Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
Prof. Sirshendu De
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 05
Adjoint Operator

Good morning everyone. So, in this session we will be looking into the, we will be looking into the, how to obtain the adjoint operator given an operator. So, that is very important and we will be seeing later on this course, that there are various operators Sturm-Liouville operators or eigenvalue operator that we have discussed in the last class, but basically the self adjoint operator. And now in order to obtain the adjoint operator, given an operator is very important in this course as we see later on. So, in this class we will be first defining an operator. Then we will be looking into the procedure how to obtain the adjoint operator so.

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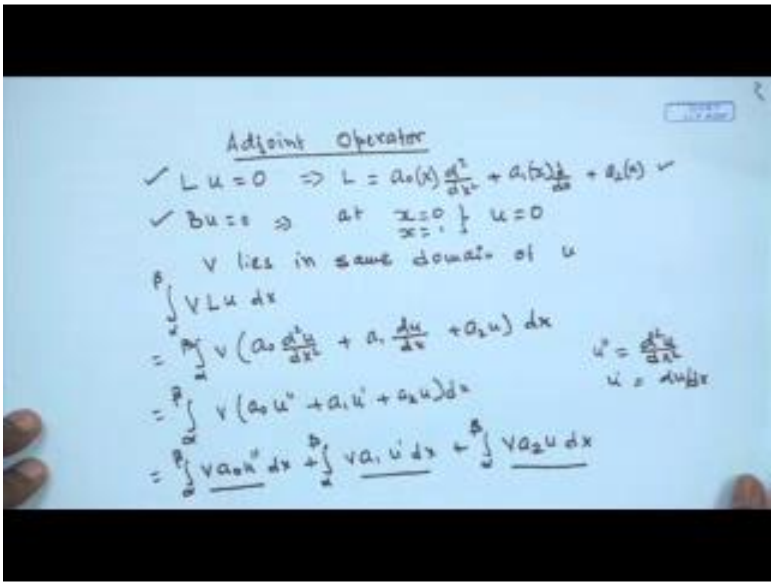

Let us define a general operator for example, l u is equal to 0 in this particular problem, l is the operator and we are talking about an operator a generalized one a second order operator, d square d x square plus a one, d d x plus a 2 which will be function of x. So, in this operator all the coefficients a 0 a one and a 2, they are function of x, and let us look into the boundary condition of this operator that and let say they are homogeneous, at x is equal to 0 and at x is equal to one both u is equal to 0. So, these defines the problem l u is equal 0 defines the problem, and b u is equals to 0 defines the boundary conditions. So, the boundary conditions are homogeneous, as we have discussed that if the boundary conditions are homogeneous then the problem is posed as the eigenvalue problem.

So, let us find out in this operator l, what is the adjoint operator, let see the procedure. So, in order to get that, first we consider another variable v lies in that same domain, in same domain of u. So, we let us take the integral v l u d x these integration is over domain of x from alpha to beta, then let us just expand the operator, a 0 d square u d x square, plus a one d u d x, plus a 2 u is equal 0 a 2 u multiplied by d x. And then we write v a 0 u double prime, plus a one u prime plus a 2 u d x where primes denote the derivative with respect to x. So, u double prime is nothing, but d square u d x square and u prime is nothing, but d u d x. Then what we will do we will be expanding we open up this brace, and v a 0 u double prime d x plus integral alpha to beta, v a a one u prime d x plus alpha to beta v a 2 u d x.

Once we write this then, what we will do next is that each of this integral will be will be integrating by parts, concentrating the first part as the first function and double prime as a second function. In this integral v a one is the first function u prime is the second function. So, let us solve this equation by opening up the integration by parts, if you really do that, what we really getting is that alpha to beta v l u is equal to first function v a 0, differential of the integral of second function. So, it will be u prime from alpha to beta, minus differential of first function v a 0 prime, integral of the second.

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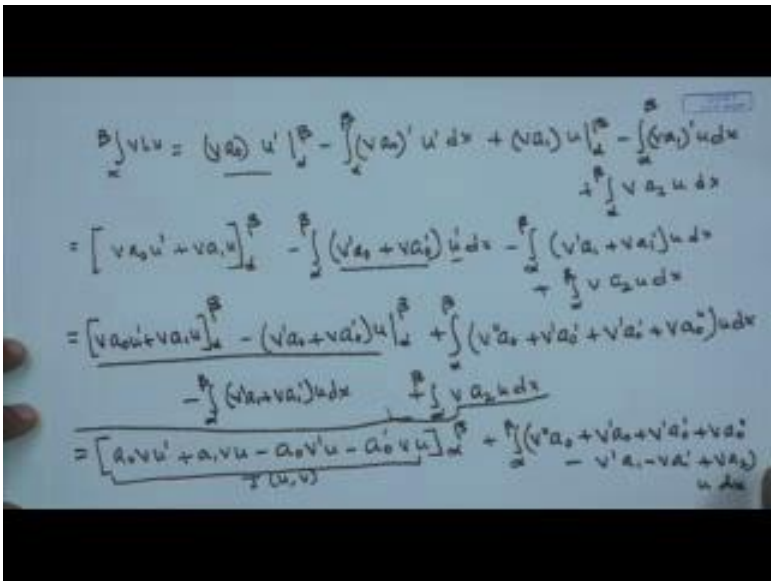

So, u prime d x so, that will be a first one the second integral, again we do first function v a one integral of the second one u alpha to beta, minus differential of first function. So, v a one prime integral of second function, alpha to beta and the third integral remain as it is alpha to beta, v a 2 u d x. So, let us know collect these terms. So, this becomes v a one, v a 0 u prime plus v a one u this will be evaluated from alpha to beta minus, again we integrate it by parts. So, this will be the off. So, before that let us consulate this one. So, this will be v prime a 0 plus v a 0 prime u prime d x, minus alpha to beta v prime a one plus v a one prime, u d x plus v a 2 u d x.

The next we will be doing integration by parts again considering this a first function these are the second function. So, I write the equation as v a 0 u prime plus v a one u lying between alpha to beta. Now first function integration of the second to 1. So, it will be v prime a 0 plus v a 0 prime, u alpha to beta, minus minus minus plus differential of the first function integral of the second function alpha to beta. So, differential of the first function will be v double prime a 0 plus v prime a 0; prime plus v prime a 0 prime plus, v a 0 double prime; u d x plus integral alpha to beta v into u d x.

Now let us see what we get. So, we collect these terms first. So, this will be a 0 v u prime plus a one v u minus a 0 v prime u minus, a 0 prime v u these will be evaluated between alpha to beta, and we call this part as bi linear concomitant, and it is notation is j u v and then we collect the term. So, this becomes plus alpha to beta, v double prime a 0, plus v prime a 0 plus, v prime a 0 prime, plus v a 0 double prime, minus v prime a one minus v a one prime, plus v a 2 u d x. In fact, this is the first term. In fact, we should write as this term as well.

So, my in the in the previous step, that is missing v prime a one plus v a one prime u d x plus this fine. So, now, let us see what we get we get alpha to beta v l u is equal to nothing, but j u v plus, alpha to beta. What will be getting is a 0 v double prime plus, 2 a 0 prime, 2 a 0 prime minus a one v prime, plus a 0 double prime minus a one prime plus a 2 times, v multiplied by u d x.

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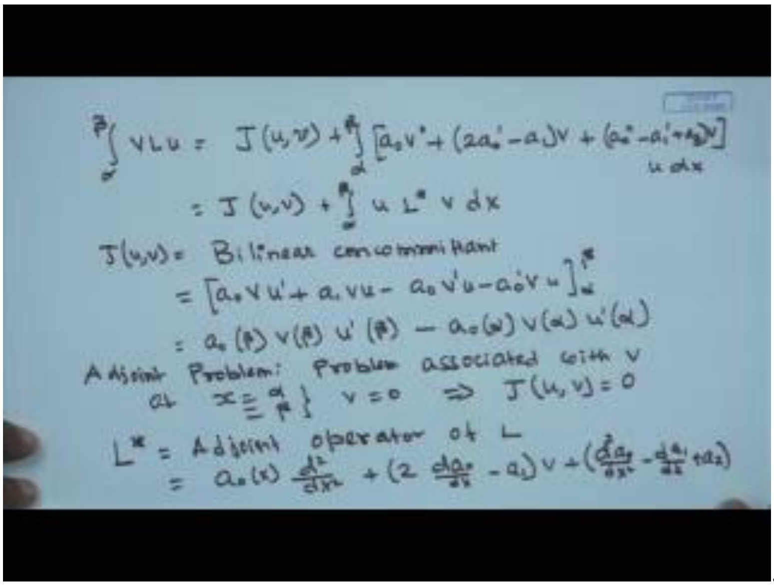

So, we will be getting a 0 prime times v. So, a 0 prime times v there will be it will be occurring twice. So, it will be 2 a 0 prime and you will be getting this. So, ultimately, these can be written as j u v plus alpha to beta u l start v d x; where j u v is known as bilinear concomitant. And it is value is a 0 v u prime plus, a one v u minus a 0 v prime u, minus a 0 prime v u, evaluated at alpha and beta. And if you remember that original boundary condition was that at x is equal to alpha u was equal to 0, at x is equal beta u is equal to 0. So, all these three terms will be gone only this term will be surviving. So, it will be a 0 beta v at beta u prime at beta minus a 0 at alpha v at alpha u prime at alpha.

So, now we can select the boundary condition of the adjoint problem. So, the adjoint problem is basically the problem associated with v. So, adjoint problem is problem associated with v. So, therefore, we have to if you select at x is equal alpha and beta, if we select v is equal to 0 then your bilinear concomitant will be equal to 0; that means, we can select the boundary conditions on the adjoint problem v. So, that the bilinear concomitant will be equal to 0 now let us look into the l star which is nothing, but the adjoint operator of l. So, what is l star, l star is nothing, but adjoint operator of l. So, therefore, this l star is a 0 x d square d x square, plus 2 d a 0 d x minus a one times v plus d square a 0 d x square, minus d a one d x plus 2. So, this is the adjoint operator and we can we can clearly see that adjoint operator is not equal, l star is not equal to l in this particular case.

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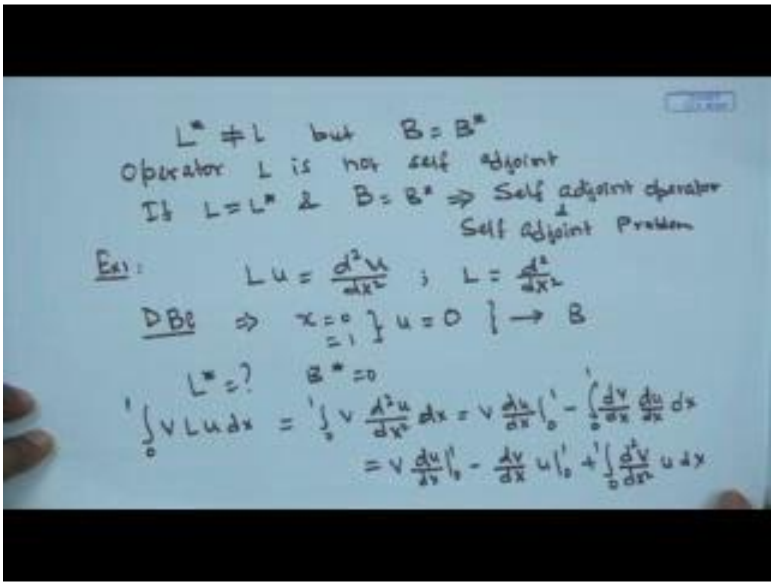

So, l is l star is not equal to l, but b is equal to b star; that means, both the boundaries of the original problem u, which was basically nothing, but the, but combined into the boundary condition b that at x is equal to alpha, and beta u is equal to 0 and we have selected for the adjoint problem v same boundary condition at x is equal to alpha and beta v is equal to 0. So, b is equal to b star, b star is the boundary condition associated with the adjoint problem so, but l is not equal to l star. So, this problem is not a self adjoint problem. Operator l is not self adjoint, if l is equal to l star and b is equal to b star, then we will be having a self adjoint operator, operator and self adjoint problem.

So, we just look into one example, that l is equal let us consider a Laplacian operator in one dimensional d square u d x square. So, my operator is Laplacian in one dimensional. So, d square d x square is my operator. Let us look into the boundary condition dirichlet boundary condition. So, dirichlet boundary condition is at x is equal to 0, and x is equal to 1 let us say u is equal to 0. Now let us find out what is l star and what is b star in this problem. So, this condition boundary condition is nothing, but the b of the original problem. So, what we to do next are, we carry out and integration of v times l u d x from 0 to 1 where, v is variable in the same domain as u. So, therefore, it is becomes 0 to one v d square u d x square is equal to into d x.

So, then we integrate by parts, it is a first function integral of the second function, 0 to one minus differential of the first function, integration of second function d x 0, to 1 then again we do integration by parts here. So, this becomes v d u d x from 0 to 1 minus first function d v d x integral of second function u 0 to 1 minus minus plus differential of the first function and integral of the second function. Now let us collect terms and see what we can do about it. So, integral 0 to one l u d x in this particular case becomes v d u prime d x minus u, v d u d x minus u d v d x evaluated from 0 to one, plus u d square v d x square d x. So, let us put the boundary conditions. So, this is v at 1 d u d x evaluated at 1, minus u at one, d v d x evaluated at 1 minus v at 0 d u d x evaluated at 0 minus minus plus, u at 0 d v d x evaluated at 0, plus 0 to 1 u.

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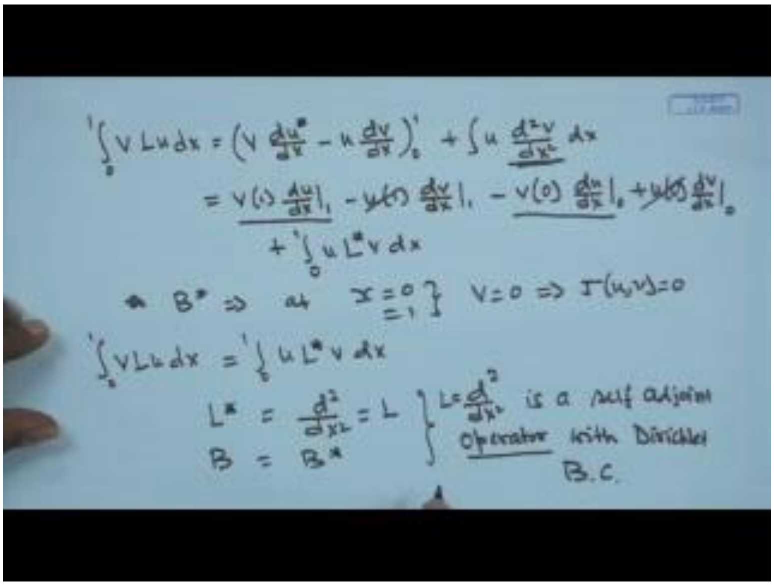

So, this is now my l star v d x. So, as we as we had the boundary condition b u at 0 is 0 u at 1 equal to 0. So, therefore, the bilinear concomitant it remains only 2 terms. We do not have any idea about u prime at x is equal 0 and x is equal 1. So, therefore, we have if we select at x is equal to 0 and x is equal 1 v is equal to 0, then this bilinear concomitant now will vanish; that means, if we select b star as at x is equal to 0 and 1, v is equal to 0 then bilinear concomitant terms will vanish. And what we get is v l u d x 0 to one is nothing, but 0 to 1 u l star v d x.

Now, let us see what is l star. If you look into this l star l star is this. So, l star is nothing, but d square d x square. This is same as l and b we have seen as this same as b star. So, Laplacian operator is a self adjoint operator, with the dirichlet boundary condition. So, d square l is equal d square d x square is a self adjoint operator, with Dirichlet boundary condition. Now let us look into the problem if the boundary condition is changed to neumann or robin mixed, whether by you know the adjoint problem is self adjoint or not.

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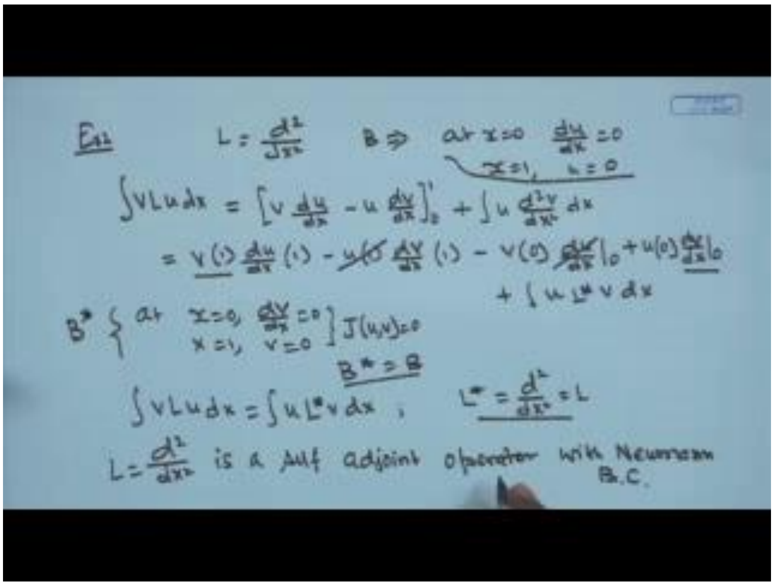

Now, let us just go into 2 quick example that l is same as one dimensional Laplacian, but the boundary conditions are now changed, at x is equal to 0, we have d u d x is equal to 0, and x is equal to one, we have let say u is equal 0. So, let us integrate v l u d x and ultimately, what we will be getting is v d u d x minus u d v d x evaluated from 0 to one, plus integral u d square v d x square d x. So, the all the algebra remains the same. Now it will start showing difference from this point one let us evaluate the bilinear concomitant first. V at one d u d x at one minus u at one, d v d x at one minus v at 0 d u equal at 0, minus minus plus u at 0 d v d x at 0 plus u l star v d x. So, at x is equal 0, d u d x is equal to 0 this term will be off, at x is equal to one u is equal to 0.

So, this term will be off now if we select that. So, we do not have any idea about d u d x so, but if we select v at x is equal to one, and d v d x at x is equal to 0 equal to 0, then this bilinear concomitant will be equal to 0. So, therefore, at x is equal to 0 my d v d x is equal to 0, and my x is equal one v is equal to 0. So, that will force my bilinear concomitant to vanish. So, if we look say these are same problem as b star. So, these as be. So, these boundary condition b star and if you see at b star is equal to b and this problem now becomes, v l u d x is equal to u l star v d x where l star is equal to d square d x square is equal to l. So, l star is equal to l b star is equal to b. So, again it is a self adjoint operator. So, d square d x square, is a self adjoint operator with Neumann boundary condition

Now, we will be redoing this problem once again, by considering a robin mixed boundary condition at x is equal to one, and we will show that again the one dimensional Laplacian is a self adjoint operator.

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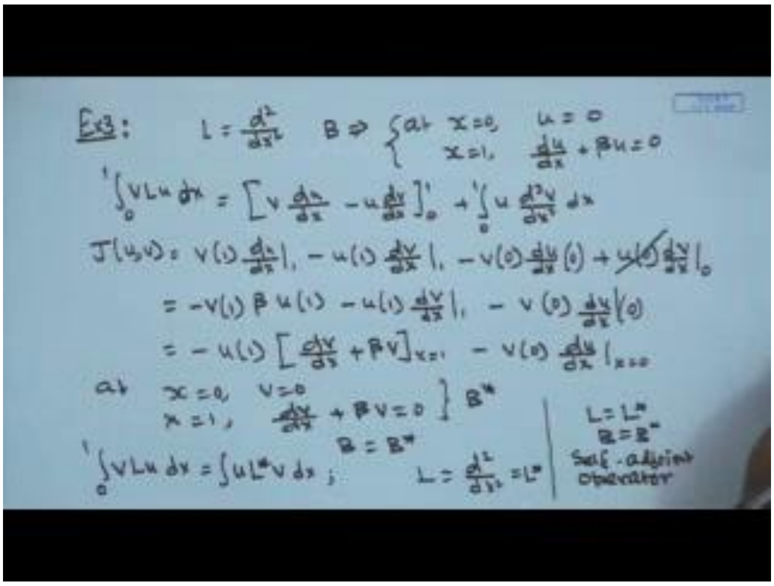

So, let us go for a third example quickly we have the same Laplacian one dimensional as the operator, d square d x square and b will be at x is equal to 0, u is equal to 0 at x is equal to one we have d u d x plus beta u is equal 0. So, we have the robin mixed boundary condition at x is equal to one. Now again there as look into the integral v times l u from 0 to one d x. So, these will be having let us look into this form v d u d x minus u d v d x from 0 to one plus u d square v d x square d x.

Now, we evaluate the bilinear concomitant term. So, let us find out what is j u v in this case. J u v in this case will be v at one d u d x evaluated at one, minus u at one d v d x at one minus v at 0 d u d x at 0 and then minus minus plus, u at 0 d v d x at x is equal to 0, now we have already the boundary condition at x equal 0 u is equal to 0. So, at x equal to 0 u is equal to 0. So, this term will be off now at x equal one, d u d x is nothing, but minus beta u. So, at x equal one, we write d u d x as minus beta u evaluated at one. So, minus v at one, beta u at one minus u at one, d v d x at one minus v at 0 d u d x evaluated at 0 we call we take u u at one common minus u at common. So, this becomes, d v d x plus beta times v evaluated at x is equal, to one minus v at 0 d u d x evaluated at x is equal to 0.

So, in order to make the bilinear concomitant value to vanish; I select the boundary condition at x is equal to 0, v is equal to 0, and at x is equal to one I have d v d x plus beta v is equal to 0. So, these are same as b. So, b star is same as b in this case. So, b is equal to b star if I select this then my bilinear concomitant will vanish, and I will be having integral 0 to one v l u d x is equal u l star v d x where l is equal d square d x square is equal to l star. So, l is equal to, l star b is equal to b star. So, this problem the operator the one dimensional Laplacian is a, self adjoint operator. Even you have a robin mixed boundary condition.

So, we have already seen that the said the Laplacian operator is a self adjoint operator. Irrespective of the boundary condition, whether there is it says, dirichlet boundary condition whether it is you have a Neumann boundary condition whether you have a robin mixed boundary condition. So, in this class whatever you have seen is that let us summarize we know given an operator a general operator how to obtain the adjoint operator and during the you know derivation of the adjoint operator the boundary conditions will appear as a you know sum as a as a you know combination of terms which is known as the bilinear concomitant by selecting appropriate conditions on the adjoint problem adjoint variable v, we can select the boundary conditions on v and can force the bilinear concomitant to be 0. There by we will be obtaining the conditions on the adjoint problem v.

And similarly we will be getting the adjoint operator l star if l is equal to l star and b is equal to b star then we will be having a self adjoint problem and operator is the self adjoint operator and we will see in the next class that the Sturm-Liouville operator or the standard eigenvalue operator, is a self adjoint operator. And we will be looking into some of the properties of the self adjoint operator. Then we will be ready for doing a separation of variable type of solution for partial differential equation.

Thank you very much.