Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables

Prof. Sirshendu De
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 07
Properties of Adjoint Operator

Welcome to this session. In the last class, we have seen that, we have derived a standard Eigen value problem in its general form, and we have seen that, this will form a self adjoint operator, and we have looked into the various properties of self adjoint operator.

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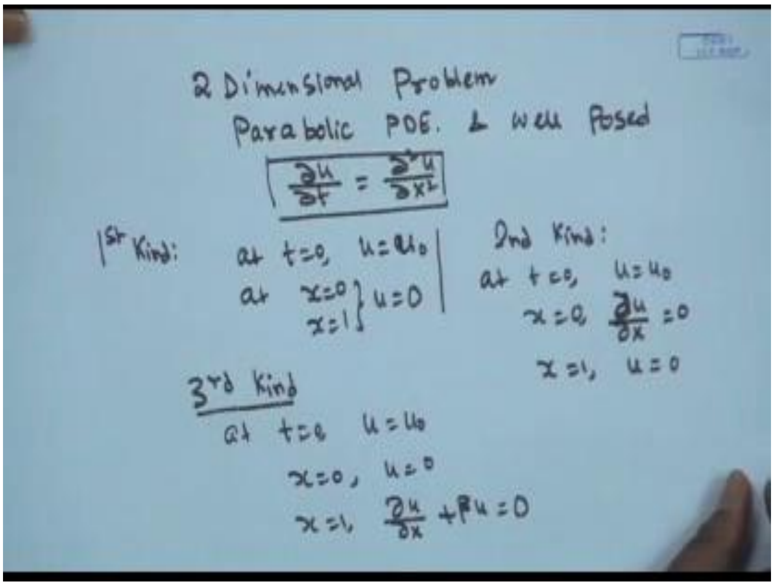

Now, we have appropriate background of going for a, as to complete solution of a, by using separation of variable. The first problem that we will be, we will be talking about; we will be talking about a parabolic partial differential equation in 2 dimension. So, we will be working with a 2 dimensional problem, parabolic partial differential equation. We will be defining three problems; first kind, second kind and third kind.

So, that will be depending entirely on its type of boundary condition we are dealing with, del square u del x square del u del t is equal to del square u del x square. And, problem of first kind is with the boundary condition at t is equal to 0, and we will be talking about a well posed problem, parabolic partial differential equation and well posed at t is equal to 0, U is equal to 0, at x is equal to 0, and U is equal to U naught, let us say, at t equal to 0.

Well posed problem means, non-homogenous initial condition; at x equal to 0, and x equal to 1, we have the Dirichlet boundary condition U is equal to 0.

Now, what is the second kind? Second kind is at t is equal to 0, U is equal to U naught; at x equal to 0, we have d u del u del x is equal to 0; at x is equal to 1, we have U is equal to 0. So, in this particular problem, we have the Neumann boundary condition at x is equal to 0. And, third kind will be, at t is equal to 0, U is equal to U naught; at x is equal to 0, U is equal to 0; at x is equal to 1, del u del x plus beta u is equal to 0. So, this will be the third kind of boundary condition. Now, let us solve one by one, and see how the solution differs, and one can go about a complete solution of using separation of variable. So, let us, let us solve for the first kind parabolic P D E and well posed.

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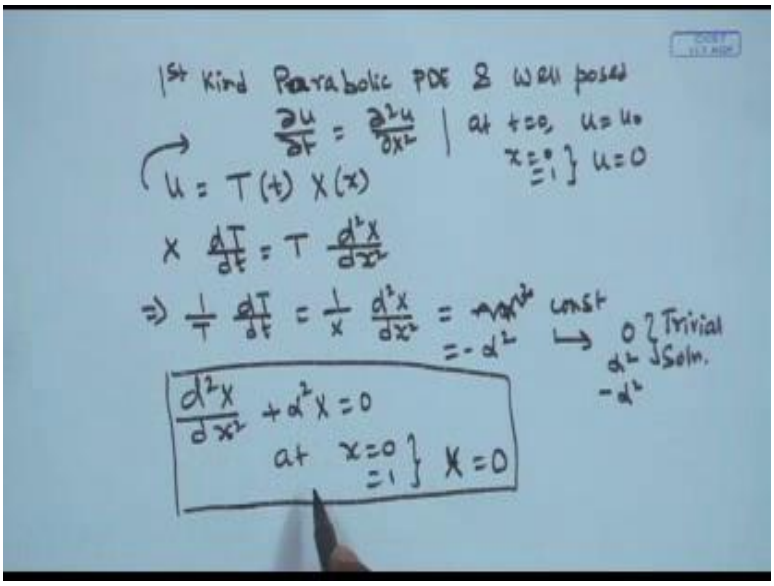

So, let us look into the complete solution of that. So, del u del t will be, is equal to del square u del x square, and, at t is equal to 0, U is equal to U naught; at x is equal to 0 and 1, U is equal to 0. So, if this is the case, then, we assume that, U is a complete function of, it is a product of two functions; one is, it will be a sole function of x; another is, it is a, it is a complete function of time, so, where U is equal to t into x. Now, in the governing equation, we are going to put this. If we put this in the governing equation, what we will be getting is, x d T d t is equal to T d square X d x square. If you divide by x t, this becomes, 1 over T d T d t is equal to 1 over X d square X d x square.

So, now, the left hand side is a function of time only; the right hand side is a function of space only; they are equal. So, therefore, they will be equal to some constant. And again, this constant can be positive, can be negative, and can be 0. So, so this will be, this has to be minus alpha square. And, we have seen earlier that, if this constant. So, this constant can be, it is a constant; and constant can be 0; can be positive; can be negative. So, if this is 0 and positive, we will be getting a trivial solution; that we have already seen earlier.

So, the constant has to be a negative constant. So, this will be equal to minus alpha square. Now, let us let us look into the special varying part. So, d square X d x square, plus alpha square x should be equal to 0, subject to boundary conditions. So, this equation now, this is an ordinary differential equation, requiring two boundary conditions to be solved, and two boundary condition of x will be on x will be basically the boundary conditions of the original problem, we will be satisfying the same boundary condition of the sub problem x.

So, the boundary condition of this problem, x will be having the same boundary conditions on the original problem on u; that means, at x is equal to 0, and at x is equal to 1, U is capital X is equal to 0. So, we have already seen the solution of this problem in the last to last class. The solution is this is a standard Eigen value problem, with a special form. If you remember that, a 0 is equal to 1 in this case; a 1 equal to 0; a 2 equal to 0; lambda is basically alpha square and a 3 equal to the weight function is equal to 1. So, therefore, we have already looked into the solution of this, the Eigen values will be n pi, and Eigen functions will be the sine functions. So, I am not going into the solution, because we have already solved in the class.

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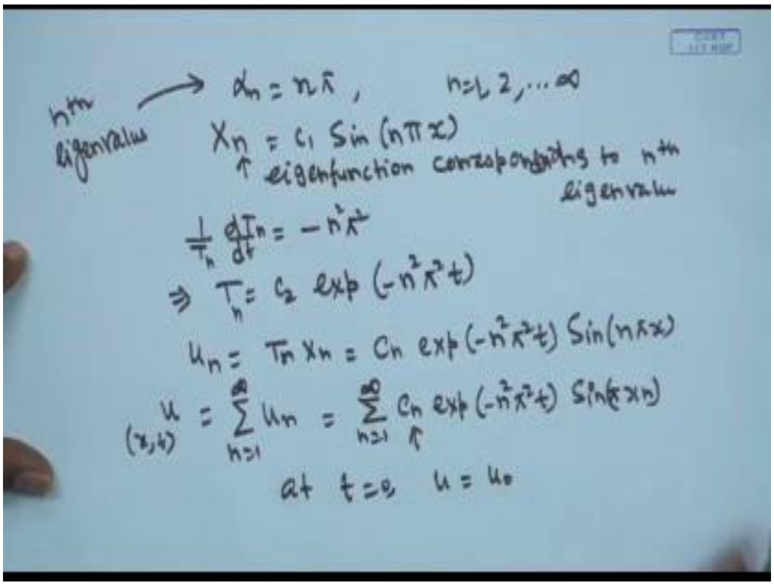

So, alpha n is equal to n pi, where n is equal to 1, 2, 3 up to infinity; there are infinite number of Eigen values; the theorem we have already seen. The corresponding solution will be X n is equal to C 1 sin n pi x. And, let us look into the, The subscript n corresponds to the Eigen function, corresponding to nth Eigen value.

What is the nth Eigen value, n pi? So, this is the nth Eigen value. So, now, let us look into the time varying part. If we look into the time varying part, 1 over T, d T d t, is equal to minus lambda n square minus alpha n square. So, it is n square pi square. So, T will be C 2 exponential minus n square pi square t. Now, we can get a complete solution. So, this T should have a subscript n, because it will be corresponding to nth Eigen value. So, therefore, the complete solution will be So, what is U n, corresponding to nth Eigen value, is T n x n, and C 1 and C 2 will be multiplied, and it will be giving a new constant C C n exponential minus n square pi square t sin n pi x. So, what will be complete solution, U will be superposing all these independent Eigen values, Eigen, independent solutions.

So, we will be adding up all the solutions, corresponding to each Eigen values, and Eigen function, and we will be formulating the complete solution. So, n is equal to 1 to infinity. So, therefore, this will be nothing, but n is equal to 1 to infinity, C n exponential minus n square pi square t sin n pi x. So, that gives the complete solution of x and t; only the constant C n has to be evaluated.

Now, if you look into this problem, we had 3, we have this equation was del u del t is equal to del square u del x square. It is order 2 with respect to x, order 1 with respect to t, where the initial condition at t equal to 0, and two boundary conditions at x equal to 0. We have already used up these two boundary conditions while formatting the solution of x varying part. So, now, only one condition is left behind; that is at t equal to 0 that is the initial condition. And, we have only one constant that has to be evaluated. So, using the initial condition at t equal to 0, we will be evaluating this constant C n.

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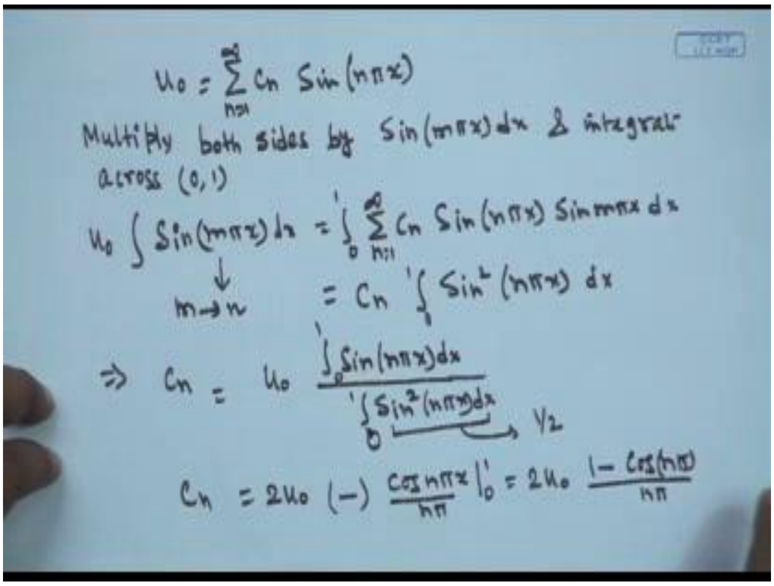

And, at t is equal to 0, we have U is equal to U naught. So, just put it, put this condition in this equation, and see what do we get. So, U 0 is equal to summation of n equal to 1 to infinity, C n sin n pi x. And, we have already proved in the last class that, the Eigen functions will be forming the orthogonal functions. They will be the orthogonal function with respect to weight function, one in this particular case.

So, what I will do next, I will multiply both side of the equation by sin m pi x d x, and integrate over the domain of x from 0 to 1. So, multiply both sides by sin m pi x d x, and integrate across domain of x 0 to 1. So, if you do that, U 0 is constant; this will be sin m pi x d x, is equal to 0 to 1 summation C n sin n pi x sin m pi x d x. Now, if you open up this summation on the right hand side, now, all the terms will vanish because of the orthogonal property of the sine function, but we are leaving behind only one term, when m is equal to n. We have already seen that, orthogonal, the Eigen functions are orthogonal to each other for two distinct Eigen values m naught is equal to n. So, when m naught is not equal to n, then, integral sin m pi x sin n pi x d x will be equal to 0, to satisfy the orthogonal property of the Eigen functions; that, we have already proved earlier.

So, only one term will survive in this infinite series on the right hand side; that will be m is equal to n. So, C n integral 0 to 1 sin square n pi x; sin n pi x, and when m is equal to n, so, sin n pi x; sin square n pi x d x. And now, we change the running variable m into n. So, what we will be getting is C n is equal to U 0 integral sin n pi x d x from 0 to 1, integral 0 to 1 sin square n pi x d x. And, if you it is a, it is very trivial to show that, this value is half. So, this will be always half. So, it will be having 2 U naught and sin n pi x. So, minus cosine n pi x, divided by n pi; then, you put the values 0 to 1. So, this will be 2 U naught cosine n pi minus cos 0 is 1. So, it becomes 1 minus cosine n pi, divided by n pi. So, that is the value of C n. So, now, we will be getting the complete solution.

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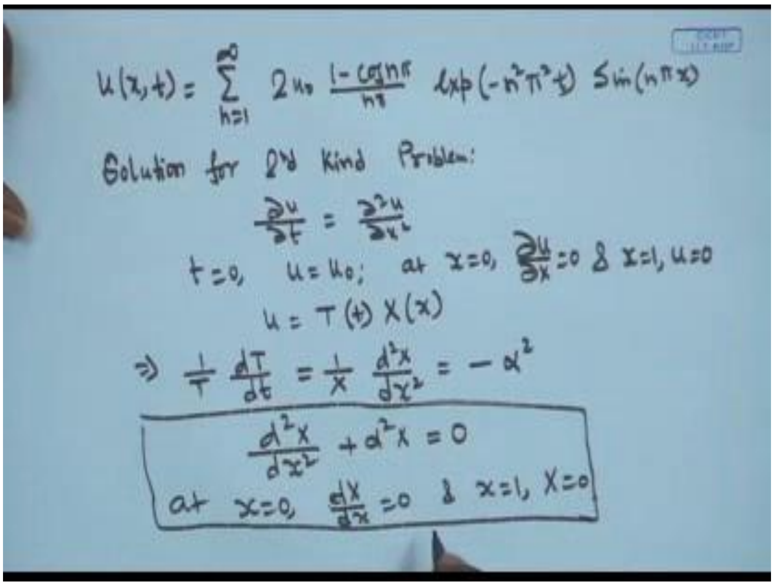

If you write the complete solution, U as a function of x and t, will be nothing, but summation n is equal to 1 to infinity, 2 U 0 1 minus cosine n pi over n pi, exponential minus n square pi square t, sin n pi x. 2 U naught can be taken out of the summation sign, because, basically, that will be a constant. So, this gives the complete solution of the parabolic, 2 dimensional parabolic partial differential equation of the first kind. Now, let us look into the solution of parabolic, well posed, parabolic partial differential equation in second kind.

Solution for second kind problem, you have del u del t is equal to del square u del x square, where t is equal to 0, you have U is equal to U naught; at x is equal to 0, you have del u del x is equal to 0; and, at x is equal to 1, you have U is equal to 0. So, again, we will be looking for a separation of variable type of solution. We assume that, the solution is composed of multiplication of two terms; one is purely a function of x; another is purely a function of time. If you do that, then, this becomes 1 over T, after separating the variables, it becomes 1 by T d T d t, is equal to 1 over X d square X d x square.

Again, this will be, for entirely, the left hand side is entirely a function of time; the right hand side is entirely a function of space. They are equal. They will be equal to some constant, and that constant can be 0, can be positive, and can be negative. We have seen earlier that, if it is 0, or negative, if it is 0 and negative, then, we will be getting a trivial solution. So, if it is 0 and positive, on the right hand side, then, it will be giving a trivial solution. So, this constant has to be a negative constant. So, this has to be equal to minus alpha square.

So, if that is the case, we can formulate the standard Eigen value problem in the x direction. So, that will be alpha square d square X d x square plus alpha square x equal to 0, and this special variation varying part will be satisfying the boundary conditions of the original problem in the direction of space. So, therefore, at x is equal to 0, we will be having d X d x is equal to 0, and at x is equal to 1, we have capital X is equal to 0.

In fact, we have already solved this standard Eigen value problem earlier, and if you remember, the Eigen values were 2 n minus 1 pi by 2, and the Eigen functions will be the cos functions, cosine functions. So, let us write that, the Eigen values lambda alpha n are 2 n minus 1 pi by 2 and Eigen functions are cosine functions, cosine, let us say alpha n x.

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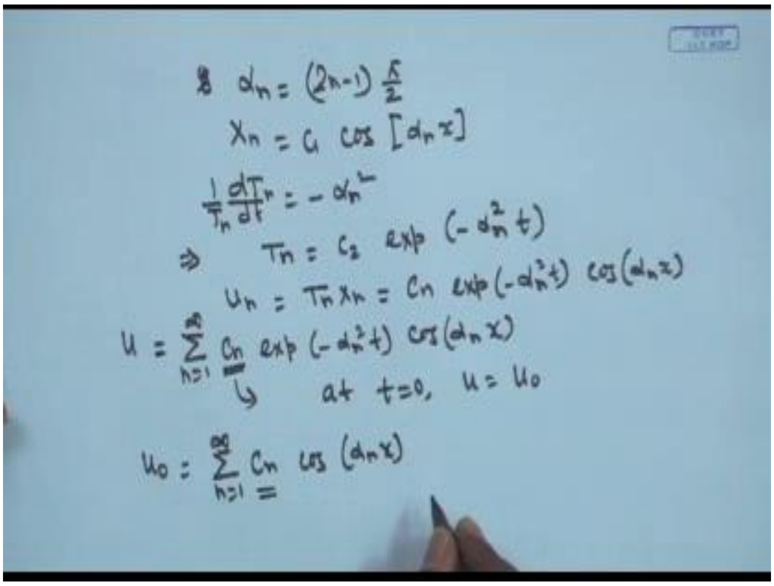

So, the corresponding to the time varying part, we will see d T d t 1 over T is equal to minus alpha n square. So, it will be T n will be nothing, but C 2 exponential minus alpha n square t. So, you can get the complete solution U of n, is equal to nothing, but T n x n, and C 2 and C 1 will be multiplied giving to a new constant C n exponential minus alpha n square t cosine alpha n x. And, we will be getting the complete solution by superposing all the solutions, n is equal to 1 to infinity, as C n exponential minus alpha n square t cosine alpha n x.

Now, we have already seen, now, we have already used up the two boundary condition in the spatial direction. Now, what is left is this boundary, this constant has to be evaluated. We will be evaluating this constant by utilizing the boundary condition at t equal to 0, which is nothing, but the initial condition; U is equal to 0 is equal to U naught is summation n is equal to 1 to infinity C n cosine alpha n x. We will be evaluating C n by utilizing the orthogonal property of the Eigen functions as we have done earlier.

So, what we will be doing, we will be multiply both side by cosine alpha m x d x, and integrate over 0 to 1 on both sides. And then, only one term will survive in the right hand side, if you open up the summation series, because, all the other terms will be equal to 0, in order to satisfy the orthogonal property of the Eigen functions for two different, for corresponding to different Eigen values lambda n, and alpha n, and alpha n.

So, therefore, we will be having only one term on the right hand side, after opening up the summation series; that will be the one which will be equal to, when lambda is equal, lambda alpha m is equal to alpha n; m is equal to n; only that will be survived. So, using the orthogonal property of the Eigen functions, we can evaluate C n as U 0 integral cosine alpha n x d x, 0 to 1, integral 0 to 1 cos square alpha n x d x. And, if we can really do this, this will be again turning up to be half. So, this becomes 2 U naught; this will be sin alpha n x divided by alpha n, from 0 to 1.

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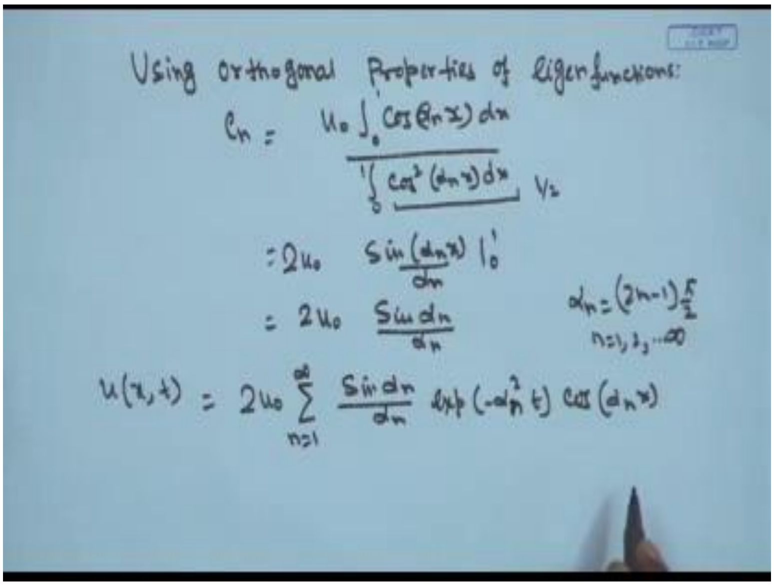

So, sin 0 is 0. So, only one term will survive here; 2 U 0 sin alpha n divided by alpha n, where alpha n is equal to 2 n minus 1 pi by 2, the Eigen values and the index n transform 1 to up to infinity, because there are infinite number of Eigen values present in the system. So, if you write down the complete solution, U x t will be nothing, but 2 U naught, summation n is equal to 1 to infinity, sin alpha n divided by alpha n, exponential minus alpha n square t, cosine alpha n x. So, this will be the complete solution, and alpha n is equal to 2 n minus 1 pi by 2. So, this is the second kind of, you know, well posed parabolic problem.

Now, let us look into the third kind of problem. So, it will be the similar type of equation. So, equation, governing equation remains the same; del u del t is equal to del square u del x square. This is the third kind, well posed problem. And, at t is equal to 0, U is equal to U naught; at x is equal to 0, we have U is equal to 0; and, at x is equal to 1, we have del u del x plus beta u is equal to 0. So, it is a robin mixed boundary condition prevailing at the boundary at x equal to 1. So, now, we will be doing again separation of variable type of solution.

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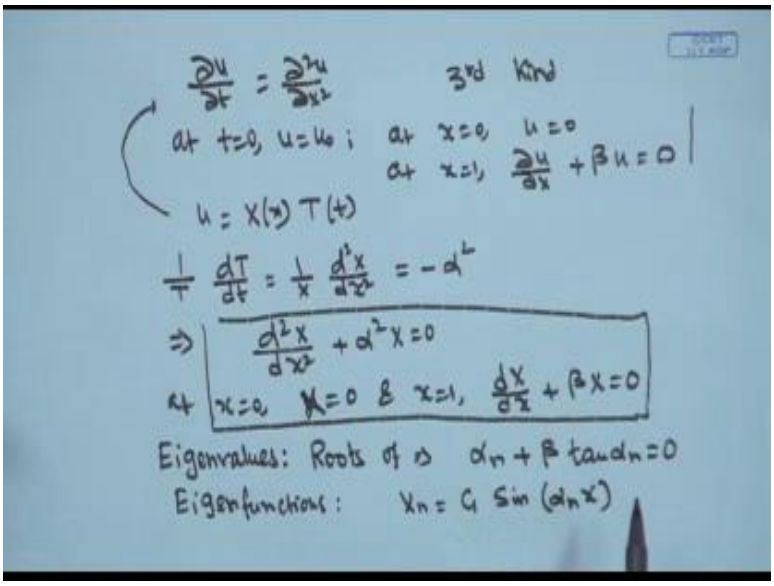

So, U is equal to X of x, multiplied by T of t. After putting this in the governing equation, and by separating of variables, one will be having 1 over T d T by d t, is equal to 1 over X d square X d x square. And again, this will be equal to minus alpha square, and we will be having the Eigen value problem in the spatial direction, in the x direction, as d square X d x square, plus alpha square x is equal to 0; and the boundary condition of the spatial varying part will be satisfying the boundary condition of the original problem.

So, at x is equal to 0, you have U is equal to 0, capital X equal to 0; and at x is equal to 1, we have d X d x plus beta X is equal to 0. Again, these standard Eigen value problem in the simplified version, we have already seen the solution earlier. The Eigen values of these problems are coming from the solution of the roots of the transcendental equation alpha plus beta tan alpha equal to 0. So, Eigen values will be roots of alpha n plus beta tan alpha n is equal to 0.

And, Eigen functions are the sine functions. So, we have already seen that, the roots of this transcendental equation will be the Eigen values. There are infinite numbers of roots present of this equation. So, therefore, therefore, there are infinite number of solutions are also present for this system, and each solution is the Eigen function, and nth solution corresponding to nth Eigen value, is the nth Eigen function.

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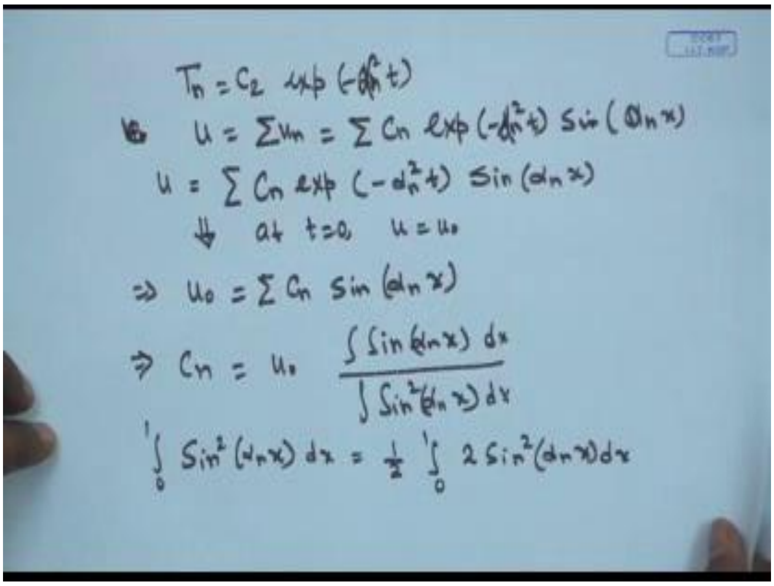

Now, the time varying part, as you have seen earlier, that remains the same. Time varying part will be T n C 2 exponential minus lambda n square t. And, we can get the complete solution as, U is equal to summation of U n, is equal to summation of C n exponential minus lambda n square t sin minus alpha n; this will be alpha n; this alpha n square t sin alpha n x. So, U let us write in the clean neat form, C n exponential minus alpha n square t sin alpha n x.

And again, this equation can be, this C n will be evaluated at the boundary condition, as initial condition at t equal to 0; at t is equal to 0, we had U is equal to U naught. So, therefore, U naught is equal to summation of C n sin alpha n x. and again, C n can be evaluated by exploiting the orthogonal property of the Eigen functions, Eigen functions sin alpha n x; that means, we have, we will be multiplying both side by sin alpha m x d x, and integrate over the domain of x, from 0 to 1. All the terms on the right hand side will be vanishing, except the term when m is equal to n, when you open up the summation series.

So, let us, let us get that. So, C n will be U 0 integral sin alpha n x d x divided by integral sin square alpha n x d x. So, let us look into the solution. So, sin square integral sin square alpha n. So, let us, first, let us look into the solution of 0 to 1 sin square; I am talking about the denominator sin square alpha n x d x. So, this becomes, just multiply it by half. So, this becomes 0 to 1, 2 sin square alpha n x d x and utilizing the function, you know, sin 2 x, or sin 2 x, sin 2 x is equal to.

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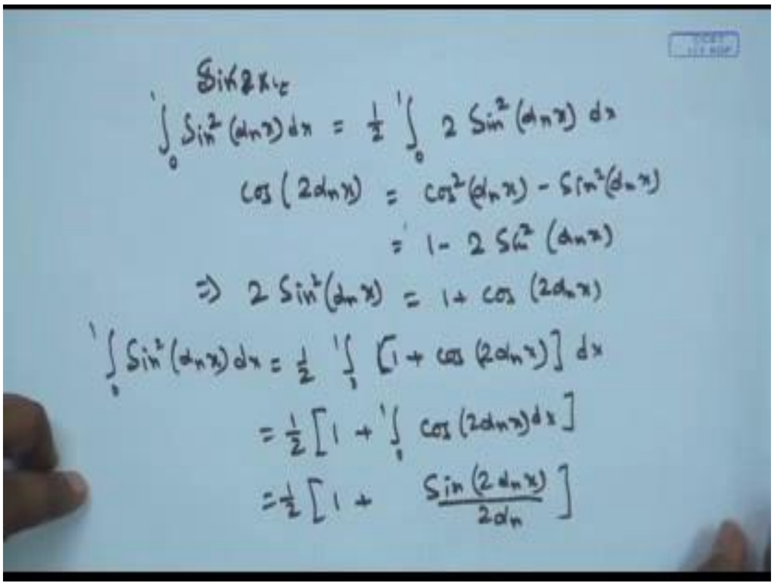

So, let us look that, look into how, how this can be evaluated. Sin square alpha n x d x, from 0 to 1, is equal to half, 0 to 1, 2 sin square alpha n x d x. So, cos 2 n x 2 2 alpha n x, will be nothing, but cos square alpha n x, minus sin square alpha n x. And, this becomes 1 minus sin square. So, this, this becomes 1 minus 2 sin square alpha n x. So, therefore, 2 sin square alpha n x will be nothing, but 1 plus cos 2 alpha n x. So, let us substitute here, 0 to 1 sin square. So, it is a denominator only; alpha n x d x will be nothing, but half, 0 to 1, 1 plus cosine 2 alpha n x d x. So, let us carry out the integration; half; this will be 1, plus integral of cosine 2 alpha n x d x is nothing, but 1 by half, 1 plus, this will be sin 2 alpha n x divide by 2 alpha n. We express sin 2 alpha n in the, in the form of tan.

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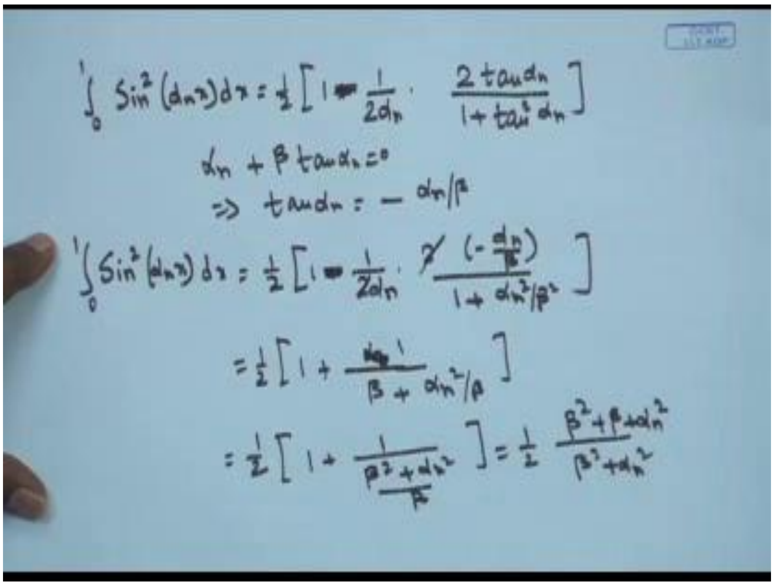

So, 0 to 1 sin square alpha n x d x now becomes half 1 plus divided by 2 alpha n; sin 2 alpha n is nothing, but 2 tan alpha n divided by 1 plus tan square alpha n. And, we have seen that, the Eigen values will satisfy the governing equation as alpha n plus beta tan alpha n is equal to 0. So, therefore, tan alpha n can be written as minus alpha n divided by beta. So, if you substitute there; this denominator now becomes sin square alpha n x d x, is equal to half 1 plus 2 alpha n; 2 tan alpha n will be multiplied by 2; tan alpha n will be minus alpha n by beta, and this will be 1 plus alpha n square by beta square. So, 2, 2 will be canceling out. So, this will be half 1 minus, this alpha n, alpha n will also be canceling out. So, it will be 1 minus alpha n; alpha n, alpha n will be canceling out.

So, it will be 1 by beta plus alpha n square by beta. So, this will be half; there is a sin here. So, 2 sin square alpha n x will be, it will be coming on the other side; this will be minus. So, this will be minus. So, this will be minus; this will be minus. So, again, we will be having, this will be minus; this will be, there will be minus sign here; minus, minus will be plus. So, it will be plus; half 1 plus one beta square plus alpha n square divided by beta. So, you will be getting half beta square, plus alpha n square, is equal to beta square plus beta plus alpha n square. So, let us look into the complete solution now. This is the denominator we will be getting. Let us look into the numerator. The numerator of the solution is 0 to 1 sin alpha n x d x.

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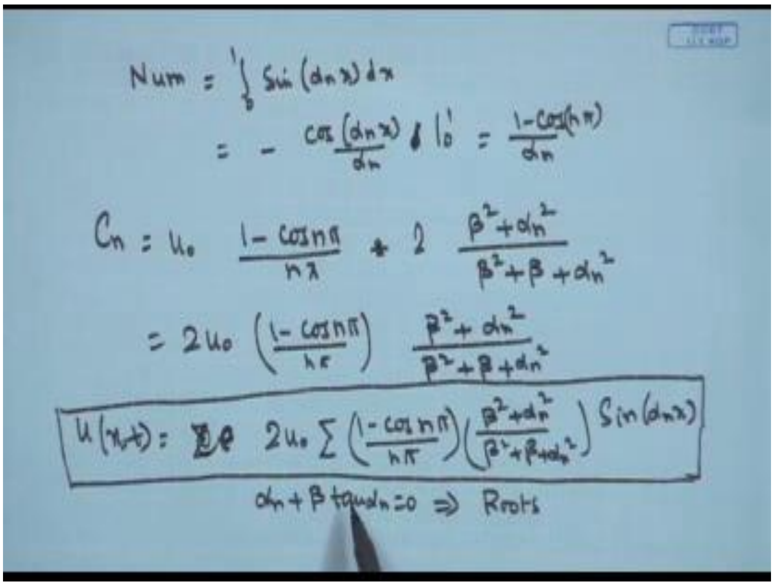

So, this will be minus cosine alpha n x divided by alpha n from 0 to 1. So, it will be alpha n 1 minus cos n pi. So, now, we will be getting the complete solution of C n. C n is equal to U naught. So, numerator is 1 minus cosine n pi by n pi; and, denominator is, there will be multiplied by, it will be reversed. So, it will be 2 beta square plus alpha n square divided by beta square plus beta plus alpha n square. So, we will be having 2 U naught 1, minus cosine n pi, divided by n pi multiplied by beta square plus alpha n square divided by beta square, plus beta, plus alpha n square.

And, we will be having a complete solution, U as a function of x and t, is summation of, it will be 2 U naught, summation of 1 minus cosine n pi, divided by n pi beta square, plus alpha n square, divided by beta square, plus beta, plus alpha n square sin alpha n x. So, this will be the complete solution of the third kind of parabolic partial differential equation, a well posed problem with the boundary condition at, as a robin mixed boundary condition.

Now, alpha n will be evaluated from the roots of the transcendental equation that we have already talked earlier; alpha alpha n plus beta tan alpha n equal to 0; alpha n plus beta tan alpha n equal to 0; alpha n at the roots of this equation; beta is a known parameter to us. So, that will be there. So, this will be giving you complete solution of U as a function of x and t. So, we have seen the, how to solve by using a separation of variable type of method, a parabolic partial differential equation of second order, which is a well posed problem.

In the next class, what will we do? We will be solving a second dimensional, you know, second, 2 dimensional problem only, which will be, but it will be an actual problem. It is not a well posed problem. It is actually a practical problem; how this problem will be non-dimensionalized, and then, it will be giving an ill posed problem. Then, we will be converting the ill posed into well posed problem, then, we will be solving it.

Thank you very much.