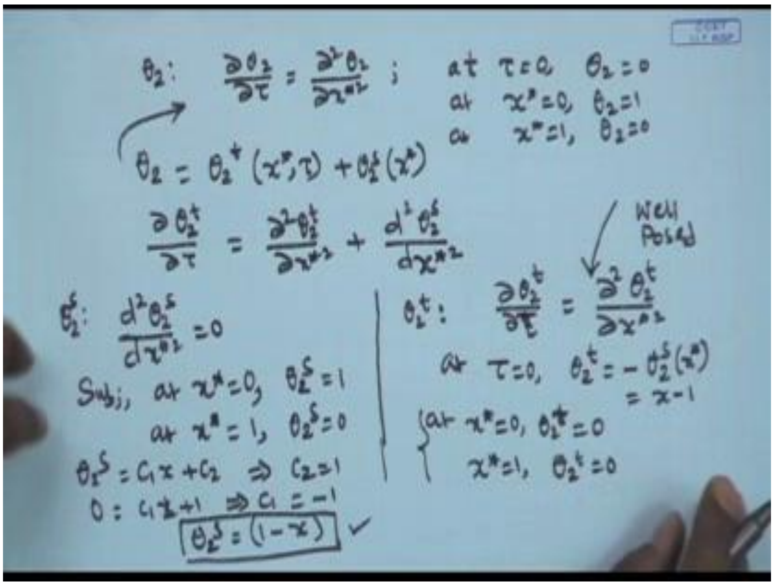

So, let us look into the second sub problem. Theta 2, where del theta 2 del tau, is equal to del square, theta 2 del x star square, at tau is equal to 0, my theta 2 is equal to 0, at x star is equal to 0, my theta 2 is equal to 1. And at x star is equal to 1 my theta 2 is equal to 0. So, since there is 1 non homogeneity, at the boundary condition and the initial condition is homogeneous, as we have discussed earlier will be breaking down this problem into 2, sub problem once again, but 1 part in these case, will be will be a steady state will be time independent part, another part will be the transient part.

So, this will be divided into 2 parts, theta 2 tau theta 2 t which will be function of x star, and tau both plus theta 2 s, which will be a function of x star only. So, if you substitute this over here. So, del theta 2 del tau will be. So, this will be a mother problem for these 2 sub problems. So, if you are substitute this in the governing equation this, becomes del theta 2 t, del tau and this part the time derivative, will be equal to 0, this will be del square theta 2, t del x star square plus, d square theta 2 s, d x star square.

Now, will be collecting the similar terms, and solve them separately. So, write down the governing equation of theta 2 s theta 2, is s, s d square theta 2 s d x star square, is equal to 0 subject to the boundary condition, at x star is equal to 0, theta 2 is equal to theta 2 s plus, theta 2 t. I associate judiciously the nonhomogeneous part with the special varying part. So, therefore, I will forced the transient part, of the boundary condition and located at x star equal, to 0 will be equal to 0, therefore, at x star is equal to 0. I write theta 2 s is equal to 1.

So, I associate the nonhomogeneous part to this boundary condition, and at x star is equal to 1, I have theta 2 s is equal to 0. So, there is the problem for theta 2 s, and let us look into define the theta 2 t. It will be del theta 2 t del t, del tau should be equal to del square, theta 2 t del, x star square, and we as, we have seen earlier that, now my governing equation, that the initial condition tau is equal to my theta 2 t will be nothing, but the minus of solution of, steady state part. And at x star equal to 0, my theta 2 t t equal to 0. Because I have selected that, and x star is equal to 1 theta 2 t is equal to 0.

So, I met the boundary conditions of the transient part to be homogeneous. And the initial condition to be nonhomogeneous. Now this problem is a well posed problem, but before that I have to solve this equation. So, if you really solve this equation. So, this becomes theta solution of theta 2 s will become c 1 x, plus c 2 if you write down the boundary condition at x, equal to 0 theta 2 s is equal to 1. So, that will be giving you c 2 is equal to 1. So, theta 2 at x star is equal to 1, theta 2 s equal to 0. So, this becomes c 1 x plus 1, and this is at x equal to 1. So, therefore, c 1 will be nothing, but minus 1. So, theta 2 s will be nothing, but 1 minus x.

So, this will be theta 2 s is basically x minus 1. So, c 2 is equal to 1. So, c 2 is equal to 1 c 1 is equal to minus 1. So, it will be 1 minus x this fine is this. So, t is the steady state part of the solution. So, the steady state part of solution is x, minus, minus will be taken in side. So, it will be x minus 1.

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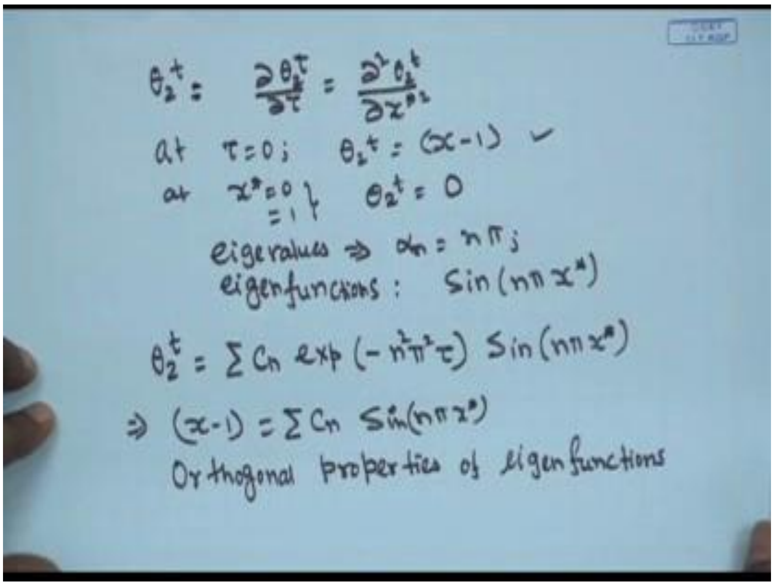

So, now this is a well posed problem. Now this problem is now a well posed problem because it is having a non homogeneous initial condition and homogeneous boundary condition. If we again now look into the solution of theta 2 t, theta 2 t, will be now del theta 2 t, del tau is equal to del square theta 2 t, del x star square at tau is equal to 0, we have theta 2 t is equal to x minus, 1 and at x star, is equal to 0, and 1 we have theta 2 t, is equal to 0. Again we have solved this problem. We have looked into this problem earlier.

If you remember again this is a problem of parabolic partial differential equation, well posed problem of the first kind, where we have where we had the dirichlet boundary conditions. So, if you look into this problem, the problem is having the eigenvalues n pi, alpha n is equal to n pi, and Eigen functions are the sin functions, are sin functions sin n pi x star. So, if you look into the solution, the final solution will be theta 2 tau will be summation c n, exponential minus n square pi square tau, sin n pi x star. So, this c n the constant has to be evaluated, from the non zero initial conditions so; that means, x minus 1, will be summation of c n sine n pi, x star and will be evaluating it, by using the orthogonal property of the sin functions or Eigen functions.

So, using the orthogonal properties of the Eigen function, functions let us see what we get will be getting c n as sin square, n pi x, dx 0 to 1 and there it will be having 0 to 1 x minus, 1 sin n pi x star dx, star will star and these we have seen this will be half. So, c n is equal to half 0 to 1. So, it will be 2. So, half will becoming now 2 to 0, to 1 x minus 1 sin n pi x star, dx star all x is will be basically stat nondimensional. So, 1 can integrate by parts. So, this can be integrated. 2 0 to 1, x star sin n pi, x star, dx star minus 2 sin n pi x star, dx star and these will be you know it is has to be integrated by parts and this will be a straight forward 1 minus cosine n pi by n pi 0 to 1.

So, so, we can we can get the explicit expression of c n. Now we can get the complete solution of theta 2, theta 2 will be theta 2 s, plus theta 2 t. So, theta 2 s is nothing, but 1 minus x star, and plus theta 2 t theta 2 t, will be basically the theta 2 t, is whatever we have obtained here. So, that will be the expression of theta 2 t and c n is obtained from this expression. So, the complete solution with the overall problem becomes theta is equal to theta 1, plus theta 2 and ah, and theta 2 is basically 2 parts, theta 1 plus theta 2 s plus theta 2 t. So, if we add up all sub parts, then that will be giving you the complete solution of a practical problem, in 2-dimensional part parabolic partial differential equation.

So, once we get that, now what we have learnt, here that if we have a well posed problem any kind of boundary condition, will be able to tackle that, if it is a well posed problem absolutely no issue. So, one can go for a first kind of dirichlet whether will be having a dirichlet boundary condition Neumann, or robin mixed, will be able to handle. If you do not have a well posed problem, if it is ill posed problem then you know how to converted into an well posed problem, and can get by sub dividing the problem into 2 sub problem, once again.

So, we have came across, we have come across an actual practical problem, where it is basically 1 dimensional transient in a conduction problem, we have solved. And we have seen that there are 3 sources of non homogeneity of this problem originally. Then what we did, we took request to the non dimensionalization, of this problem judiciously, then we are able to reduce the 1 source of non homogeneity into. So, the ultimately we have you know dealt with 2 sources of non homogeneity.

This problem was divided into 2 sub problems. 1 is theta 1 and theta 2. And theta 1 was considering one non homogeneity at a time. So, therefore, 1 of this problem was a well posed problem. We have seen that we have solved that problem earlier also. So, we got the complete solution of that. The second sub problem was theta 2, which will be having a homogeneous initial condition, and then you have divided this problem into 2 sub problems once again. 1 is time dependent part another 1 is time independent part. And we have seen how the time dependent part will be, and the nonhomogeneous boundary condition will be associated, to the time um independent part. So, therefore, the time dependent part the boundary conditions, are become homogeneous in space, and nonhomogeneous in time. So, it has become a well posed problem. So, an ill posed problem has been converted into the well posed problem, and individually we have looked into the solution of the steady state part and the transient part as well.

Then we have constructed the complete solution, of the original problem. By adding up all 3 sub problems together, and then we got the complete analytical solution of a problem, which will be of a parabolic partial differential equation of 2 dimensions; which will be having you know lots of non homogeneity in it is boundary and initial conditions.

So, next what will be doing is, that will be formulating a problem of 3 dimensional, but we will be looking into the well posed problems, only because, we know how to handle the ill posed problem, we know how to tackle the ill posed problem, by dividing the problem into sub problems, that the ill posed problem can be converted into well posed problem, we can go ahead with the similar type of solution. So, now, will be looking into a 3 dimensional problem; 3 dimensional parabolic partial differential equation and well posed.

So, the governing equation will be like this. Del you del t is equal to del square, u del x square plus del square u, del y square. So, will be having the above will be constructing the initial and boundary condition, at t is equal to 0, we are having u is equal to u nought, and all the 4 boundaries located at x is equal to 0, and x is equal to 1, we have u is equal to 0, all all boundary conditions are Dirichlet and they are homogeneous. At y is equal to 0 and y is equal to 1, u is equal to 0. So, these are well posed problems. So, therefore, you will be having a non 0 initial condition and all 4 boundaries located at x equal to 0 and x equal to 1, y equal to 0 and y equal to 1 all are homogeneous and they will be equal to 0.

So, again will be going for a separational variable type of solution, and we assume that the complete solution u is a, function is a product of 3 functions 1 is entirely function of time and another is entirely function, of x third 1 is entirely function of y. So, we substitute these in the governing equation. What we get is x y divided by d t d t, divided by 1 over, this will be x y d t d t, plus t y d square x d x, square plus, t x d square y d y, square. I divide both sides by x y t. So, we can separate the variables out 1 over t, d t by d t divided by 1 over x, d square x d x square, plus 1 over y d square y d y square.

So, left hand side is a function of time entirely. The right hand side is a function of space is a function of x and y. So, they are equal and they will be equal to some constant. Because of the shortage of time, in this class this will be taking a bit more time. So, I will be taking this problem of completely in the next class. And will be solving a 2 dimensional parabolic, 3 dimensional parabolic partial differential equation, which is a well posed problem in Cartesian coordinate system. Then will be taking up other you know example of 4 dimensional parabolic partial differential equation. And which will be in Cartesian coordinate system.

Then will be taking up the parabolic partial differential equations, in the in the spherical polar coordinate system and cylindrical polar coordinate system, as well as the elliptical partial differential equations, as well as the hyperbolic parabolic partial differential equations. So, I will stop in here in this class. In the class I will be completing this problem of 2, dimensional of 3-dimensional parabolic partial differential equation, which is well posed and in Cartesian coordinate system.

Thank you very much.