Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables

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Lecture – 20
Examples of Application Oriented Problems (Contd.)

Welcome to the session. Now we will be solving the problem, that it generalized 3 dimensional problems that we are solving we have taken up in the last class will be completing it. We have already completed the sub problem theta 1. In the sub problem theta 2, we have completely solved the study state part which is nothing, but an elliptical partial differential equation.

Now, let us look into the time varying part.

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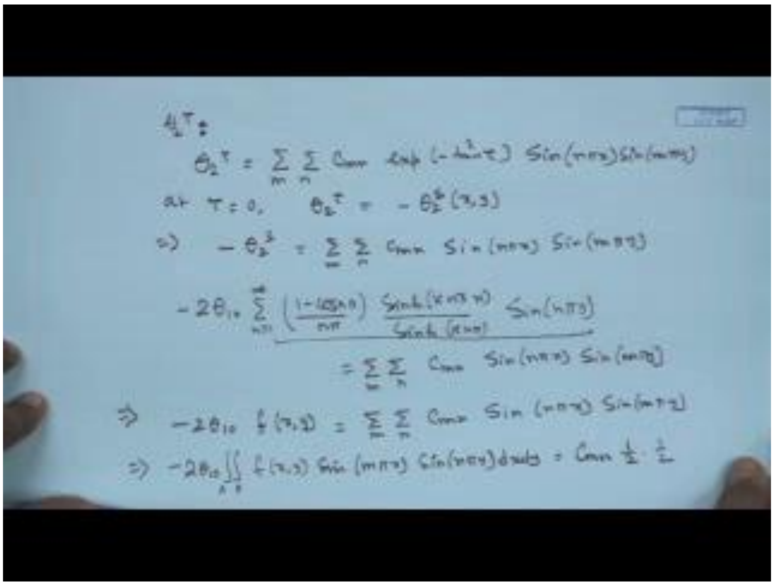

If you look into the solution of the time varying part, it will be having a non 0 initial condition and the rest boundary conditions are all homogenous and dirichlet. So, therefore, we already saw this type of problem in the first sub problem, and I will be directly writing the solution of that. So, theta 2 tau will be nothing, but summation of m, summation over n c m n exponential minus lambda m n square tau, Sin n pi x Sin m pi y, and at tau is equal to 0 we have the initial condition theta tau is nothing, but minus theta 2 s, which will be the solution of the steady state part.

So, we put it there. So, these becomes minus theta 2 s, is equal to summation over m, summation over n c m, n Sin m pi x, Sin m pi y. And we will be evaluating the constants c m n is by using orthogonal property of the Sin function and the Eigen functions we have done already before, that what is the solution of theta 2 s, theta 2 s is minus, 2 theta 1 naught, n is equal to 1 infinity, 1 minus cosine n pi, divided by n pi, Sin hyperbolic, k n pi x, divided by Sin hyperbolic k n pi, Sin n pi y.

So, that was the solution of the steady state part. That should be equal to double summation 1 over m, another over n, c m n Sin m pi x, and Sin n pi y. So, we multiplied by both side by Sin m pi x, and Sin m pi y dx dy, integrate after opening the summation series only 1 term will survive in the right hand side, and that will be, and other will be lost, will be gone. So, it will be minus 2 theta 1 naught, some function of x and y. So, I write this as some function of x and y is equal to, summation m n c m n, Sin m pi x, Sin m pi y.

So, this will be nothing, but minus 2 theta 1 naught, double integral over x, over y, f of x y, Sin m pi x, Sin n pi y, dx dy, and the right hand side we will be getting, c m n there will be 1 integral Sin square pi x will be half, integrals you know integral sign m Sin square pi y d y will be half. So, there will be half-half there. So, if we do that then, we will be getting an expression of, c m n and c m n will be, minus 8 theta 1 naught, from 0 to 1 integral from y is equal to, from 0 to 1 f of x y, Sin m pi x, Sin n pi y dx dy, and where the index m was associated with y, and index n was associated with x, in this particular problem. So, we convert them back to n and m, our normal variable.

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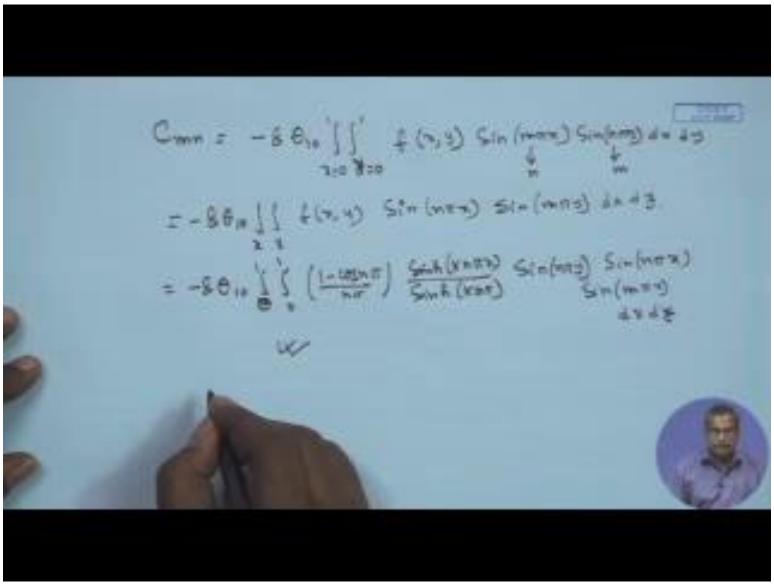

So, these becomes 8, theta 1 naught, x y f of x y, Sin m pi x, Sin n pi y, dx dy, this will be minus 8 theta 1 naught 0 to 1, 0 to 1. Now let us put f of x y f of x y is nothing, but 1 minus cosine n pi divided by n pi, Sin hyperbolic k n pi x, Sin hyperbolic, k n pi Sin n pi y, Sin n pi x Sin m pi y, dx dy. So, we will be getting. So, this will be giving you know as some constant some value.

So, you will be we can obtain the c m n quite accurately. So, once we will be getting the estimating the c m n, similarly the other sub problems, which will be having a 0 initial condition and non 0 boundary conditions, they can be converted into a steady state part, and it will be divided into steady state part and a transient part and that will be we will be you will, be able to solve them.

So, once we do that then let us so, that completes the problem that we are dealing with now let us go to the, other problem. Now I will be taking a typical problem that we have not talked about earlier. So, this is a 2 dimensional problem.

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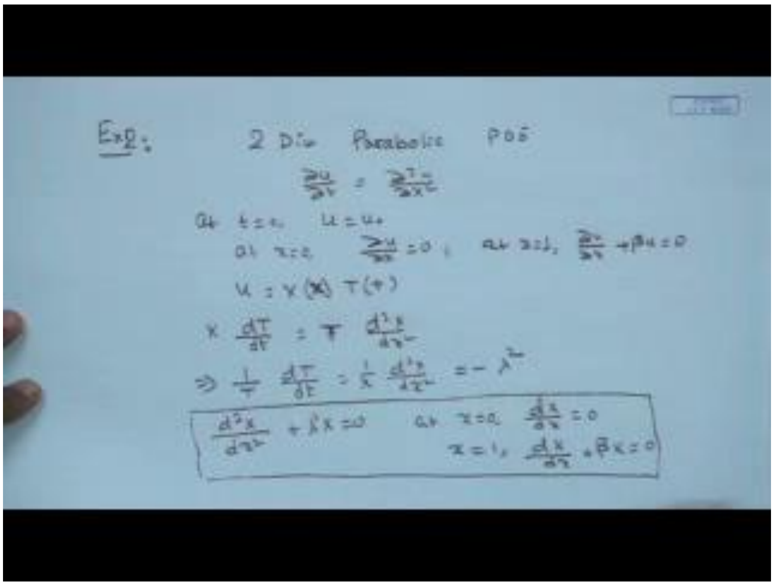

So, we take up another example, let say a 2 dimensional problem, a parabolic one, 2 dimensional parabolic partial differential equation like del u del t, it will be is equal to del square u del x square, and at t is equal to 0 we have let say, u is equal to u naught, and the boundary at x is equal to 0, it is insulated. So, it will be del u by del x will be equal to 0 and boundary at x is equal to 1, we have a mixed boundary condition del u del x, plus beta u is equal to 0.

So, if we remember, that, in the previous we have looked into the problem, where the both x is equal to 0 and x is equal to 1 we have a Dirichlet boundary condition. That is the problem of first kind problem of the second kind, was at x is equal to 0 del u del is equal to 0, and x is equal to 1 u is equal to 0 and problem of third kind was at del was at x is equal to 0 u was equal to 0, and del u del x plus beta u will be 0, x is equal to 1. So, it is a mixed boundary condition that is prevailing at x is equal to 0.

Now, this particular problem we have a Neumann boundary condition at x is equal to 0 and they have mixed boundary condition at x is equal to 1. So, that we have not attempted this problem earlier. So, let us look into the solution of this particular problem. So, again we will be going ahead with a separation of variable type of solution, u is a function of space and it is a sole function of time. So, if you put it there. So, these becomes x dt dt, and x and t d square x d x square. Now if you derived by x t and separate out the variable this becomes 1 over t dt dt, is equal to 1 over x d square x d x square. The left and side of function time and the right hand side is a function of space. They are equal and they will be equal to some constant, in order to have a non trivial solution.

Now, let us constitute the eigenvalue problem in the x direction, the eigenvalue problem solution is d square x d x square, plus lambda square x is equal to 0. And at z is equal to 0 dx dx is equal to 0, and at x is equal to 1, we have dx dx plus beta x is equal to 0. So, these formulate the standard eigenvalue value problem in the x direction, and let us look into the solution of this. Now as you all of us know the solution of x varying part will be, constituted by combination of Sin and cosine function.

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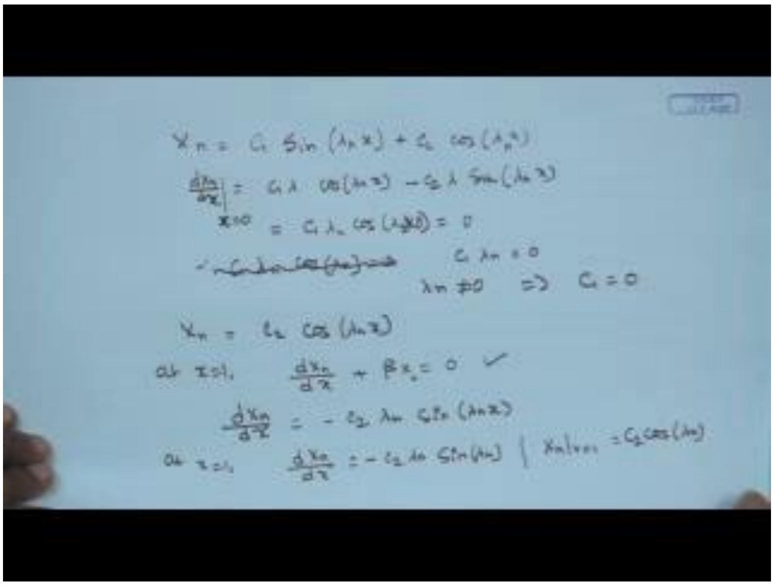

So, x n is equal to, c 1 Sin lambda n x, plus c 2 cosine lambda n x.

So, 8 correspond to nth eigenvalue, we will be getting this solution. So, now, dx n, dx will be nothing, but c 1 lambda cosine lambda n x, minus c 2 lambda Sin lambda n x and dx n dx, evaluated at x is equal to 0 Sin 0 is 0. So, the term that will survive this c 1 lambda cosine lambda, that will be equal to 0 at this boundary, therefore, lambda n c, 1 lambda n, cosine lambda n will be equal to 0.

So, in order to have, cosine at x is equal to 0, cos 0 is 1. So, this will be multiplied by 0. So, you will be having cos 0. So, c 1 lambda n will be equal to 0. So, for a non trivial solution lambda n cannot be equal to 0, if it is equal to 0, then we will be landing up with a trivial solution. So, the solution is c 1 is equal to 0, if c 1 is equal to 0 let say what is the solution of this, solution is c 2 cosine lambda n x.

Now, these are the Eigen function of this particular problem. Now let us look into the eigenvalues. How the eigenvalues are obtained. The eigenvalues are obtained by the boundary condition, we will be putting at x is equal to 1. So, at x is equal to 1, it should satisfy the original boundary condition of the original problem. So, it will be d dx dx plus beta x is equal to 0 if you look into the nth eigenvalue. So, there will be dx n dx plus beta x n value equal to 0 or x n equal to, this now let us say what dx n dx dx n dx will be nothing, but minus c 2 lambda n Sin lambda n x, and at x is equal to 1, this becomes dx n d x is equal to minus c 2 lambda n Sin lambda n and x n at x is equal to 1 becomes c 2 cosine lambda n.

So, then we will be substituting this into the boundary condition at x is equal to 1, and let us see what we get. What we will be getting is that minus c 2 lambda n Sin lambda n, plus beta c 2 cosine lambda n is equal to 0.

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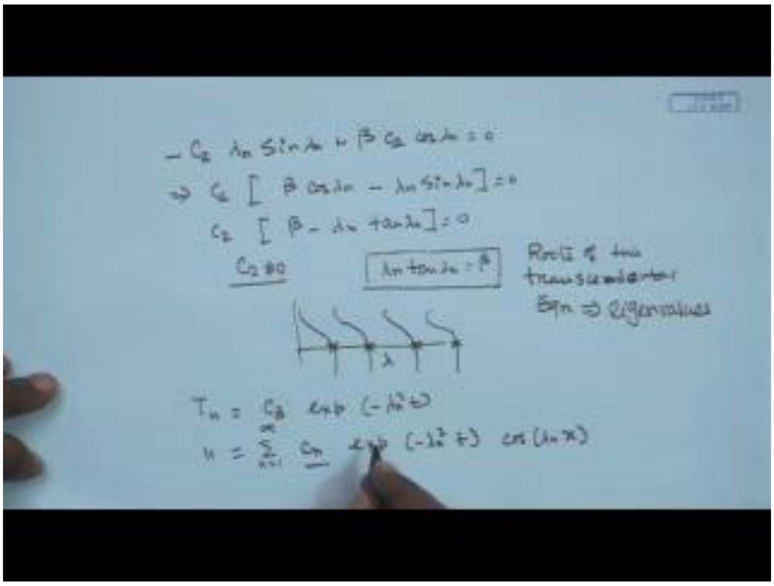

So, therefore, c 2 is in common. So, beta cosine lambda n minus lambda n Sin lambda n, will be equal to 0. So, c 2 is equal to beta, minus lambda n, tan lambda n will be equal to 0. So, in order to have a no trivial solution, c 2 must not be equal to 0. So, the only option is, lambda n tan lambda n is equal to beta. So, lambda n is nothing, but the eigenvalues of this transcendental equation, roots of this transcendental equation, equations are the eigenvalues.

Now, this lambda n tan lambda n equal to beta; it will be intersecting the lambda n axis at n number of locations. So, if you really plot it. So, this will be, there will be it will be cutting; the x axis at infinite point, and each of the intersection point is a root of this particular equation. This can be also solved numerically by using a Newton and Raphson method. So, typically this type of equation, the roots are appearing in an arithmetic progression of common difference three. So, 1 can take request to Newton and Raphson and having an initial guess, and then once that will be Newton-Raphson for a particular guess, is conversed then 1 can put a outer loop, where the initial guess will given an increment, plus 3 the outer loop will be completed.

So, if you calculate this 4 times, one will be automatically landing up with 4 roots of this governing equation. So, by bring by taking a request to a numerical method one can really obtain the you know first 5 roots of first ten roots of this particular transcendental equation. So, once we get that then we will be formulating the time varying part, t n is equal to c 3, exponential minus lambda n square t, and u will be as a function of x and t will be nothing, but n is equal to 1 to infinity, c n exponential minus lambda n square t, cosine lambda n x ok.

Now, the constant c n has to be evaluated from the initial, non 0 initial condition that at t is equal to 0, u is equal to u naught. So, let us go ahead with that if you really do it.

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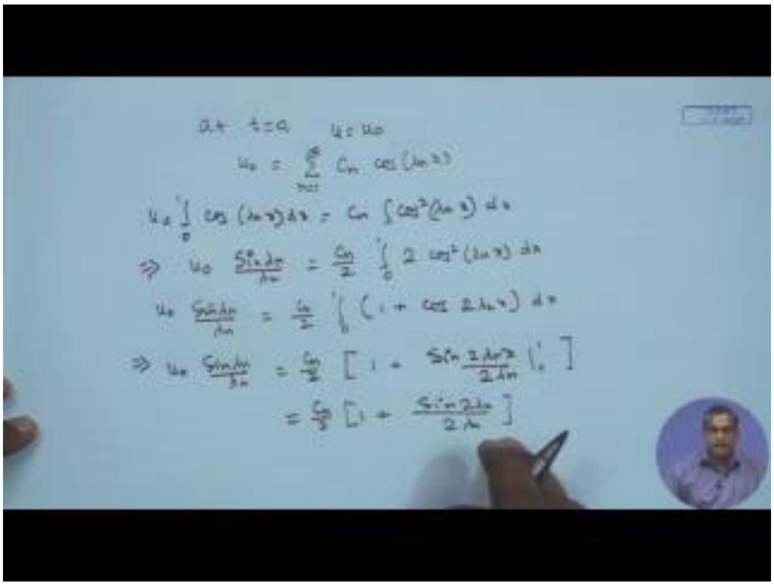

Then you will be getting at t is equal to 0, u is equal to u naught, and therefore, u naught will be is equal to summation cn, cosine lambda n x where n is equal to from 1 to infinity. So, you utilize the orthogonal property of the Eigen functions. So, u 0 zero to 1 cosine lambda n x d x is equal to c n cos square lambda n x d x. So, left hand side will be getting u, 0 Sin lambda n divided by lambda n, and the right hand side we are having c n by 2, 0 to 1, 2 cos square lambda n x d x.

So, we can further simplified it Sin lambda n by lambda, n c n by 2, this can be written as 0 to 1, 1 plus cosine 2 lambda n x d x. If that is the case then we can integrate it out on the right hand side, and what will be getting is c n by 2 is equal to 1 plus Sin 2 lambda n x divided by 2 lambda n 0 to 1. So, we will be getting c n by 2, 1 plus Sin 2 lambda n, divided by 2 lambda n. Now we can put Sin 2 lambda n in terms of we can express into Sin lambda n into tan lambda n, then we have the equation lambda n tan lambda n equal to beta, and we can substitute that and will be getting a simplified version of the coefficient c n. If you do that we will be getting u 0 Sin lambda n divided by lambda n, is equal to c n by 2, 1 plus 1 over 2 lambda n is equal to into 2 tan lambda n we are substituting cos lambda Sin 2 lambda n, by tan tan lambda.

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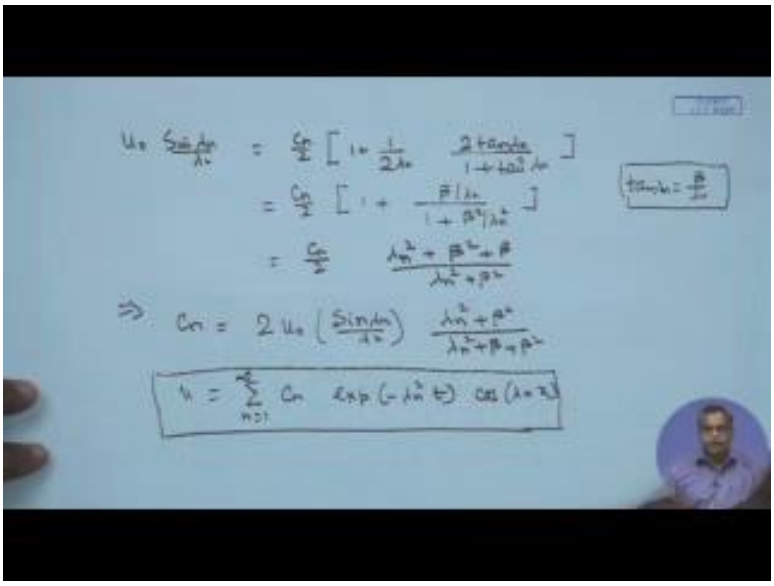

So, 1 plus tan square lambda n. So, this is will be c n by 2, 1 plus 2 will be canceling out. So, it will be beta tan lambda n will be, beta by lambda n that is the transcendental equation the eigenvalues must satisfied. So, beta by lambda n divided by 1 plus beta square by lambda n square. So, this becomes c n over 2 lambda n square plus beta square plus beta divided by lambda n square, plus beta square and we can get the expression of c n as 2 u naught Sin lambda n divided by lambda n, multiplied by lambda n square plus beta square divided by lambda n square, plus beta plus beta square.

So, this the constant and the complete solution is u is equal to summation n is equal to 1 to infinity, c n exponential minus lambda n square, t cosine lambda n x where n can transform 1 to infinity. So, that solves this problem completely now let us looking into the some more problems. So, this completely solves this problem. It is we can call it a fourth kind of boundary condition where the boundary at x is equal to 0 is Neumann, and boundary at 1 is a robin mixed.

Next, while we are taking up one more example in order to demonstrate the various types of solutions using the separation of variable. The problems that I will be talking about it will again a practical problem. Where we will be having you know robin mixed boundary condition.

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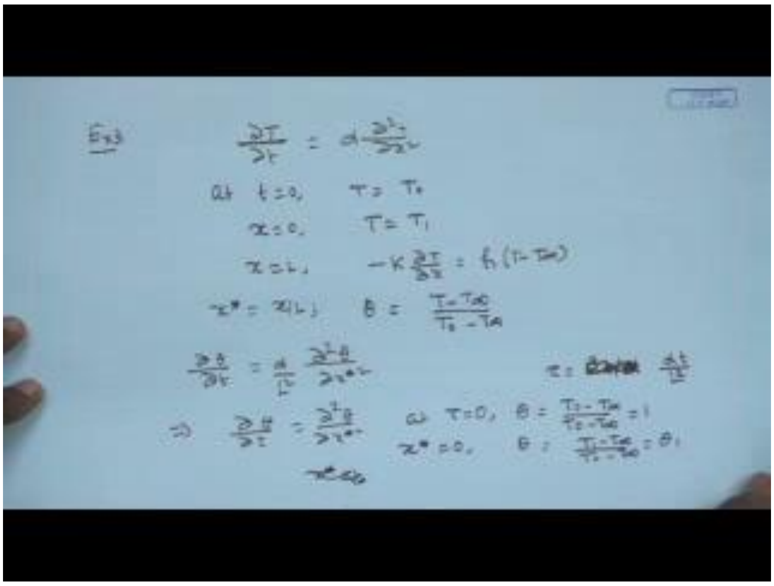

It is a 1 dimensional heat conduction transient problem this will be alpha. So, this will be another example 3 let us say alpha del square t del x square, at t is equal to 0 we have t is equal to t naught at x is equal to 0, t is equal to t 1, and at x is equal to l, we have robin mixed boundary condition k del t del x is equal to h t minus infinity; that means, at this boundary it is opened the atmosphere.

Whatever the amount of heat that has been transported here by conduction, that will be taken by convection where h is the heat transfer coefficient and t infinity is the, you know the ambient temperature. Now we make this equation 1 dimensional, such that 1 of the non homogeneity will go. So, x star will be x by l, and theta the non dimensional at temperature will be defined t minus t infinity divided by t 0 minus t infinity. So, let us put it in the governing equation. So, this becomes del theta, del t is equal to alpha, del square theta, divided by l square, del x star square.

So, this will be nothing, but del theta del tau, is equal to delta square theta del x star square, where tau is equal to l square t over alpha alpha t over l square. So, tau is equal to nothing, but. So, this will be l square will be on the other side. So, it will be alpha t over l square is the non dimensional time. Now let us put in the boundary condition at tau is equal to 0 means at, t is equal to 0 means at tau is equal to 0 theta is equal to theta naught, minus t naught minus t infinity, divided by t naught by t infinity, that will be equal to 1. At x star is equal to 0 theta is equal to t minus t t is equal to t 1. So, t 1 minus t infinity, divided by t 0 minus t infinity, so that will be theta 1, and at x star is equal to 1 we have a mixed boundary condition.

So, I think we will discuss it in detail.

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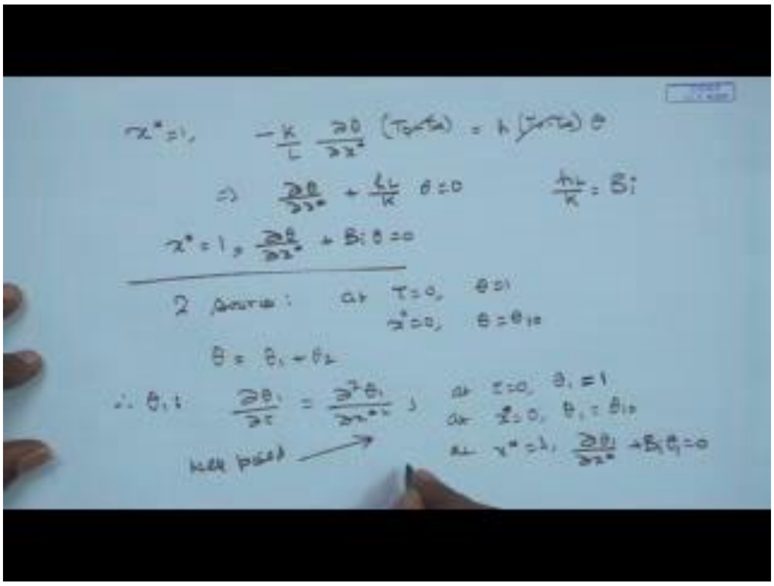

So, at x star is equal to 1 we have minus k by l, this will be del x star, and this will be del theta and that will be multiplied by t naught minus t infinity. So, this will be equal to h t minus t infinity. So, this will be. So, what is the definition of the theta, theta is equal to t minus t infinity, by t naught minus t infinity. So, it will be t naught minus t infinity times theta. So, this will be canceling out.

So, what will be getting is, del theta del x star, plus h l over k, times theta and what is h l by k, this called the biot number. It is a non dimensional number. It is a biot number. It is a ratio of heat transfer coefficient divided by thermal conductivity. It is ratio of the thermal resistance by diffusion, divided by thermal resistance by convection. So, we will be having del theta del x star plus, b i theta is equal to 0, at x star is equal to 1. So, in this particular problem, we have 2 sources of non homogeneity. So, the non homogeneity 2 sources of homogeneity.

So, let us look into what are the 2 sources. So, 1 is at initial condition at tau is equal to 0, theta is equal to 1, another is at boundary condition at x is equal to 0, theta is equal to theta 1. So, I divide this, we call this as theta 1, naught in order to make it consistent. So, I divide this problem into 2 problem 1 is theta 1 another is theta 2, considering 1 non homogeneity at a time as we have done earlier. So, therefore, theta 1 sub problem 1, will be basically del theta 1, del tau, is equal to del square theta 1 del x star square, at tau is equal to 0, theta 1 is equal to, you will put it, put the boundary, you, keep the non homogeneity intact, and force the other to be vanished at tau is equal to 1, and theta is equal to 1, and at x star is equal to 0, theta 1 is equal to theta 1 naught, and at x star is equal to 1.

We have the mixed boundary condition; del theta 1 del x star, plus bi theta 1 is equal to 0. Now this particular sub problem these are well posed problem well posed partial parabolic partial differential equation, and we have already solved in great detail how to solve this sub problem earlier. Now let us look into the other sub problem, the other sub problem is for theta 2.

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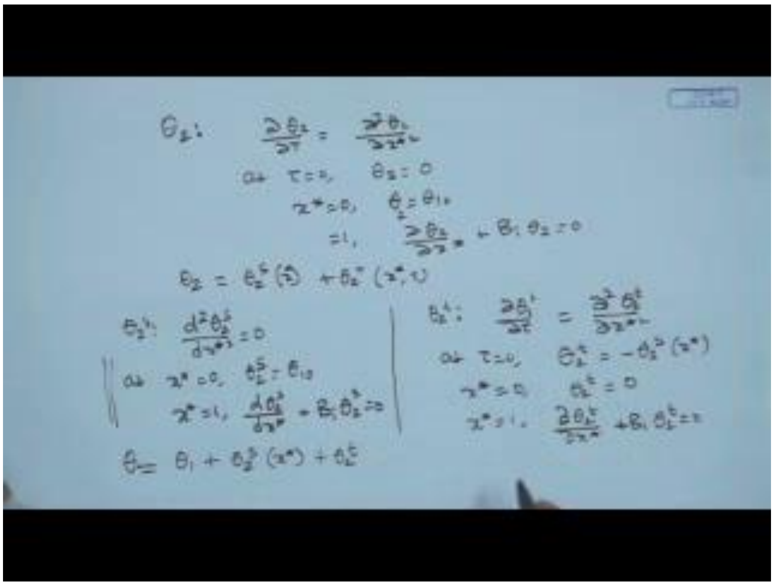

So, it will be del theta 2 del tau is equal to del square theta 2 del x star square, and at tau is equal to 0, we keep this initial condition to be homogeneous, and substitute the other 1, x star equal to 0, theta is equal to theta 1 naught, and x star is equal to 1 we have the robin mixed boundary condition, del theta 2, del x star plus bi theta 2 is equal to 0.

Now, again since we have a 0 initial condition, this problem as to be divided into 2 sub problems, one is the time dependent part another is the time independent part. So, we can constitute the governing equation of theta 2 s where s is time dependent part d square theta 2 s d x star square will be equal to 0, at x star is equal to 0, we have theta 2 s is equal to theta 1 naught. We associate the non homogeneous part in the steady state solution, and x star is equal to 1 d theta 2 s, d x star, plus bi theta 2 s, is equal to 0.

Similarly we formulate the other part the transient parts, it will be del theta 2 t del tau, is equal to del square, theta 2 t del x star square, at tau is equal to 0 theta t 2, will be nothing, but minus theta 2 s, x star it is a steady state solution, solution of the steady state part, and at x star is equal to 0, since we have associated the non homogeneous governed boundary condition, with the steady state solution. So, theta 2 t will be become homogeneous, and at x star 1 equal to 1, this becomes del theta 2 t, del x star plus bi theta 2 t will be equal to 0.

Now, again this well posed problem, of well posed of third kind. Where the Eigen functions will be the Sin functions and eigenvalue will be coming from the transcendental equation that we have already looked before. And this will be a straight forward part, we will be getting the straight forward solution, of the steady state part, and the solution of the steady state part will be substituted here, and again it is a well posed problem and we know the solution of this. So, the complete solution will be obtained as theta is equal, to theta 1 plus theta 2 s, which will be a function, of x star only plus theta 2 t, which will be the solution of this part.

So, we will be constituting this we will be construct the, we will able to construct the complete solution of this problem. So, I am also almost come to the end of my, end of our course. So, in this course we have we have learnt about the solution of partial differential equations, for the engineering problems. And we have already looked into the in detailed the, how to form the what are the classifications of the partial differential equations, they are you know how to define how to identify and define various boundary conditions, various partial differential equations, what is the principle of a linear super position.

We have developed various theorems related to the standard eigenvalue problem, at joint operator and the properties of the standard eigenvalue problem, which will be having a infinite number of eigenvalues and the Eigen functions of orthogonal to each other various properties we have looked into, then we have looked into the solution of you know separation of, using separation of variable method. For the rectangular coordinate, we have defined different kinds of sub problems, depending on the boundary conditions.

For example, first kind, second kind, third kind, and how to divide the problems into sub problems, in order to take care of non homogeneities in the boundary conditions. Then we have looked into the 3 types general solutions, of the 3 types of partial differential equations that are parabolic, elliptical, and hyperbolic. Then we have looked into the 1, 2 dimensional problem, 3 dimensional problem as well as the 4 dimensional problem. And then we have covered the, you know second dimensional 2 dimensional 3 dimensional problems in cylindrical coordinate system, as well as the spherical polar coordinate systems, which the engineers will be they will be becoming across quite often.

So, these course, gives a basic fundamental background, of a offer the engineers, of how to tackle the you know partial differential equation, those will be appearing for different engineering applications, and how to solve them using separation variables, if the operator is a linear operator. Now in actual case the operator may not be linear the problem may not be a linear. So, 1 can take request to the numerical techniques, which will be the time consuming, as well as the computational intensive, but as a first case 1, can go ahead with the, one can linearize the problem assuming the, you know constant thermal physical and the transport coefficients, and one can go ahead with the solution of the separation using separation of variables, as a first as a first and information of the complicated system.

So, I hope this course will be useful to you, for all the engineering students and particularly for you know chemical engineering, mechanical engineering, aerospace engineering students.

Thank you very much.