**Partial Differential Equations (PDE) for Engineers:**
**Solution by Separation of Variables**
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**Lecture – 04**
**Standard Eigenvalue Problem and Special ODEs**

Welcome to this session as we have discussed in the last class we will be discussing the non trivial solution of a standard Eigenvalue problem and I will be taking we have already looked into the solution by considering the durational boundary condition. So, in this class we will be looking into how the solution evolves in case of Neumann and Robin mixed boundary conditions.

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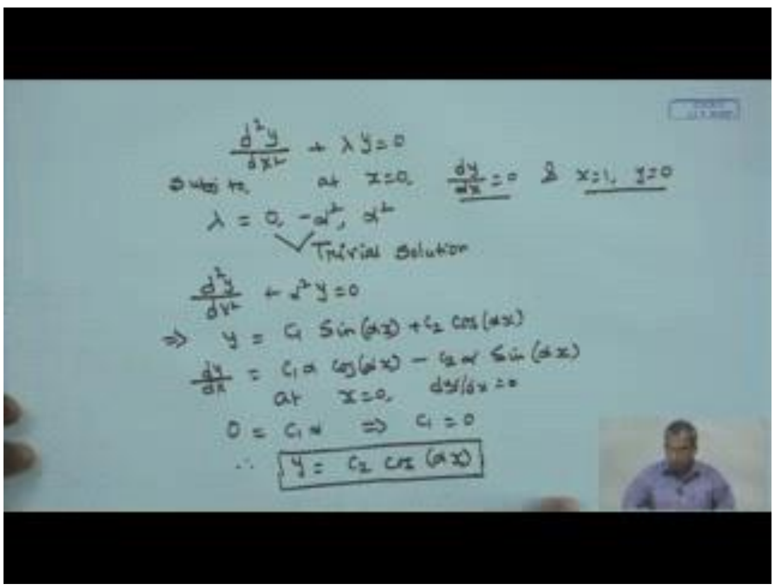


So, d square y d x square plus lambda y equal to 0 subject to at x is equal to 0 d y d x is equal to 0 and at x is equal to 1 y is equal to 0. So, both are homogenous boundary condition, but the condition at x equal to 0 is a Neumann and the condition at x equal to 1 is a Dirichlet.

Now, let us look into these type solution of these type of equation now again as we have discussed earlier lambda can be 0 can be negative can be positive. Now, I am not going into detail by solving the sub cases 0 and minus alpha square negative lambda and 0 because they will be giving a Trivial Solution. So, it is up to the reader just take it as an assignment to show exactly going by the same way whatever we have done earlier to show that if lambda is equal to zero we are going to land into a trivial solution. Similarly, if lambda is negative then also we will be getting a Trivial Solution.

So, we will be solving a case where we will be getting a Non-Trivial Solution for lambda equal to positive or plus alpha square. If you do that we will be getting d square y d x square plus alpha square y is equal to 0 and these as you know that this are the solution y is equal to c one sin alpha x plus c 2 cosine alpha x. Now, let us write down the first boundary condition at x is equal to 0 d y d x equal to 0. So, therefore, we take the derivative of y with respect to x. So, this becomes c 1 alpha cosine alpha x minus c 2 alpha sin alpha x.

Now, d y d x equal to 0 at x is equal to 0. So, the first boundary condition at x is equal to 0 d y d x will be equal to 0 this will be 0 x equal to 0 cosine 0 is 1 and sin 0 is 0. So, we will be having c 1 alpha is equal to 0 for alpha Non-Trivial Solution alpha is not equal to 0; that means, c 1 is equal to 0. So, therefore, we will be having y is equal to c 2 cosine alpha x as the solution.

Next, what we will be doing we will be utilizing the other boundary condition that is residing at x is equal to 1.

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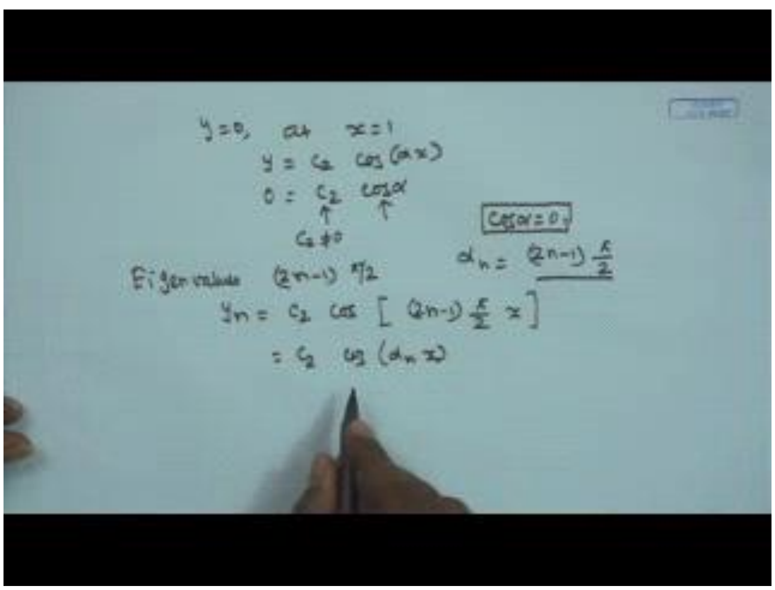


So, y is equal to 0 at x is equal to 1 as well. So, therefore, what is the y, y is equal to c 2 cosine alpha x. So, therefore, this becomes 0 is equal to c 2 cosine alpha. Now there are two options as you have detailed we have discussed earlier either c 2 equal to 0 or cosine alpha is equal to 0 if c 2 is equal to 0 then we will be again getting into a Trivial Solution. So, c 2 is not equal to 0 therefore, what is the option left is cosine alpha is equal to 0 and what is the general solution of cosine alpha equal to 0 alpha n is equal to 2 n minus 1 pi by 2. So, these are the routes where the cosine alpha equal to 0. So, Eigenvalues are 2 n minus 1 pi by 2 and what is the corresponding Eigen functions were c 2 cosine 2 n minus 1 pi by 2 times x. These are the Eigen function or we can write it at c 2 cosine alpha n x where alpha n is equal to 2 n minus 1 pi by 2.

In the next example we will be taking up the boundary condition as a mixed boundary condition at x equal to 1 it is a Robin Mixed Boundary Condition and see how the solution evolves out d square y d x square plus lambda y is equal to 0 at x is equal to 0 we have y is equal to 0 and at x is equal to 1 we have a mixed boundary condition that is d y d x plus beta y is equal to 0 both boundary conditions are homogeneous.

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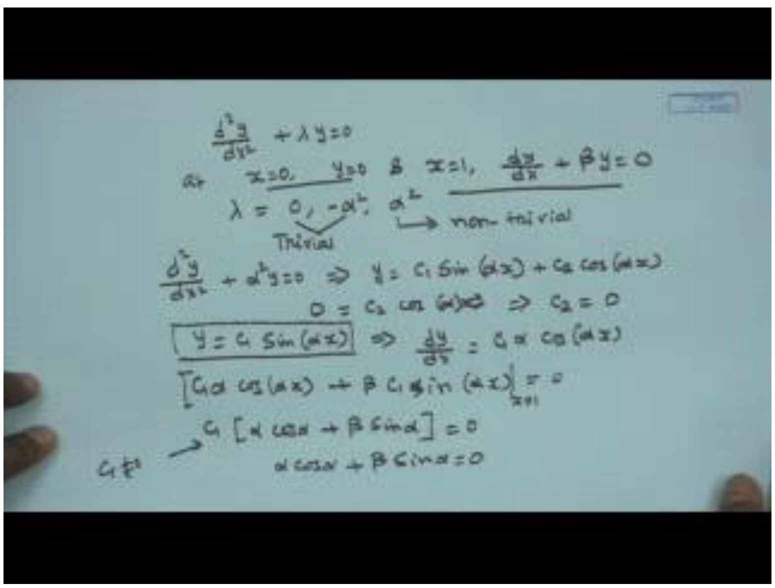


But in this case the boundary at x is equal to 1 is replaced by you know the Robin Mixed Boundary Condition. So, again you can have three cases of lambda it can be 0 it can be negative it can be positive as you have discussed earlier zero and negative value of lambda will be leading to a trivial solution we are not interested into a Trivial Solution we are interested in a Non-Trivial Solution.

Let us look into how this Non-Trivial Solution will arise in this case. So, d square y d x square minus alpha square y is equal to 0 plus alpha square y equal to 0 it is a positive. So, therefore, we know the solution y is equal to c 1 sin alpha x plus c 2 cosine alpha x and x is equal to 0 y equal to 0 let us apply the first boundary condition y equal to 0 at x equal to 0 this sin 0 is zero. So, this will be c two cosine alpha x cosine alpha. Now cosine alpha will be not equal to 0. So, therefore, c 2 will be equal to 0. So, you have the solution as c 1 sin alpha x now let us put the other boundary condition at x is equal to 1 d y d x plus beta y equal to 0. So, what is d y d x? D y d x will be nothing but c 1 alpha cosine alpha x.

So, at x is equal to 0 d y d x will be c 1 alpha cosine alpha x plus beta y beta times c 1 sin alpha x. So, this is the solution of y now beta times c 1 sin alpha x should be equal to 0 when the full thing is evaluated at x is equal to 0. Let us do that c 1 will be common alpha cosine alpha plus beta sin alpha is equal to 0. Now, if c 1 is equal to 0 the solution is y is equal to c 1 sin alpha x if c 1 is equal to 0 then again we will be landing up in a Trivial Solution that is, that has to be avoided. So, in order to do that for non zero c 1 the other term in the bracket will be equal to 0.

So, c 1 cannot be equal to 0. So, therefore, alpha cos alpha plus beta sin alpha should be equal to 0. So, what is that this becomes now.

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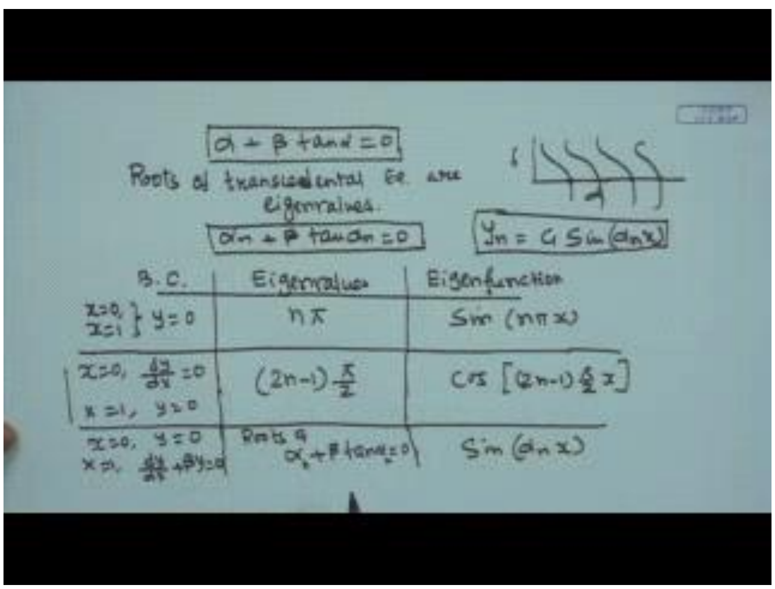


Alpha plus beta tan alpha is equal to 0. Now unlike the earlier two cases where we had a distinct expression we have an explicit expression of eigenvalues for example, in case of the in the first case we had n pi as the eigenvalues and a cosine n pi x at the Eigen functions where the distinct solution we have at the explicit solution of the eigenvalues the first case the eigenvalues had assume the values n pi in the second case in the case of when we had a Neumann boundary condition x equal to zero we had the 2 n minus 1 pi by two at the eigenvalues.

Now, in this case we are not getting an explicit expression for the eigenvalues, we will be getting a transcendental equation Roots of this transcendental equations will be the eigenvalues of this problem. If you really plot this function as a value of alpha this transcendental equation you will be having the solutions like this, there will be infinite number of solutions where they cut the x axis or alpha axis each of these alpha's will be the eigenvalues for this particular problem. So, Roots of this transcendental equation and are eigenvalues. So, we denote as them as alpha n plus beta tan alpha n equal to 0. So, how these roots will be evaluated these roots can be evaluated by using a Newton-Raphson Technique.

So, what are the corresponding eigen functions corresponding eigen functions will be c n corresponding to n'th eigenvalue is c 1 sin alpha n x where, alpha n are the roots of this eigenvalue problem. Now let us, look into the now let us summarize whatever we have done for a standard eigenvalue problem. So, let us put down the boundary conditions and then we put down the eigenvalues in the next column in the third column we write down the Eigen functions. If we have Dirichlet boundary condition at x equal to 0 and x is equal to 1, we have y equal to 0 then eigenvalues are n pi and Eigen functions are sin n pi x.

If the boundary condition at x equal to 0 d y d x is equal to 0; that means, a Neumann and at x equal to 1 we have a Dirichlet then 2 n minus 1 pi by 2 at the eigenvalues and cosine functions are the Eigen functions. In case of boundary condition at x equal to 0 is y equal to 0 and x equal to 1, we have a mixed boundary condition d y d x plus beta y equal to 0 the eigenvalues will be the Roots of transcendental equation of alpha plus beta tan alpha is equal to 0 alpha n basically n'th eigenvalue and the corresponding Eigen functions are sin alpha n x where alpha n's are the roots evaluated from this transcendental equation.

So, this table summarizes the various values and of various eigenvalues and Eigen functions how they altered into of a different types of boundary conditions. So, these are the standard values if you have let us say a boundary condition instead of x equal to 0 you will be having at x is equal to 0 y is equal to 0 and x is equal to 1 d y d x is equal to 0. So, in that case suppose these boundary conditions are reversed for example, the boundary at x is equal to 1 is insulated; that means, d y d x equal to 0 there and boundary at x is equal to 0 is 0 is homogeneous where is y equal to 0 at that boundary. So, in that case we will be having a change of coordinate by defining a new x star as 1 minus x. So, that again it will be of in the transformed coordinate system it will be the boundaries will be falling into the standard boundary conditions like this and we will be having the eigenvalues and Eigen functions in this form.

So, these are the various types of boundary conditions we will be encountering and the eigenvalues and Eigen functions will be changed accordingly. So, in the next we will be looking into the some forms of special functions which will quite you know frequent in various separation of variable type of solution of partial differential equations.

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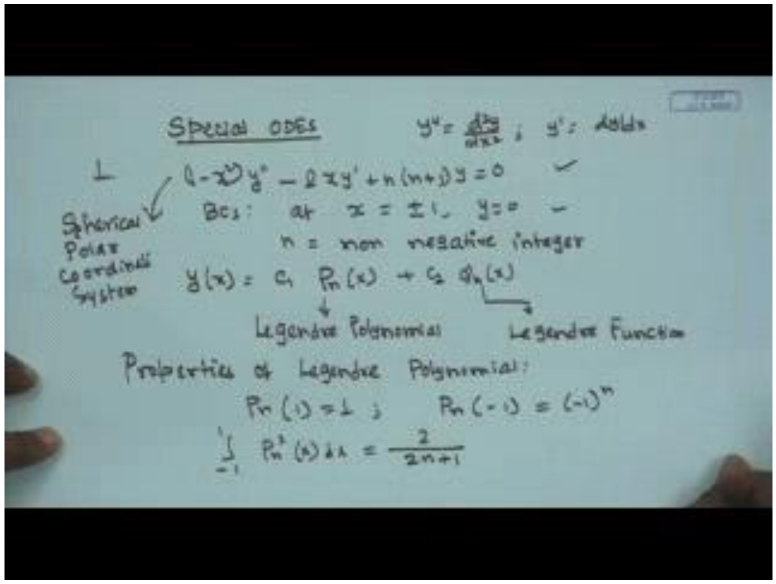


Now, let us look into the special ODE's some more special ODE's for example, the first one is the ODE's like 1 minus x square y double prime y double prime is basically d y d square y d x square and y prime is d y d x 1 minus x square y double prime minus 2 x y prime plus n into n plus 1 times y is equal to 0.

So, boundary conditions are at x equal to plus minus 1 y equal to 0 and n is non-negative integer. If there is the case if this is a type of equation we are going to have and this is the type of equation boundary conditions we are going to have then the solution is in the form of c 1 p n x plus c 2 q n x where p n is known as the Legendre Polynomial and q n is known as the Legendre Function. Now these type of equations one will be encountering whenever we are talking about the Spherical Polar Coordinate System. So, the previous one that we have discussed d square y d x square plus lambda square equal to 0 those will be one will be encountering whenever we are talking about a Cartesian coordinate System.

So, if you talk about the Spherical Polar Coordinate System you will be encountering this type of equations now the solution will be consisting of p n the Legendre Polynomial and Legendre Functions then let us look into some of the properties of Legendre Polynomial. These are p n at 1 will be is equal to 1, p n at minus 1 is equal to minus 1 raise to the power n minus 1 to plus 1 p n square x d x is equal to 2 divided by 2 n plus 1 p n the Legendre Polynomial it satisfies the Orthogonal Property.

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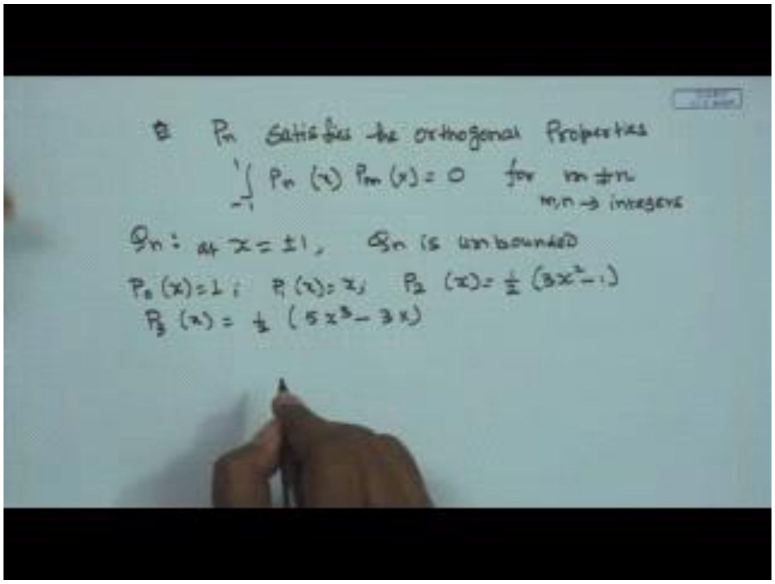


P n satisfies the Orthogonal Property that is minus 1 to plus 1 P n x and P m x will be equal to 0 for m not equal to n and m n are integers both m and n are integers, the property of Q n is such that at x equal to plus minus 1 Q n is unbounded.

So, what does a imply this implies if you look into the solution of this since the domain of x is varying from minus 1 to plus 1 and at the boundaries minus 1 to plus 1 Q n is unbounded this simply physically implies that the associated constant because we are going for a bounded solution if bounded solution is divided into two parts and one part goes to unbounded or you know at a particular boundary then the associated the whole solution becomes unbounded in that case. So, therefore, the associated constant must be equal to 0; that means, c 2 must be equal to 0 in order to have a realistic solution because at the boundaries the solution goes unbounded as I said earlier that the solution of the governing equation is valid within the control volume at every point of control volume as well as on the boundary.

If the boundaries the solution becomes unbounded that is not tenable or agreeable therefore, the associated constant has to be equal to 0 in that case. Then we will look into the several type of you know expressions of p 0 and others you know high rated Legendre Polynomials P zero x they may be of use P zero of x is equal to 1 P 1 of x is equal to x P 2 of x is equal to half 3 x square minus 1 P 3 of x is equal to 1 by 2 5 x cube minus 3 x. So, there are various types of you know one can go into any standard mathematics book and into the tables which will be appearing in the first year for example, (Refer Time: 20:38) there you can get into the chapters of Legendre Polynomials and can know about various other properties of the Legendre Polynomial and Legendre Function I have just doted down or noted some of the important properties of Legendre Polynomial Legendre Function will be encountering with.

Next we are going into the Cylindrical Coordinate System because in most of the Engineering Applications we will be talking about the Heat Transfer Mass Transfer Momentum Transfer or any transfer processes occurring in a tube or in a pipe. So, in that case the Cylindrical Polar Coordinate System becomes very very important and in the Cylindrical Polar Coordinate System we will be encountering the Bessel Equation and the corresponding Bessel Function's as a solution now let us look into the Bessel Equation.

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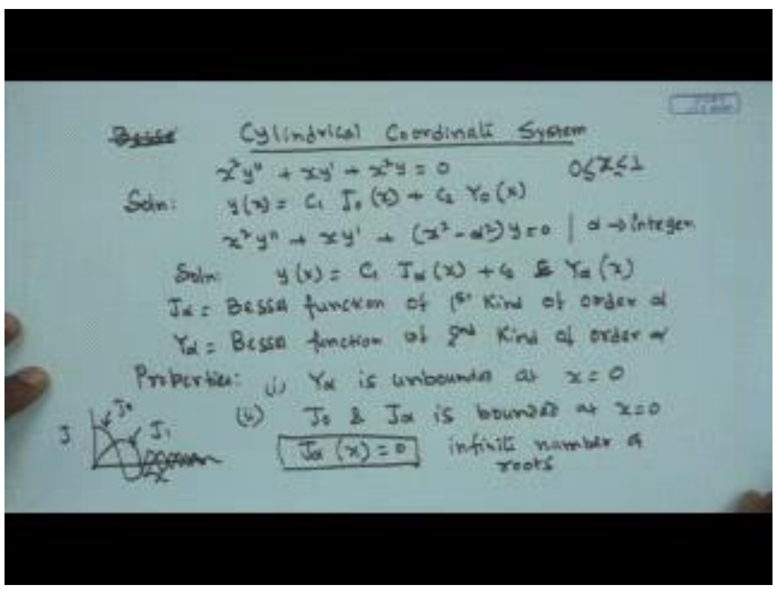


This will be basically for Cylindrical Coordinate System the differential equation that one will be encountering in this form x square y double prime plus x y prime plus x square y will be equal to 0 and if that is the case the solution will be in the form of C 1 J 0 x plus C 2 Y 0 x if a for differential equation if one encounter as x square y double prime plus x y prime plus x square minus alpha square times y where alpha is an integer.

So, therefore, solution is in the form of y of x is C 1 J alpha x plus C 2 J minus alpha or we call it y alpha x where alpha is an integer. So, J alpha is known as the Bessel Function of first kind of order alpha and y alpha is known as the Bessel Function of second kind of order alpha. So, if this is the form this is a we are talking about the zero'th order Bessel Function and zero'th order of first kind and second kind of Bessel Function now let us, look into the some of the properties of the Bessel Function those will be quite important the first property is that Y alpha is unbounded at x is equal to 0 and therefore, the whenever you will be coming across to these type of equation where x is basically varying from 0 to 1 in the domain 0 to 1. So, the associated constant must vanish.

So, in case of solution in the Cylindrical Polar Coordinate System second important property of Bessel Function is that J zero and J alpha J zero and in general J alpha is bounded particularly we having a finite value bounded at x is equal to 0 so; that means, if you plot a j zero or j alpha as a function of x this will be the plot of j zero and there will be plot of J 1; that means, in general J alpha will be oscillating about the x axis n number of times. So, therefore, whatever the every point J alpha whether its 0 in order of alpha 0 1 2 3 of anything they will be oscillating about the x axis and cutting the x axis infinite number of times and each and every cutting point is basically a Root of the transcendental equation or the eigenvalue of the system.

So, J alpha x will be having 0 and since it will be cutting infinite number of times. So, there will be infinite number of roots are obtained of this equation. So, each of this root is basically an eigenvalue to the system. So, next we will be just looking into one more type of a Bessel Equation.

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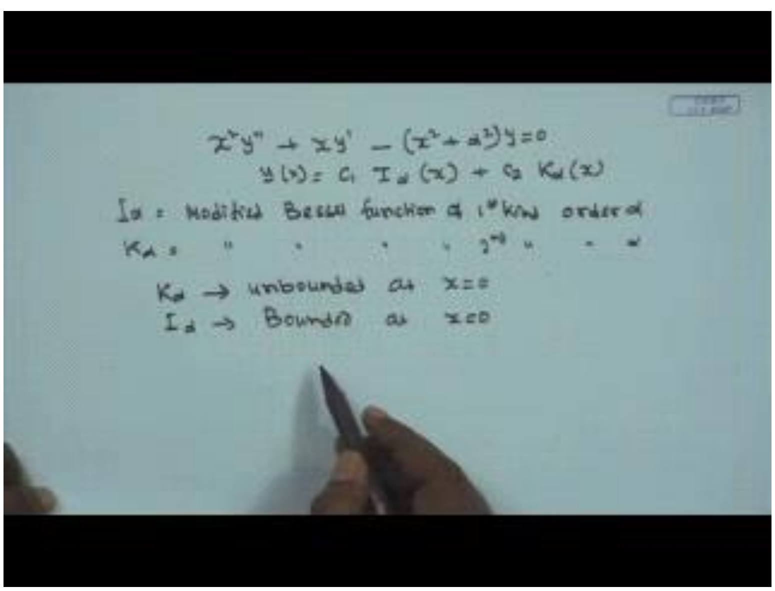

X square y double prime plus x y prime minus x square plus alpha square times I what times y equal to 0, the solution is y x is equal to c 1 I alpha x plus c 2 k alpha x then I alpha is known as modified Bessel Function of first kind order alpha and k alpha is modified Bessel Function of second kind of order alpha.

The important property is that K alpha is unbounded at x is equal to 0 and I alpha is bounded at x is equal to 0; that means, for a domain x lying between 0 and 1 the corresponding you know constant of k alpha will vanish and you will be getting the solution of y x is equal to c 1 I alpha x and I alpha again it cuts the x axis in n number of times or corresponding to n roots or the eigenvalues of the system the final special function that we will be talking about is Euler's Equation in some of the mathematical problems.

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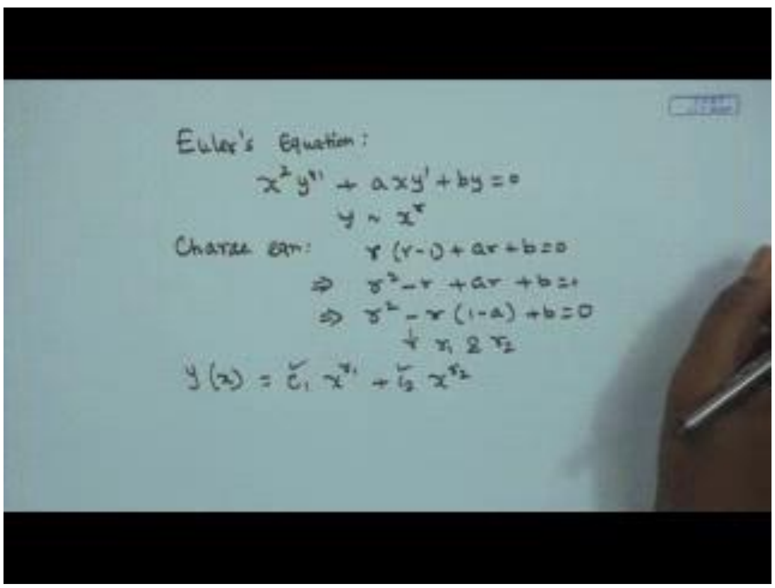


You will be encountering this type of equation if you have an equation in the form x square y double prime plus a x y prime plus b y is equal to 0 then the solution will be in the form of y should be in the order of x to the power r and you will be having a characteristic equation or just put x to the power r in the governing equation r r minus one plus a r plus b is equal to 0 and you will be getting a quadratic in r. So, r square minus r plus a r plus b is equal to 0. So, it will be r square minus r 1 minus a plus b is equal to 0. So, you will be having two equations r one and r 2 and you will be having the solution in the form of c 1 x to the power r 1 plus c 2 x to the power r 2.

So, again by putting the boundary conditions one can evaluate the constant c 1 and c 2. So, these are the various type of special functions those one will be encountering in solving the partial differential equations using the separation of variables and different types of eigenvalue problems will be appearing based on the Cartesian Coordinate System Cylindrical Coordinate System Spherical Polar Coordinate System and various kinds of you know boundary conditions one will be encountering the solution will differ and the Eigen functions and eigenvalues will differ.

So, I stopping in this class in the next class we will be looking into how to get the adjoint operator given an operator and how they are connecting to eigenvalue problems or problems and we will be looking into the various properties and characteristics of the problem before going into the separation of variable type of solution.

Thank you very much.