Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture – 06
Generalized Sturm – Liouville Problem

Welcome to this session. In this class, we will be looking into the standard Eigen value problem, or the Sturm - Liouville problem and its property; and, we will be looking into a generalized problem.

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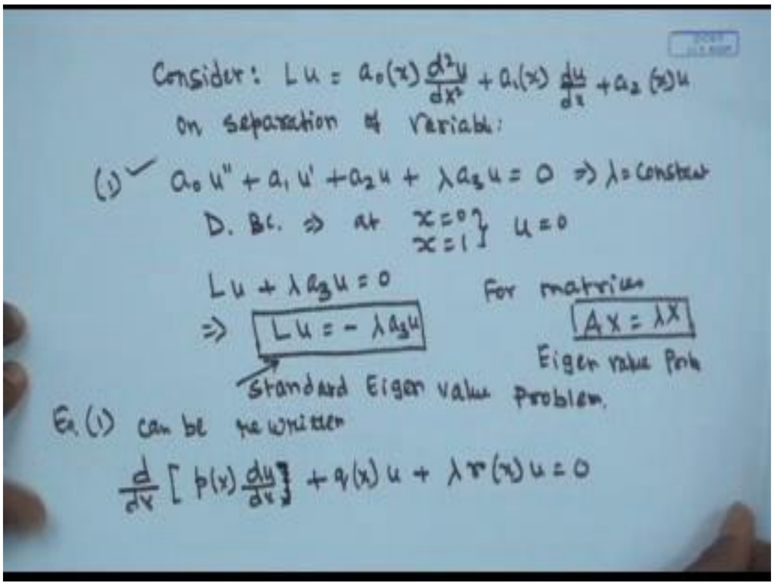

Let us consider the operator, general operator L as L u equal to a 0 x d square u d x square, plus a 1 x d u d x, plus a 2 x u. So, let us consider this operator, and on separation of variable, we may be getting a relationship like this; a 0 u double prime, plus a 1 u prime, plus a 2 u, plus lambda a 3 u is equal to 0, and where lambda is a constant.

Let us assume the digital boundary condition that, at x is equal to 0, and x is equal to 1, we have u is equal to 0. Now, this equation can be written in a compact form as L u plus lambda a 3 u is equal to 0; or, it will be written in the form of L u is equal to minus lambda a 3 u.

Now, if you look into the formation of this equation, the form of this equation, the equivalent notation in the matrix algebra in the discrete domain, this is the continuous domain; we are talking about the functions, continuous functions. Now, in the discrete domain, in the matrix, if you look into the similar form of equation, similar form corollary is that, for matrices, the similar type of equation, we can see as A x is equal to lambda x; and, this is known as Eigen value problem. Therefore, in case of the continuous domain for functions, this is a standard Eigen value problem.

So, let us consider this is as equation 1. Now, this equation 1 can be rewritten in this form; means, equation 1 can be rewritten as d d x of p x d u d x. So, this is the end of it, plus q x u, plus lambda r x u, is equal to 0. Now, this equation can be written in this form, where we will be defining p x, q x and r x, in terms of the original coefficient a 0, a 1, a 2 and a 3.

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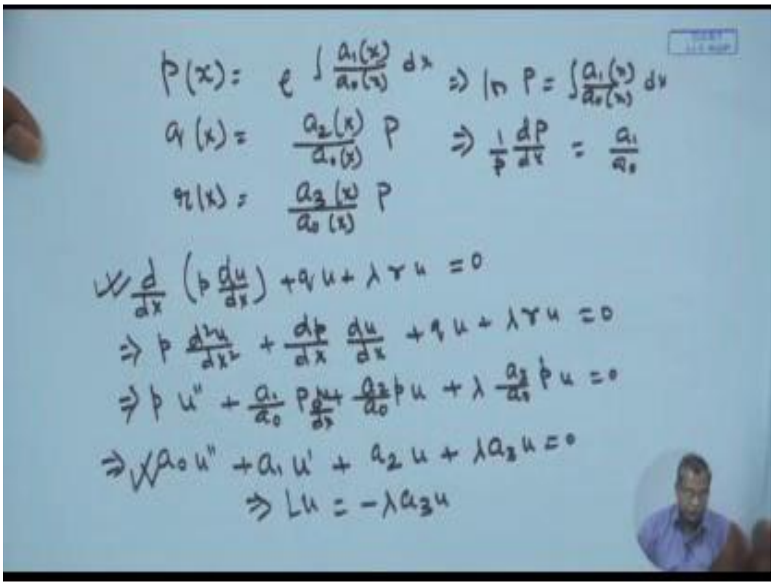

So, let us define that, p x is nothing, but e to the power integral a 1 x divided by a 0 x, d x; q x is equal to a 2 x over a 0 x, times p; and, r x is equal to a 3 x divided by a 0 as the function of x, times p. So, if you define, this equation can be, if you define this quantities p, q and r in the equation number 1, then you will be getting back the equation 1. This is very simple, very, you know, simplified case. So, if we take the logarithm of both side, and then, differentiate; so, l n p is equal to integral a 1 x divided by a 0 x d x. So, if you differentiate it, you will be getting, d p d x 1 over p is equal to a 1 by a 0.

And similarly, q x is given as a 2 by a 0 times p, and if you just put it back; you will be getting of the governing equation. So, I just establish the identity of both the equation to be same, p d u d x, plus q u, plus lambda r u. This becomes p d square u d x square, plus d p d x d u d x, plus q u, plus lambda r u. So, just substitute p; the p is, just as leave it as p. So, if this is the equation, right, this is the equation, then, you will be getting p d square u double prime, plus d p d x; d p d x is nothing, but a 1 by a 0 times p, plus q is a 2 by a 0 times u, plus there will be a p there, q in the a lambda; r is a 3 by a 0 times p u is equal to 0.

So, p will be canceling out from each step; and then, you multiply both sides by, all sides by a 0. So, what you will be getting is that, a 0 u double prime, plus a 1, there will be a d u d x here. So, a 1 u prime, plus a 2 u, plus lambda a 3 u is equal to 0. So, these two equations are identical. So, this will be giving you L u is equal to minus lambda a 3 u. So, substituting p, q, r, by defining p, q and r, this equation, equation number 1 is equivalent to this equation.

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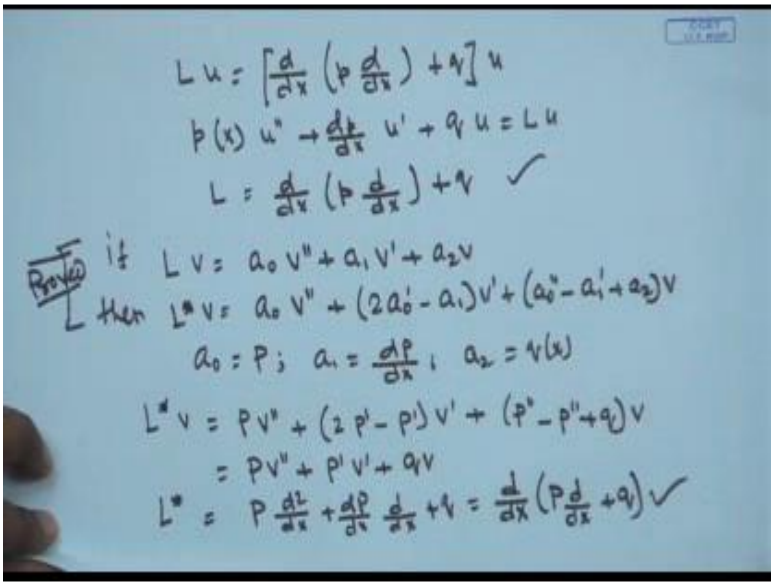

So, these two equations are identical. So, let us see what we get out of it. So, L u is now becomes d d x of p d d x, plus q multiplied by u. So, therefore, p x u double prime, plus d p d x u prime, plus q u is equal to L u. So, L is now becomes d d x of p d d x, plus q. Now, earlier, we had, if L of the v is equal to a 0 v double prime, plus a 1 v prime, plus a 2 v, then, L star v will be nothing, but a 0 v double prime, plus 2 a 0 prime, minus a 1 v prime, plus a 0 double prime, minus a 1 prime, plus a 2 times v. This you have already proved earlier.

In the last class, we have proved this. Now, if you compare a 0, a 1 and other things. So, if you compare this one, this equation number, equation number 3 with 4, that we have already seen earlier that, a 0 is nothing, but p of x. Therefore, a 1 is nothing, but d p d x, and a 2 is equal to q of x. So, L star v is nothing, but p v prime double prime, plus 2 p prime, minus p prime, times v prime, plus p double prime, minus p double prime, plus q is equal to, into v. So, therefore, this becomes p v double prime, plus p prime v prime, plus q times v.

So, therefore, you can always see that, L star is equal to L. So, L star will be nothing, but what is L star? L star is p d square d x square, plus d p d x times p prime d d x, plus q. So, this will be nothing, but d d x of p d d x, plus q. So, if you see that, L is equal to nothing, but L star. So, L is equal to this, and L star is equal to this.

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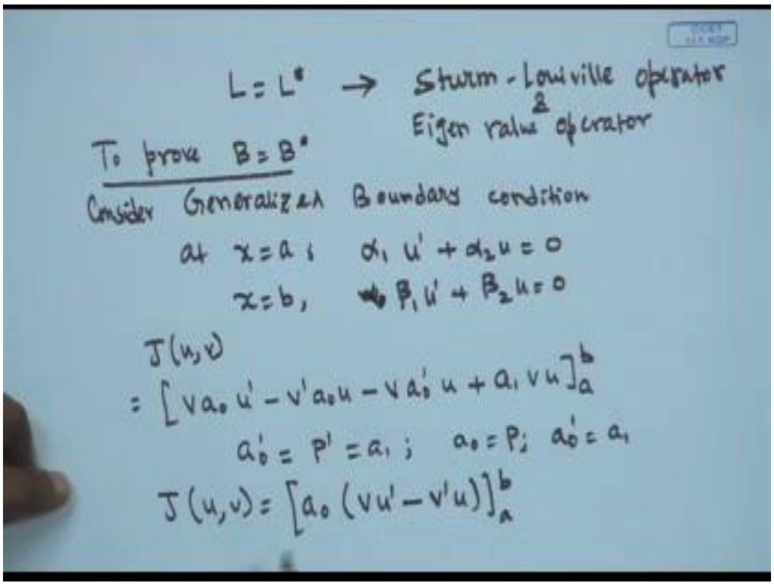

So, L is equal to L star, and our general Sturm - Liouville operator or Eigen value operator is a self adjoint operator; operator, in general, is where L is equal to L star. Now, we will see that, to prove B is equal to B star, and if we, if we want to prove B is equal to B star, we have to, we have to check, let us say, a generalized boundary condition. Consider a generalized boundary condition. This will be no longer a (Refer Time: 11:13) boundary condition; consider a generalized boundary condition that, at x is equal to a, we have alpha 1 u prime, plus alpha 2 u is equal to 0; and, at x is equal to b, we have beta 1 u prime, plus beta 2 u is equal to 0.

And then, we will be looking into the bilinear concomitant. If we look into the bilinear concomitant, this becomes v a 0 u prime, minus v prime a 0 u, minus v a 0 prime u, plus a 1 v u, evaluated on the two boundaries a and b. Now, we have already proved that, a 0 prime is equal to d p d x, is nothing, but p prime is equal to a 1, and a 0 is equal to p, and a 0 prime is equal to a 1. So, bilinear concomitant now becomes only, you know, it boils down into a 0 times v, multiplied by u prime, minus v prime, multiplied by u, whole thing evaluated between a and b.

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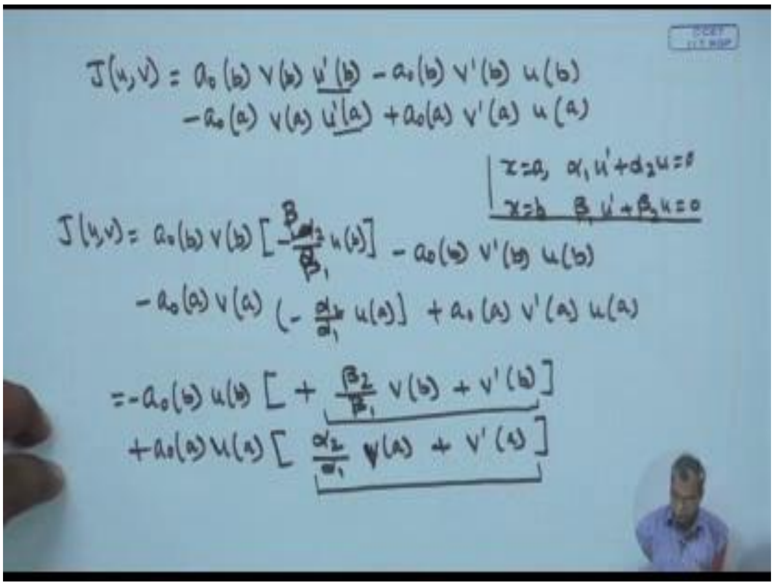

Now, if you are really do that, and substitute the values, let us see what the bilinear concomitant will give us. So, J u v is nothing, but a 0 at b, v at b, u prime at b, minus a 0 at b, v prime at b, u at b, minus a 0 evaluated at a, v at a, u prime at a, minus minus plus, a 0 at a, v prime at a, and u at a. We already had the boundary condition of the original problem as that, at x is equal to a, we had alpha 1 u prime, plus alpha 2 u is equal to 0. So, we can substitute at x is equal to a u prime at a. So, u prime at a, we can substitute; and then, at x is equal to b, we have beta 1 u prime, plus beta 2 u is equal to 0. So, you can substitute these two.

So, J u v will be nothing, but a 0 at b, v at b; and u prime at b will be nothing, but minus alpha 2 by alpha 1 u at b, minus a 0 at b, v prime at b, u at b, and minus a 0 at a, v at a, and u prime at a; we substitute, this was u prime at b. So, this will be beta 2; this will be nothing, but this will not be beta; alpha 2; alpha 2, and this is beta 2 and beta 1, and, u prime at a will be minus alpha 2 by alpha 1 times u at a, minus plus a 0 at a, v prime at a, and u at a. Now, we can collect the terms.

For example, we can take a 0 b and u b common from this one, from the first two terms. So, if you really do that, a 0 b and u at b, we take it common. So, what is remaining is, minus beta 2 by beta 1, v at b, minus; you can take minus also common, from first two, first two terms. So, it will be plus beta 2 by beta 1, v at b, plus v prime at b. And, we can take a 0, a as common; minus, minus, plus. So, it will be plus a 0 at a common and v and u at a common. So, if you do that, what you will be getting is alpha 2 by alpha 1, u at a, plus v, sorry, v at a, then, plus v prime at a.

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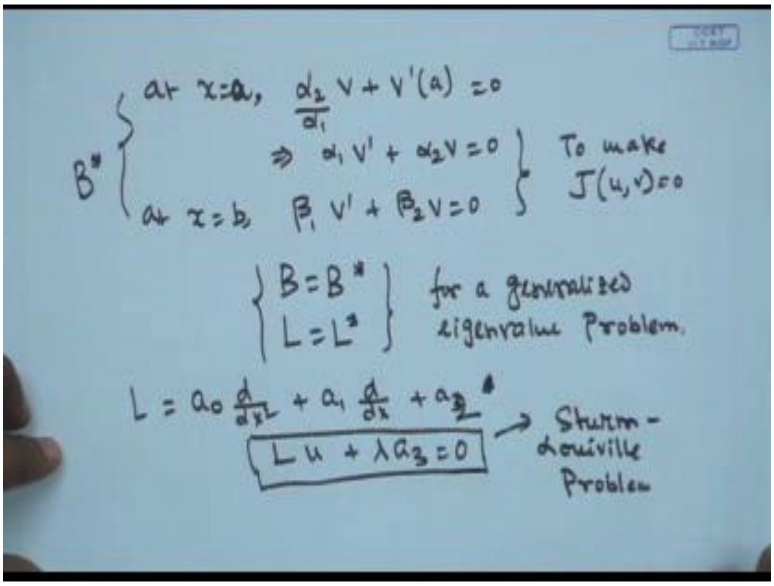

Now, if I select this equal to 0, this boundary, at this boundary, x is equal to b, and at this boundary x equal to a, this whole thing is common. Let us see what you get in order. So, if you really select that, then bilinear concomitant will vanish. So, at x is equal to a, if you select alpha 2 v, plus v prime a divided by alpha 1 is equal to 0. So, that will give me alpha 1 v prime, plus alpha 2 v is equal to 0 at x is equal to a. and, at x is equal to b, I will be getting beta 1 v prime, plus beta 2 v will be 0, to make bilinear concomitant to vanish. So, these will be now my B star, or boundary condition of the adjoint problem.

Now, we have to see that, B is equal to B star, and we have already proved L is equal to L star for a generalized Eigen value problem. So, the standard Eigen value problem operator is a self adjoint operator. So, a 0 d square d x square, plus a 1 d d x, plus a 3. So, this will be, if this is the form, then, L u is equal to plus lambda a 1; this will be a 2 plus lambda a 3 will be 0. So, this is a standard Eigen value problem, and in this standard Eigen, these are generalized form of standard Eigen value problem, or it is also known as the Sturm - Liouville problem.

And, we have proved that, a Sturm - Liouville is nothing, but a self adjoint problem. So, L is equal to L star, and B is equal to B star in this particular case, and we are having a self adjoint operator with us. Now, by establishing the Sturm - Liouville problem, or standard Eigen value problem is a self adjoint problem, is self adjoint operator, then, let us see. What are the various properties of the self adjoint problem we can exploit, for using separation of variable? Now, there are certain salient problems, of the standard Eigen value problem are there, for the self adjoint operator, the important properties of self adjoint operators, perator we should look into.

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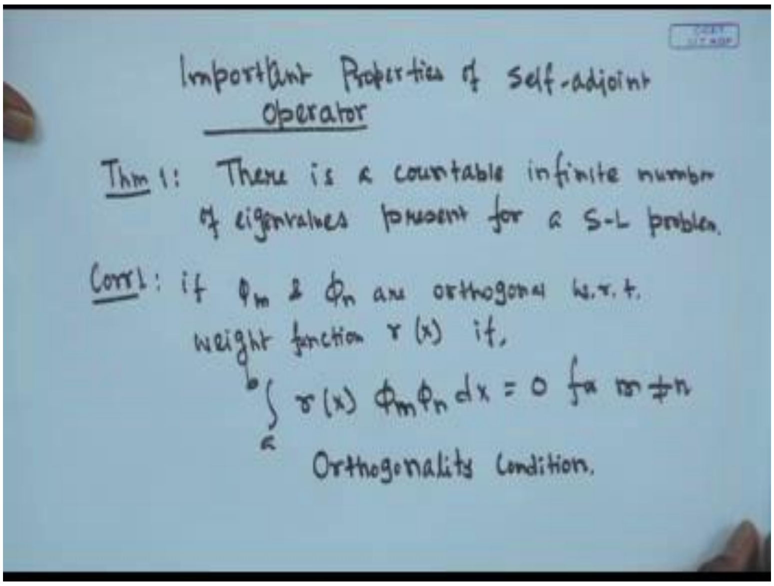

So, the first property that goes as a theorem number 1 is that, there is a countable infinity, infinite number of Eigen values present for a Sturm - Liouville problem. So, the number of Eigen values present in the system is infinite in number. So, therefore, we will be talking about infinite dimension on space, and it is basically a fun continuous function having infinity, infinite dimension. So, number of Eigen function, Eigen values will be infinite, and number of Eigen functions will be infinite.

Then, the next important theorem is that, the Eigen with the; if phi m, this is the corollary 1 is that, if phi m and phi n are orthogonal with respect to weight function r x. So, this is a definition of orthogonal functions; phi m and phi n are the two functions which are said to be orthogonal with respect to each other, and with the weight function r, then, if this equation is satisfied, integral a to b r x, multiplied by phi m, multiplied by phi n d x is equal to 0. So, for m naught is equal to n. So, this is known as the orthogonality condition.

So, next, what we will be showing, next theorem that, for if lambda m and lambda n are the two distinct Eigen values of the Sturm - Liouville problem with the Eigen functions y m and y n, then y m and y n are orthogonal functions with respect to weight function r.

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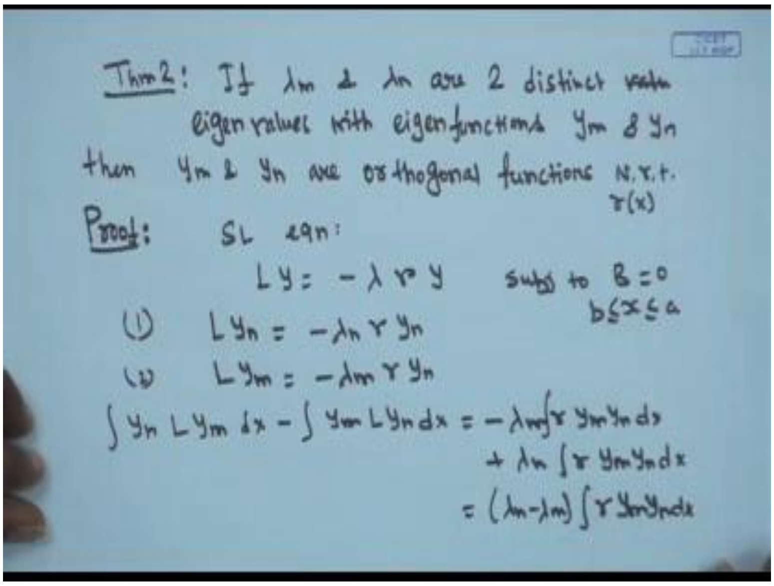

So, let us look into the theorem, and look into its proof. So, next theorem, which will be quite important to us is that, if lambda m and lambda n are two distinct Eigen values, this is important, two distinct Eigen values, with Eigen functions y m and y n, then, y m and y n are orthogonal functions with respect to weight function r x. So, let us prove this theorem. So, proof goes like this. The Sturm - Liouville equation becomes L y is equal to minus lambda r y, where r is a function of x, subject to boundary condition b is equal to 0, where the domain of x is lying between b and a.

Since lambda m and lambda n are the Eigen values with the Eigen functions y m and y n, they will be satisfying this equation. So, I can write L y n is equal to minus lambda n r y n; L y m should be is equal to minus lambda m r y n. then, what we will be doing, we will be multiplying equation 1 with respect to y m; multiplying equation two with respect to y n, and integrate, and subtract. So, what I will do next is, integral of y n L y m d x, minus y m L y n d x. So, this becomes minus lambda m r y m y n d x; minus minus plus, lambda n r y m y n d x. So, we will be getting on the right hand sides as lambda n minus lambda m integral r y m y n d x.

So, next, what will we, we invoke the property of the adjoint operator, and if you look into the property of the adjoint operator that, integral of u L v is nothing, but integral of L star u L star u v, plus J u v.

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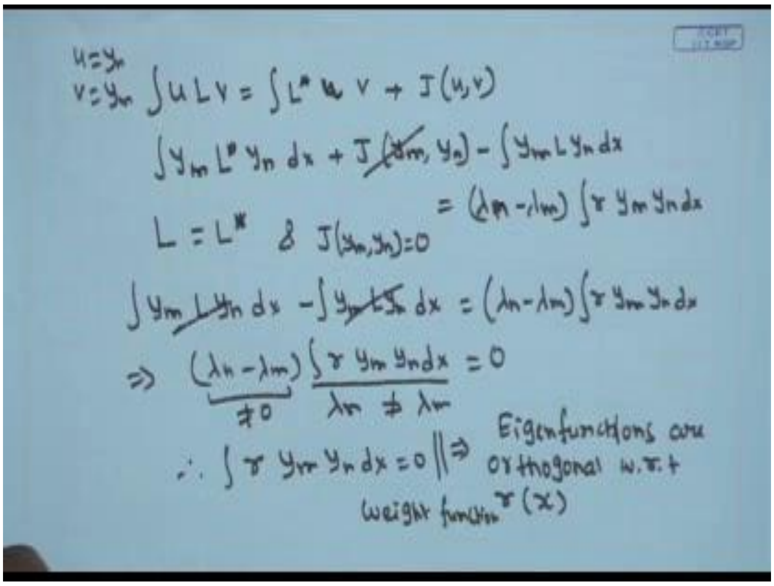

We utilize this property, and we put down the first integral y n L y m d x as integral y m L star y n d x, plus J y m y n, minus integral y m L y n d x is equal to, right hand side is lambda n, lambda n minus lambda m, integral r y m y n d x. So, what I do? I put y u is equal to y n; u is equal to y n, and v is equal to y m. So, then, expanded the first function in terms of its adjoint variable. So, in terms, I just wrote this one in terms of adjoint operator, in the form of this and we will be getting this. So, we have already proved that Sturm - Liouville operator is L is equal to L star; it is a self adjoint operator.

So, and, J y m and y n is equal to 0, that we have already proved earlier that, Sturm - Liouville operator is a self adjoint operator. Using that condition, we can write integral y m L is equal to L star. So, y n d x, minus integral y m L y n d x, is equal to lambda n minus lambda m integral r y m y n d x. And, this was of because of this, and these two will be equal and opposite. So, you will be having lambda n minus lambda m, multiplied by r y m y n d x is equal to 0. So, lambda n and lambda m are two distinct Eigen values. So, lambda n is not equal lambda m. So, this cannot be equal to 0. So, what is left is, this integral has to be equal to 0.

So, therefore, integral of r y m y n d x is equal to 0. So, what this means, this means that, Eigen functions are orthogonal functions; are orthogonal with respect to weight function r x. This is known as the weight function. So, let us summarize whatever we have done, we have looked. We have defined a standard Eigen value problem, and we have looked into the properties of the standard Eigen value problem, and we have proved that standard Eigen value problem, or Sturm - Liouville problem is a self adjoint operator, or self adjoint problem. Here, L is equal to L star, and B is equal to B star. Then, we have looked into the properties of the standard Eigen value problem, which we will be using quite often in your course.

Number one property is that, it will be having infinite number of Eigen values, and so, it will be talking about infinite dimensional space. So, it will be a, almost a continuous space. So, therefore, second important property is that, the Eigen, if the Eigen values are distinct corresponding to Eigen functions, and if Eigen values are distinct, then, the Eigen functions will constitute a set of orthogonal functions with respect to the weight function r. if the equation is in the form of L u is equal to minus lambda r u, then r can be identified as the weight functions, and the corresponding Eigen functions will be orthogonal to each other, with respect to the weight function r.

Now, in the Cartesian coordinate system, we have seen the corresponding equation is d square y d x square, plus lambda y is equal to 0. So, here, the weight function r is equal to 1, and this equation is a special form of standard Eigen value problem, where a 0 is equal to 1, a 1 is equal to 0, and a 2 is equal to 0. So, we got the corresponding equation in the Cartesian coordinate system as a subset of the standard Eigen value problem in its general form. So, we have talked about the, we have considered the general form of the standard Eigen value problem, or the Sturm - Liouville problem, and established that, it is a, it forms a self adjoint, self adjoint system, and we have looked into the various properties of the self adjoint operator. And now, we are in a position to start with the actual problem using separation of variable type of solution.

Thank you very much.