Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables

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Lecture – 03
Principle of Linear Superposition

In the last class, we have seen that how we have got, we can categorize the problems; get down the problems, into sub problems, by considering one homogeneity at a time. So, in fact, we have seen the how the sub problems can be divided, in case of a parabolic partial differential equation as a example that we have discussing in the last class. And we have found out that, by in the process of breaking down we have identified 3 sources of non homogeneity in the governing equations. And we have we broke down the problem into 3 sub problems, considering one on homogeneity time. That is resulted into one sub problem that will be, that was well posed. And 2 sub problems those are the ill posed problem.

So, all the sub problem, sub problems containing the non 0 or 1 on homogeneous initial condition, and the homogeneous boundary conditions are called the well posed problem. And the sub problems having the homogeneous initial condition and non homogeneous at least 1 on homogeneous boundary condition is known as the ill posed problem. Now in this class we will be looking into, how the ill posed problem converted into a well posed problem, and then we can directly go ahead with the separation of variable type of solution. So, let us consider 1 ill posed problem.

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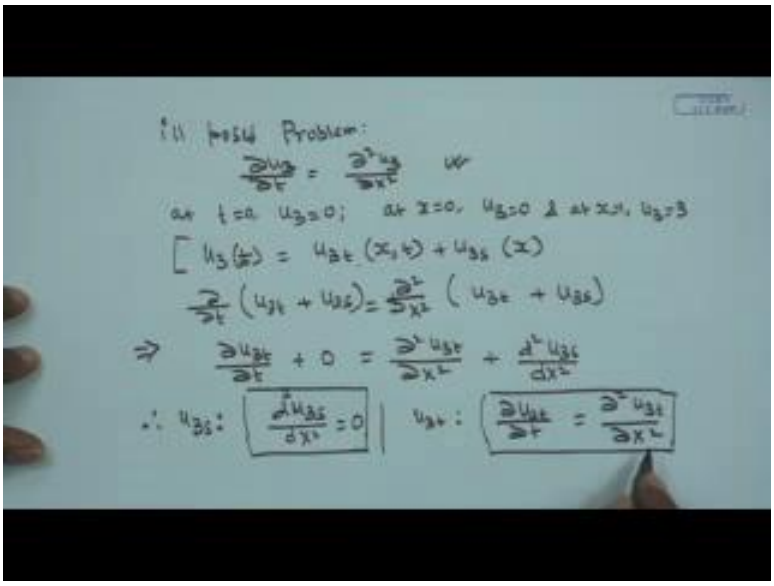

Sub problems that we have considered last time. So, del u 3 del t is equal to del square u 3 del x square. And at t is equal to 0, we had u 3 equal to 0, at x is equal to 0, we had u 3 equal to 0. and at x is equal to 1, we had u 3 is equal to 3. So, again this problem has to be divided into 2 sub problems, 1 is the, 1 is time dependent another is the time independent solution. So, therefore, u 3 will be divided into, u 3 t will be divided into the into 2 sub problems, 1 is u 3 t transient, which will be function of both x and t, and u 3 study state which will be function of x only.

So, if so, we divide this problem into 2, this sub problems into another 2 sub parts, 1 is the time independent part, which will be function of x only, another is the time dependent part, which will be function of x and t both. So, in this case, what will be doing, we will be we substitute, this solution into the governing equation. So, this is the mother problem for this particular 2 sub problems u 3 t u 3 s. So, how will be getting the governing equation of u 3 t and u 3 s, we will be substituting back, this equation into the mother problem.

So, if you really do that this becomes del del t of u 3 t plus u 3 s, plus del is equal to del square del x square and u 3 substitute u 3 t, plus u 3 s. now let us look into this what it is. So, del del t, this becomes del u 3 t del t, because u 3 is the function of x and t both, plus u 3 is which is a sole function x it is time independent part. So, it is a sole function of x. So, time derivative will be 0. So, this will be 0 is equal to del square u 3 t del x square, that term remain same because a u 3 t is the function of x and t both, plus u 3 is is a sole function of x, it is not a function of time. So, therefore, this partial derivative will be a total derivative in this case. So, it will be d square u 3 s d x square.

Now, next what we will be doing, we will be collecting the similar terms and formulate the governing equation of u 3 t and u 3 s. So, there is only u 3 t, u 3 s is coming. So, therefore, the governing equation of u 3 s, is d d square u 3 s, d x square it will be equal to 0. and we need to have 2 boundary conditions, and what is the governing equation of u 3 t. So, governing equation of u 3 t is, del u t, 3 del t, is equal to del square u 3 t del x square. So, this is the governing equation of u 3 s, and this is the governing equation of u 3 t.

Next what we will be doing, we will be setting up the boundary conditions and the initial conditions of u 3 s, and u 3 t. So, first we will be looking into the boundary conditions of u 3 s. what will be doing the boundary conditions of u 3 s, and u 3 t will be satisfying the original boundary condition of the mother problem. So, let us look into the original problem.

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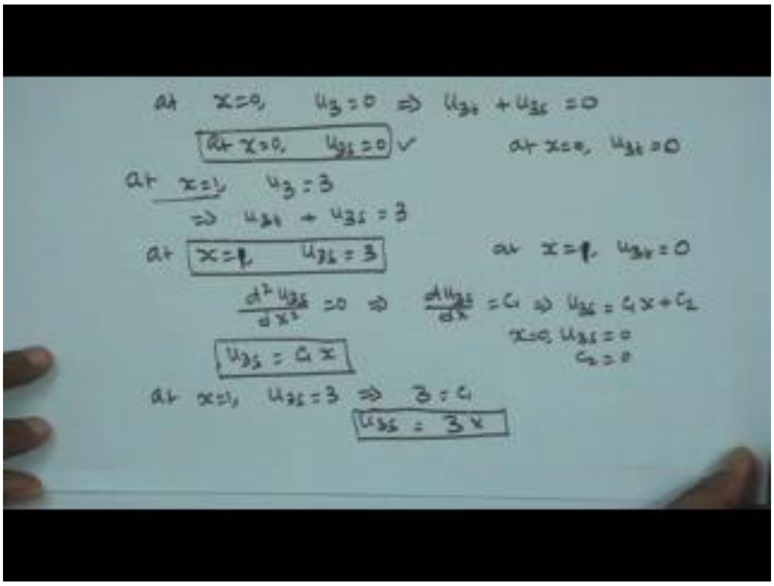

In the original problem, at x is equal to 0. Our u 3 is equal to 0. So, what is u 3, u 3 is nothing, but u 3 t plus u 3 s, is equal to 0; that means, at x is equal to 0, both of this quantities will be equal to 0 so; that means, x is equal to 0, I have u 3 s equal to 0, and at x is equal to 0, I have u 3 t is equal to 0.

So, that sets the boundary condition at x is equal to 0. then at x is equal to 1, in my original problem u 3 was 3, u 3 was 3; that means, u 3 t plus, u 3 s is equal to 3. Now what I will do, I will be judiciously selecting I will be associating the non homogeneous part, with the time independent solution. And if I do that, then my u 3 t will be 0, at the boundary at x equal to 0, x equal to 1.

So, what is our my aim, why we have doing this, our aim is to get a non homogeneous initial condition and homogeneous boundary conditions for a partial differential equation to make it well posed. In order to achieve that I will be judiciously associating the non homogeneous part of the boundary condition, to the time varying part only making or forcing the boundary condition of the transient part, or the partial differential equation part to be homogeneous. So, if I do that then I can write at x is equal to 0, my u 3 s equal to 3, at x is equal to 0, my u 3 t is equal to 0.

So, that makes that satisfies the boundary condition of the original problem as well. So, these 2 boundary conditions will be good enough, to solve the space varying part. So, therefore, what is the space varying part? If you look into the solution of governing equation of the space varying part d square u 3 s d x square is equal to 0. So, therefore, what is the solution of this? D u 3 s d x is a constant, and 1 more integration u 3 s, is equal to c 1 x plus c 2. So, if you put this to boundary condition that x is equal to 0, u 3 s is equal to 0, the first boundary condition x is equal to 0 u 3 s is equal to 0, that will give you c 2 equal to 0; that means, u 3 s should be is equal to c 1 x.

Now, we apply the second boundary condition, at x is equal to 1. So, this is x is equal to 1, this is also at x is equal to 1, because this is at x equal to 1. So, there is a mistake there. So, u 3 s is equal to c 1 x. So, at x is equal to 1 u 3 s is equal to 3. So, if you put that, then it becomes 3 is equal to c 1 and therefore, u 3 s is nothing, but 3 x this is the solution of u 3 s. So, u 3 is equal to 3 x. So, you have got the complete solution of the time, of the space varying part or the time independent part.

Now, with this we will be formulating the governing equation of the, initial condition of the time varying part or u 3 t. So, what is u 3 t?

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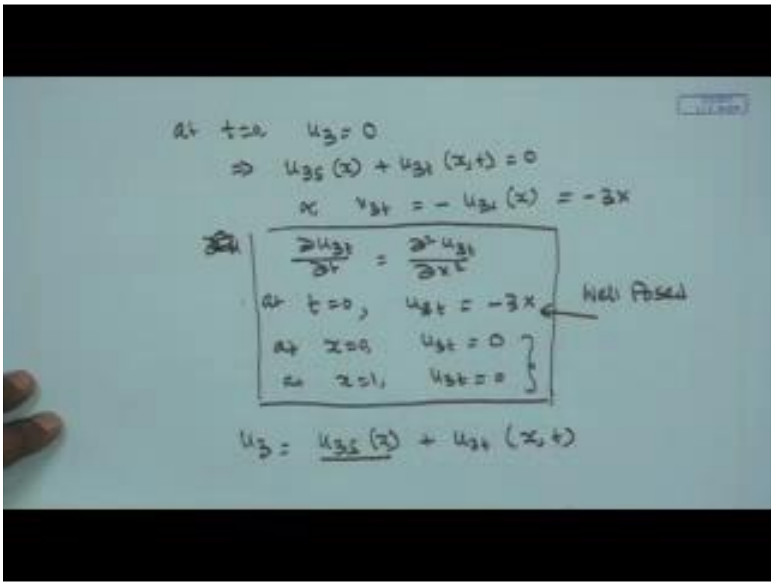

In the original problem at t is equal to 0, u 3 was equal to 0. So, therefore, u 3 is nothing, but u 3 s which will be a sole function of x, plus u 3 t, which will be a function of both x, and t must be equal to 0. So, we write u 3 t should be is equal to minus u 3 s, there is x. now what is u 3 s x that is the steady state solution. And we know the steady state solution. What is the steady state solution we just found out, that u 3 is equal to nothing, but 3 x. So, u 3 t will be nothing, but minus 3 x. So, that makes my boundary condition the initial condition to be non-homogeneous, and the problem is now well posed problem.

Now, we define the problem write down the problem completely. Del square, del u 3 t del t is equal to del square, u 3 t del x square at t, is equal to 0 my u 3 t, is equal to minus 3 x, at x is equal to 0 u 3 t is equal, to 0 and at x is equal to 1 u 3 t is equal to 0. So, this makes the problem completely well posed, with a non homogeneous or non 0 initial condition and homogeneous boundary conditions.

So, that is how an ill posed problem can be converted into a well posed problem, by further sub dividing in into 2 sub problem. 1 is the time dependent part another is the time independent part. So, once we do that ah, we can use the principle of linear super position. So, u 3 will be now the solution, u 3 will be a solution of u 3 s, which will be the function of x, plus u 3 t which will be function of x, and t both by solving this. We already saw this part this is nothing, but 3 x in our example. And u 3 t is the complete solution of this by using separation of variable, that will be look into later on, but before into going into going down to the separation of variable, we have to define the you know some special functions.

Now, we have to define the eigenvalue problem or Sturm-Liouville problem and then we will be looking into the several properties and theorems to satisfy this Sturm-Liouville problem or standard eigenvalue problem, will be satisfying. Then, we will be going to the separation of solution by separation of variable directly. So, what will be doing next is that, we will be looking into a special type of function.

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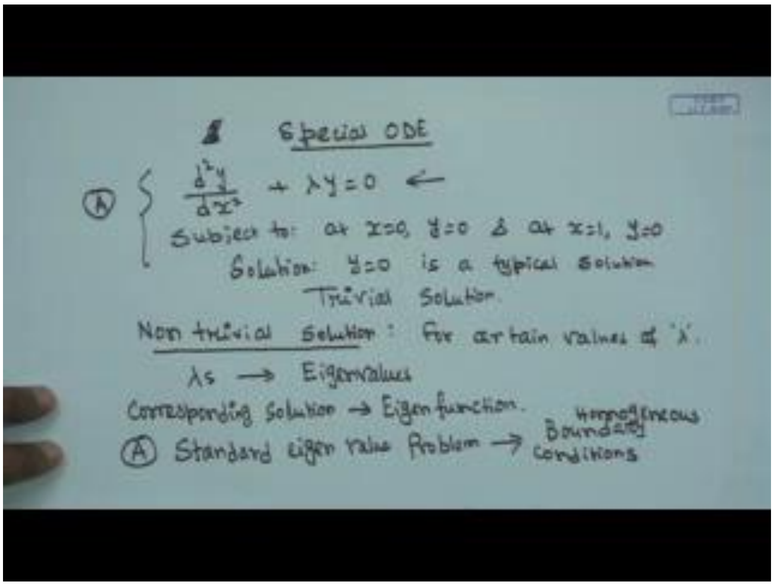

For example d square u, special ODE, first we will looking into a ODE in the form of d x square y d x square plus lambda y is equal to 0. Subject to at x is equal to 0, y is equal to 0 and at x is equal to 1, y is equal to 0.

So, both boundary conditions are homogeneous, in this particular type of ordinary differential equation. So, what is the solution of this ordinary differential equation, the solution is y equal to 0, is a solution. Solution must be satisfying the governing equation as well as the boundary condition. So, if y is equal to 0, that will be satisfying the governing equation as well as the boundary condition, but this is not a solution, this is known as a trivial solution. So, y equal to 0 is known as a trivial solution.

So, what is a next question? The next question is, is their existence of non trivial solution. So, if so, so there may be a certain value of lambda the parameter that is a constant parameter that will be appearing in the governing equations. There may be some kind of lambda, that will be existing for which the governing equation, becomes you know satisfied, that is called a non trivial solution. So, let us look into the non trivial solution. Non trivial solution can be obtained for certain values of lambda, the constant. These values of lambda are known are the eigenvalues of the problem, and the corresponding solutions are known as the Eigen functions, and whenever we will be we will be we will be talking about these type of problem, d square d x square plus lambda y lambda is a constant. And the boundary conditions are homogeneous this is a, these type of problem is known as a standard eigenvalue problem.

So, this lambda s is known as the eigenvalue. And the corresponding solution is known as the Eigen function. And whenever we talk about an equation like this, and the homogeneous boundary conditions, then this is known as a standard eigenvalue problem. So, this is is a standard eigenvalue problem, or a Sturm-Liouville problem this is a special type of Sturm-Liouville problem. We will be looking into the generalized Sturm-Liouville problem in the next class itself.

So, in a standard eigenvalue problem, the bounded, the necessity is that the form of the equation will be like this. And the boundary conditions have to be homogeneous. So, in a standard eigenvalue problem we should have the boundary conditions homogeneous. So, we should have homogeneous boundary conditions. So, next what we will be doing, we will be looking into the solution of this type of equation. And we will be identifying the eigenvalues and Eigen functions, and sees how these eigenvalues and Eigen functions, they change for different kinds of boundary conditions.

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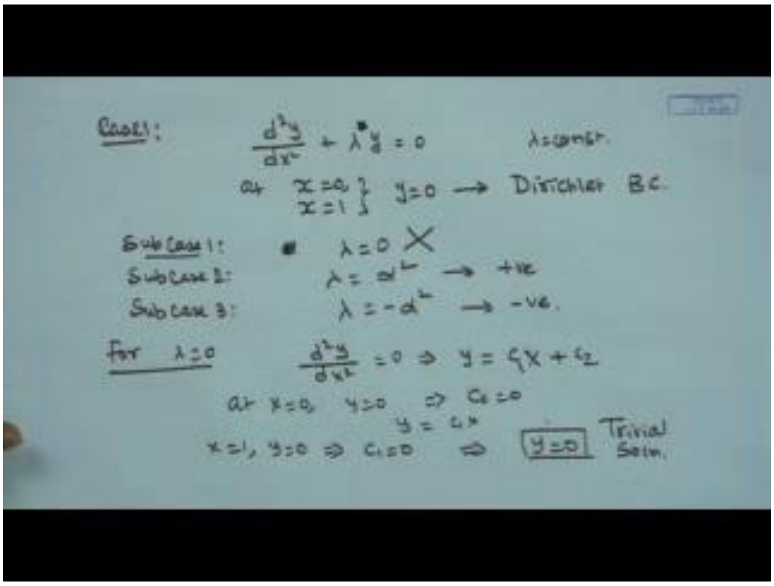

Now, first type of problem, we solve is a case 1, case 1, will be d square y d x square plus lambda square y is equal to 0. let say lambda; lambda y lambda is the constant. And at x is equal to 0, we have and x is equal to 1, both the boundaries we have y is equal to 0, this is a case of dirichlet boundary condition. Now the parameter lambda be will have 3 options, lambda can be 0, lambda can be positive, lambda can be negative.

So, we will be looking all the cases 1 by 1. So, sub case 1 lambda is equal to 0. So, let us look into in the sub case 2, will be lambda is positive lambda is equal to alpha square, and sub case 3, and will be lambda is negative. So, lambda is minus alpha square. So, lambda 0, positive, and this are negative. So, let us go by case by case, for lambda is equal to 0. Let us see what happens to a solution. Our solution is d square y d x square is equal to 0 in that case, d y d x. So, y is equal to c 1, x plus c 2.

So, now let us put the boundary condition at x is equal to 0, y is equal to 0. So, first boundary condition x is equal to 0, y is equal to 0, this gives you c 2 equal to 0. So, solution is y is equal to c 1, x at x is equal to 1, y is equal to 0 indicates c 1, is equal to 0. So, y is equal to 0. So, this gives a trivial solution. So, trivial solution is not accepted. So, the sub case is not tenable it is not agreeable. So, therefore, lambda cannot be equal to 0. So, sub case 1 is ruled out, and lambda cannot be equal to 0.

So, next let us look into the other one lambda is negative let say let us look into the sub case 3.

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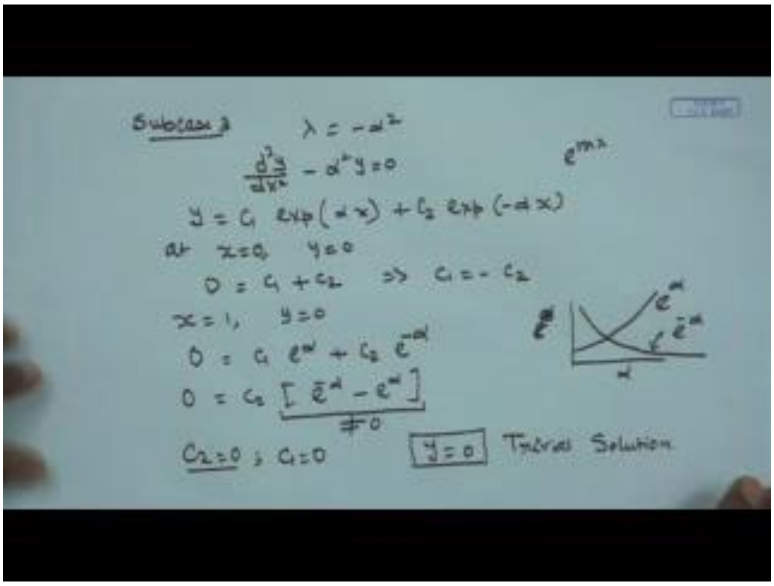

Lambda is equal to minus alpha square. So, it becomes d square y d x square, minus alpha square y is equal to 0. So, we know the solution will be in the form of m square, minus alpha square is equal to e to the power solution, will be in the form of e to the power m x, where m square minus alpha square is equal to 0. So, m is equal to plus minus alpha. So, the solution will be c 1, exponential alpha x, plus c 2 exponential minus alpha x.

Now, if you put the value boundary condition. Let us evaluate c 1 and c 2 at x is equal to 0, y is equal to 0. So, 0 is equal to c 1, e to the power 0. So, that will be 1 plus here also it will be 1. So, c 1 is equal to minus c 2. in the next case, what will we do we put the other boundary condition, at x is equal to 1, y is equal to 0. So, 0 is equal to c 1, e to the power alpha plus c 2, e to the power minus alpha. So, c 1 is equal to minus c 2. So, therefore, c 2 will be common, e to the power minus alpha, minus, e to the power plus alpha.

Now, e to the power, if you an alpha is a constant, and be a positive, it will be a constant. And if you know what is the variation, of e to the power alpha, for what is this alpha. So, alpha will be forever positive number. So, this will be e to the power alpha, what will be e, to the power minus alpha, this will be this will be e to the power alpha and e to the power minus alpha will be like this. So, that difference will be a finite value, and it will be never 0; that means, in this equation, this part can never be equal to 0.

So, what is the option we have left, the option is left, in order to satisfy this equation c 2 must be equal to 0. Now if c 2 is equal to 0 c 1 is minus c 2. So, that will also lead c 1 is equal to 0. So, if c 1 and c 2 both are equal to 0, then my solution is 0. So, it is basically again we are landing up with a trivial solution. So, trivial solution is not sort for. So, therefore, lambda cannot be negative as well. So, what is the, what is left now the option now left, is lambda has to be a positive value, or lambda has to be equal to alpha square.

So, let us look into the sub case 2.

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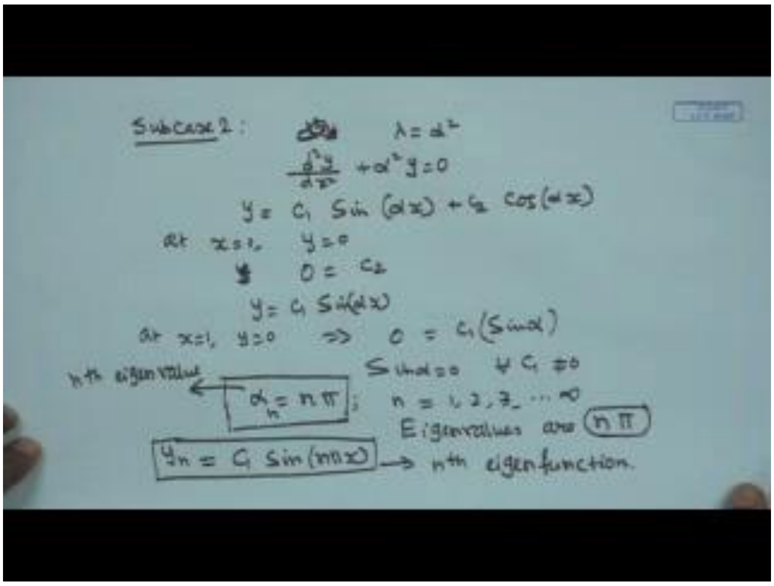

That d square y, lambda is equal to alpha square, and governing equation now becomes d square y d x square, plus alpha square y, is equal to 0. So, as you know, all of us know about it that this particular type, of ordinary differential equation, will be having the solution in terms of the, sin and cosine functions. So, y will be is equal to c 1, sin alpha x, plus, c 2, cosine alpha x.

Now, let us put down the boundary conditions, and see and evaluates the constants c 1, and c 2, and let us look into the what happens to the boundary the solution of y. at x is equal to 0, my y is equal to 0. So, if you put that 0, is equal to c 1, sin 0 is 0. So, that will be gone. So, c 2 cause 0 is 1. So, c 2 equal to 0. So, my solution is y is equal to c 1 sin alpha x. now at x is equal to 1, put the other boundary. At x is equal to 1, my y is equal to 0. So, therefore, 0 is equal to c 1 sin alpha. Now in this particular problem if there are 2 options, c 1 is 0 or sin alpha is equal to 0. if c 1 is equal to 0 then again will be getting a trivial solution that we are not looking for. So, the other option is for not equal to c 1, if we in order to survive the solution, the c 1 has to be non equal non 0 therefore, sin alpha to be equal to 0. So, from this sin alpha should be equal to 0, for c 1 not equal to 0.

So, what is the solution of sin alpha equal to 0, alpha is equal to n pi, where the index n should runs from 1, 2, 3 up to infinity. In the case n is equal to 0, should be avoided because that will be leading to the first sub case, and will be getting the trivial solution. So, therefore, we write alpha n is equal to n pi, a subscript n to the alpha, in order to find out, order to indicate that, we are talking about the nth root. So, in this equation the root is n pi, and these are the eigenvalues of the system. So, eigenvalues are n pi, and alpha n is known as n th eigenvalue, nth eigenvalue.

So, what is the nth Eigen function, y n? So, we substitute put a substitute n in order to indicate this is the n th Eigen function, corresponding to n th eigenvalue. So, y n will be nothing, but c 1 sin n pi x. So, this is the n th Eigen function. So, so we have seen the in this particular problem, that if we have dirichlet boundary conditions, on the both boundaries, then for alpha is equal to n pi, we will be having the eigenvalues, and for corresponding Eigen function, will be sin n pi x. if we have this alpha n is equal to n pi, then only you will be having a non trivial solution for this particular system.

Next what we will be doing, we will be formulating the problem by changing the boundary conditions into Neumann, and into the robin mixed, and see how the eigenvalues and the Eigen functions will involve.

So, I will stop in this class. In the next class what I will be doing, I will be taking up this problem again, with by changing the boundary conditions into Neumann. And then I will be taking up a problem another problem, with by changing the boundary condition into robin mixed. And see how the Eigen functions and eigenvalues will be taking up the shape, and 1 will be getting the various types of Eigen solutions, for such a well posed, Eigen standard eigenvalue problem.

Thank you very much.