Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture – 02
Classification of PDE

Welcome to the session of our course and here in the last class we have defined the partial differential equation. We have looked into the various you know types of equations linear, non-linear, homogeneous, non-homogeneous order of equation and we have looked into detail the different types of boundary conditions 1 will encounter. Now in this class we will be looking into the classification how you will be doing the classification or category categorization of the Partial Differential Equations. Now let us look into the classifications.

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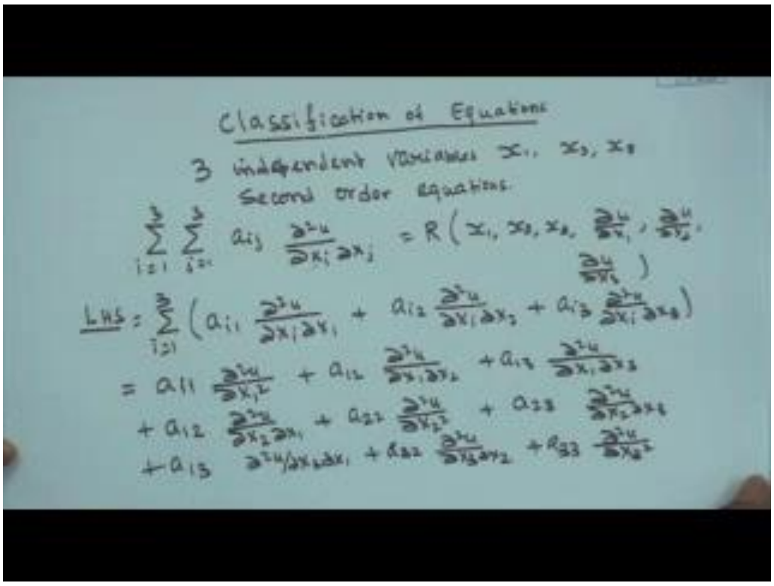

Classification of equations in most of the engineering applications 1 will be dealing with the three independent variables; x 1, x 2 and x 3. Now typically we will be dealing with the second order equations in most of the cases. So, that can be written in general as summation of i is equal to 1 2 3 summation of j is equal to 1 2 3 a i j del square u del x i del x j is equal to r x 1 x 2 x 3 del u del x 1 del u del x 2 del u del x 3 in general. So, all the typical the secondary equations can be represented in this particular form.

Now let us look into the, let us write the details on the left hand side what is that nothing, but the coefficients will expand the summation series and we will be writing down the all coefficients of you know order two terms. So, any partial differentials that will be of order two that will be recast in this form particular form and one can just open up the summation on the left hand side and can get the all the terms containing the order two. So, let us do that. So, if you do that I just open out the left hand side of this equation.

So, i is equal to 1 2 3. So, it will be a i 1 del square u del x i del x 1 plus a i 2 del square u del x i del x 2 plus a i 3 del square u del x i del x 3. Now I open out. So, I opened up the first summation over j next I will be opening up the first summation over i, if I do that this will be a i 1 a 1 1 del square u del x 1 square plus a 1 two del square u del x 1 del x two plus a 1 3 del square u del x 1 del x 3. Next term will be a 1 2 del square u del x 2 del x 1 plus a 2 2 del square u del x 2 square plus a 2 three del square u del x 2 del x 3. The third term will be a 1 3 del square u del x 3 del x 1 plus a 3 2 del square u del x 3 del x 2 plus a 3 3 del square u del x 3 square. So, we open up both the summation terms over i and over j and we have obtained the nine terms. Now nine terms of order two of the partial differential equation now let us write down the coefficient matrix of this nine terms.

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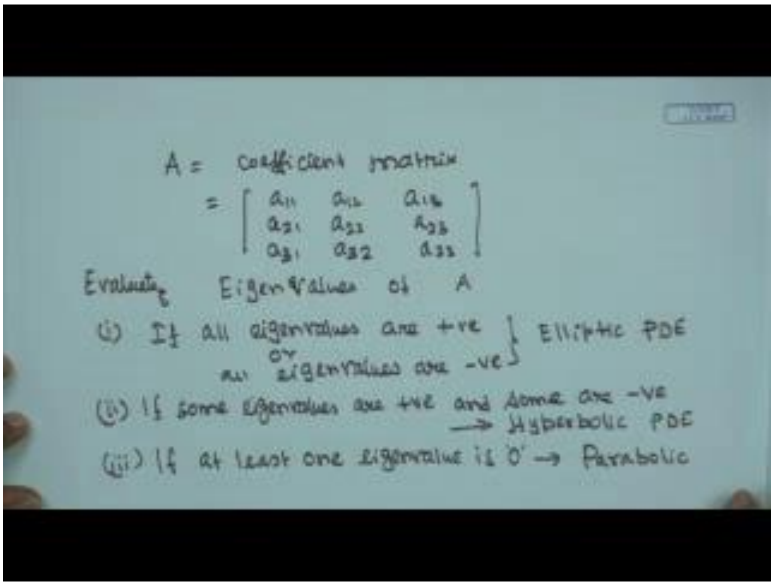

If you write down the coefficient matrix, a becomes the coefficient matrix the coefficient matrix will be nothing, but a 1 1 a 1 2 a 1 3 a 2 1 a 2 2 a 2 3 a 3 1 a 3 2 a 3 3. So, we formulate the coefficient matrix for various for different terms those will be appearing in the order two terms of the partial differential equation. Now we will be evaluating the eigenvalues of this matrix and by looking into the values and sign of the eigenvalues we can classify which the partial differential equation we are dealing with. Now if so, we evaluate the eigenvalues of the coefficient matrix of a evaluate eigenvalues of coefficient matrix a now, if all eigenvalues are positive or all eigenvalues are negative then the partial differential equation we are dealing with is the Elliptic Partial Differential Equation.

If some eigenvalues are positive and some are negative; that means, there is a eigenvalues which are some mixture of positive and negative numbers. So, this is known as the Hyperbolic Partial Differential Equation. If at least one eigenvalue is 0 then it is called a Parabolic then say Parabolic. So, will be by looking into the coefficient matrix of the order two terms in a three variable problem we can get the coefficient matrix by evaluating the eigenvalues of the coefficient matrix we can come to know the by looking into the sign of the eigenvalues and the their values we can come to know about the nature of the partial differential equation we are dealing with. Now we will be going through some of the examples in order to you know explain this further. So, the first example I will be taking about will be equation like u del square u del x square plus del square u del y square plus del square u del z square is equal to 0.

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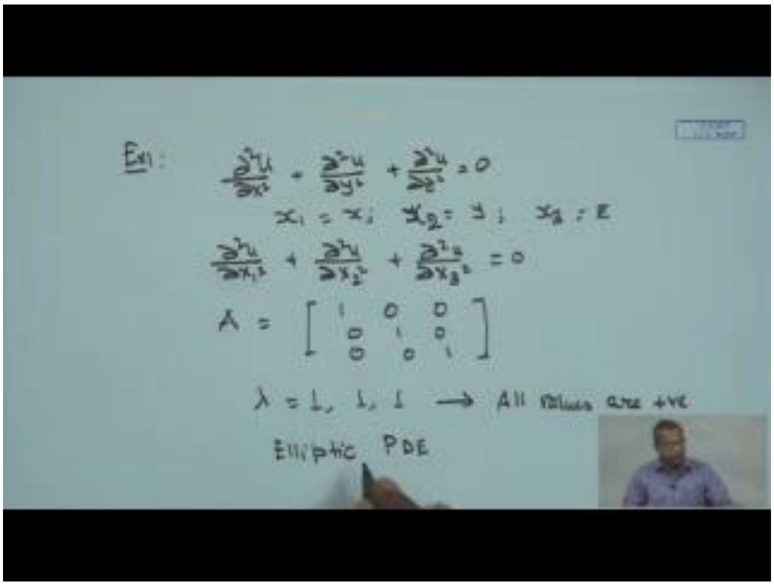

So, in this problem x 1 is basically x y 1 is basically y x 2 is basically y and x 3 is basically z in the nomenclature that we have utilized always. So, it is basically del square u del x 1 square plus del square u del x 2 square plus del square u del x 3 square is equal to 0.

So, if you now by get the coefficient matrix the coefficient matrix will be 1 0 0 0 1 0 0 0 1. So, if you really find out the eigenvalues of this problem of this matrix the eigenvalues will be 1 1 1. So, all eigenvalues are positive, all values are positive and we are talking about an Elliptic Partial Differential Equation. Next we will be taking an example of you know let us say Parabolic Partial Differential Equation.

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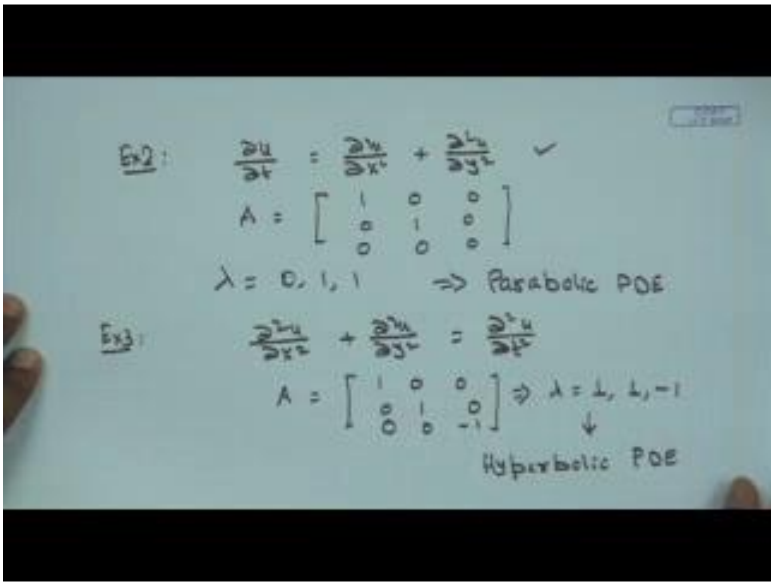

That will be example two it will be del u del t is equal to del square u del x square plus del square u del y square if you recast the equation in the form that we have discussed earlier then and then prepare the coefficient matrix. the coefficient matrix will be 1 0 0 0 1 0 0 0 0. So, therefore, the eigenvalues will be 0 1 1 since one of the eigenvalue is 0 then we are dealing with a Parabolic Partial Differential Equation. So, this equation is a Partial Parabolic Differential Equation.

Next example we will be talking about a hyperbolic one. So, del square u del x square plus del square u del y square is equal to del square u del t square. So, in this equation again we have to recast in the form that we have already discussed that we take all the second order term on the left hand side rest of the term in the right hand side and then if we formulate the coefficient matrix the coefficient matrix becomes 1 0 0 0 1 0 0 0 minus 1. So, now if you get the eigenvalues, the eigenvalues will be 1 1 and minus 1. So, we have landed into a Hyperbolic Partial Differential Equation. So, broadly these are the ways to evaluate the you know nature of the partial differential equation and there are there is a simpler version of doing that as well if we are talking about the two independent variables if there are two independent variables which are quite also in common .

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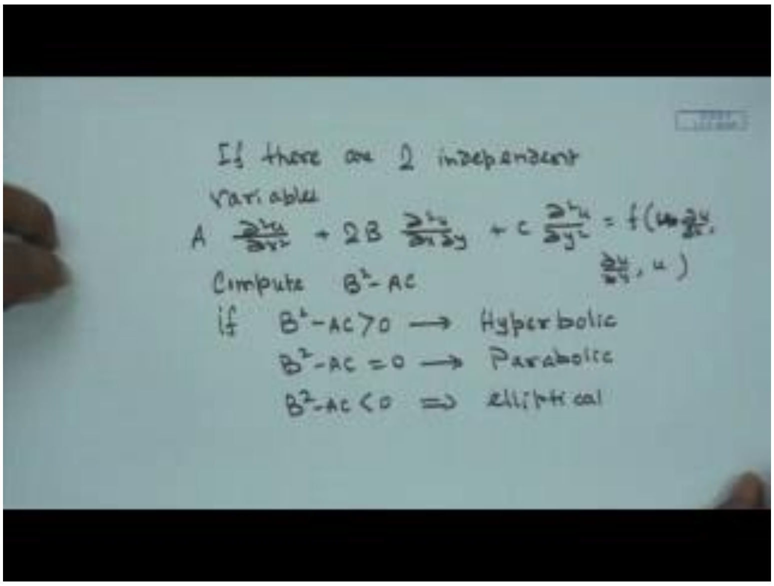

Two independent variables instead of three, in the earlier case we have talked about a three dimensional problem where we are dealing with the three independent variables now in this particular problem we are dealing with a two dimensional problem where we are talking about a about two independent variables.

Now, that will be the what is the general form, general form will be a del square u del x square plus 2 b del square u del x del y plus c del square u del y square is equal to f of u x means del u del x del u del y u. So, any other terms, if we express it this way the two variable problems and then will be compute computing b square minus a c. So, if b square minus a c is greater than 0 then we are talking about a Hyperbolic Partial Differential Equation. If b square minus a c is equal to 0 then we are talking about a parabolic partial differential equation. If b square minus a c is less than 0 then we will be talking about Elliptical Partial Differential Equation. Now sometimes you may not be having the constant coefficients constant valued coefficients there numerical values like a 1 2 3 like that in sometimes there will be the there will be variables as well.

So, in that case I will be defining the equations. So, that will be depending on the nature of the variable that we are talking about.

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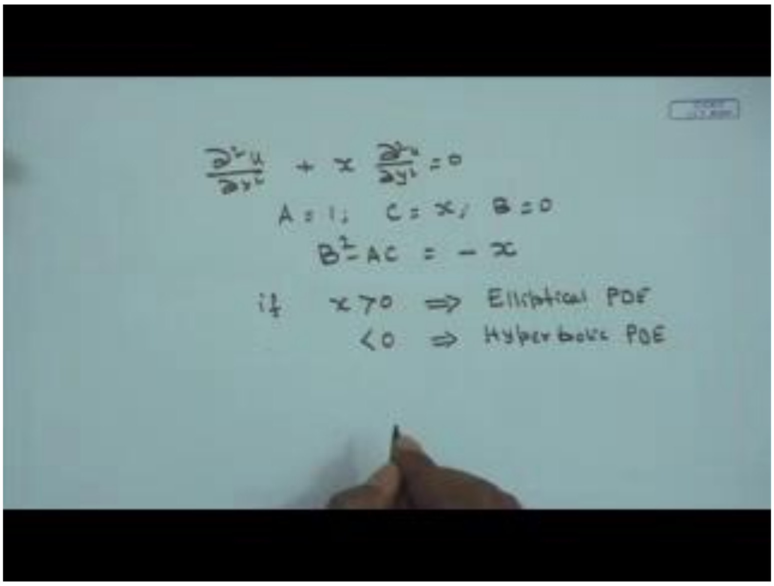

.For example if you have a equation del square u del x square plus x del square u del y square is equal to 0, in this case a is equal to 1 c is equal to x and b is equal to 0. So, we compute b square minus a c and b square minus a c is minus x. So, now, if x is positive then we will be getting an Elliptical Partial Differential Equation and if x is negative then will be getting the Hyperbolic Partial Differential Equation. So, there will be the various ways to. So, these are the ways to define to characterize the Partial Differential Equation in the in case of two independent variable problem.

So, I have looked into the classification of categorization in three dimensional problems and two-dimensional problem which are quite common in the engineering applications. Next what we will be doing we will be defining some of the terms which will be quite important in our in formulating an solution of Partial Differential Equation into the by separational variable for that we have to define some of the term or develop some the theorems which will be quite handy and useful for us. Then we will define what is called a Linear Operator.

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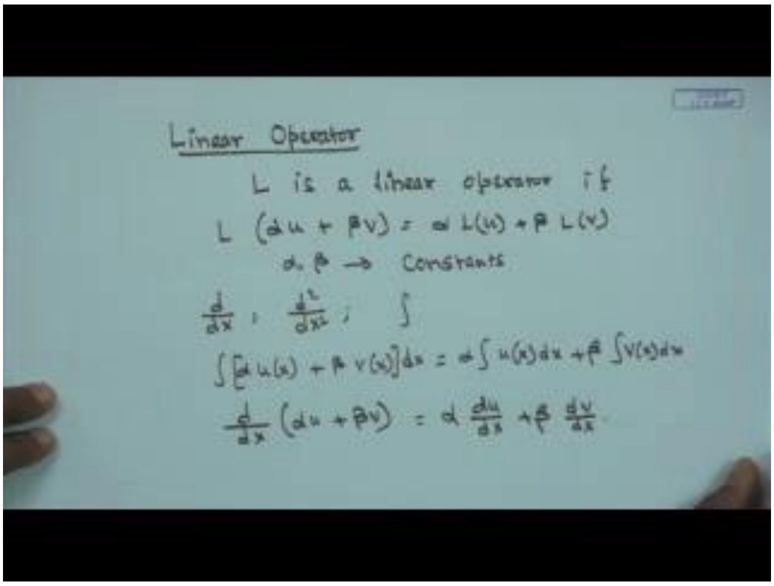

This is very important. So, the Linear Operator is the operator l is a linear operator, if l of alpha u plus beta v is equal to alpha l u plus beta l v, where alpha beta are two constants. Now d d x what is the example of Linear Operator, d d x is a linear operator d square d x square is linear operator integration is a linear operator. For example, integration of you know let say alpha u x d x alpha u x plus beta v of x d x is nothing, but alpha integral u x d x plus beta integral v x d x. So, for example, d d x of alpha u plus beta v is nothing, but alpha d u d x plus beta d v d x. So, if these are the cases then we can these are called Linear Operator and any linear operator can have a can will obey the principle of Linear Superposition. What is the principle of Linear Superposition?

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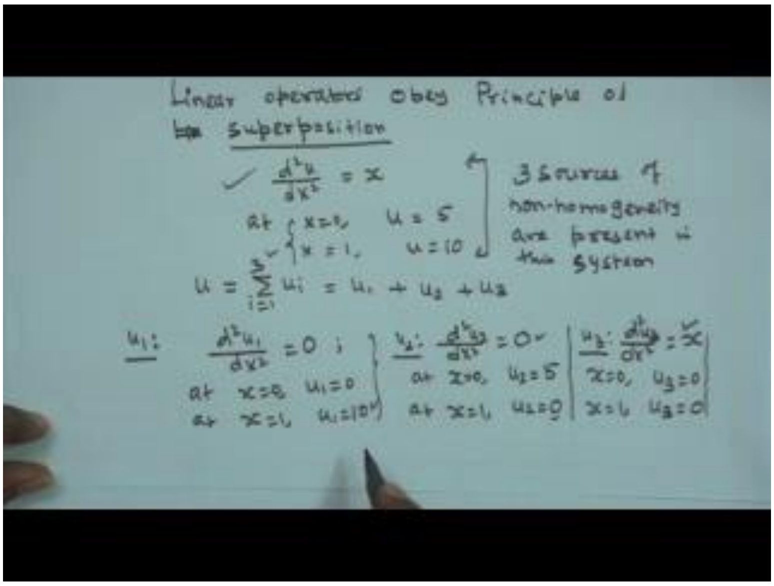

So, Linear operators obey principle of superposition. What is the principle of superposition? Suppose on having a I am solving a an ordinary differential equation which is non-homogeneous, d square u d x square is equal to x let us say and boundary conditions are also non-homogeneous at x is equal to 0 u is equal to let us say 5 and at x is equal to 1 u is equal to 10. So, this equation is equivalent to if I break down this into three sub problems keeping one on homogeneous type.

So, let us find out how many non-homogeneous are there. There are three sources of non-homogeneity are present in this equation in this system, system includes the governing equation as well as the boundary conditions. So, there are three sources of non-homogeneities are present in this system and what will be doing next will be break down this problem into three sub problems considering one on homogeneity at a time and then we will be adding up all the three solutions and then we will be getting the final solution; that means, if there is a non-homogeneous set of system then we have to identify the sources of non-homogeneity.

Suppose there are three sources of non-homogeneous in this particular problem we have to break down this problem into three sub problems. So, that one non-homogeneity will be appearing you know in the particular problem. So, therefore, the solution of all the three sub problems will be giving you the actual solution. For example, I will be defining this problem and the solution will be the linear superposition of all the sub problems. So, i will be from 1 to 3 this will be u 1 plus u 2 plus u 3. So, what is the definition of u 1 u 1 will be nothing, but d square u 1 d x square is equal to 0.

So, in this case I will be forcing the boundary conditions the forcing the non-homogeneity in the governing equation to vanish and one of the boundary condition to vanish. So, x equal to 0 u 1 is equal to 0 at x is equal to 1 u 1 is equal to 10. So, this is the definition of u 1. So, only one non-homogeneity is kept intact and rest two is forced to be zero. So, what is u 2, u 2 will be d square u 2 d x square will be equal to 0 at x is equal to 0 u 2 is equal to 5 and at x is equal to 1 u 2 is equal to 0. So, in this case I am forcing the non-homogeneous term in the governing equation to vanish and the boundary at x is equal to 1 to vanish keeping the non-homogeneity at x is equal to 0.

Similarly u 3 will be what d square u 3 d x square will be equal to x and I will be forcing the other two non-homogeneous the boundary conditions to vanish. So, x is equal to 0 u 3 is equal to 0 and x is equal to 1 u 3 is equal to 0. So, in this case I am keeping the non-homogeneity in the governing equation intact forcing the non-homogeneous in the boundary conditions. So, we will be solving if you solve you know individually each of this sub problems. So, this is my first sub problem this is my second sub problem this is my third sub problem if I get a solution of this sub problems individually and then add up the solution all in linearly so; that means, if you linearly superpose the solution of each sub problem will be getting the complete solution. So, just take this as an assignment and solve directly this equation with this boundary condition and check whether that solution is coming equal or not by solving these three sub problems one at a time and then adding them up by doing a linear superposition.

So, that give take that as an assignment and complete this problem and check that whether principle of linear superposition is valid for in case of an ordinary differential equation. Next actually why this is important this will be important because will be quite a often will be utilizing these type of solution in case of partial differential equation and will be extending this for the partial differential equation as you go down along the along the our course. So, this will be very very important principle if the operator is linear the operator the linear operator always satisfies the principle of linear superposition. Next we will be looking into a set of you know equations. So, for example, I will be talking about if if we extend the principle of linear superposition in case of partial differential equation let us see lets give an example how one can go ahead with it.

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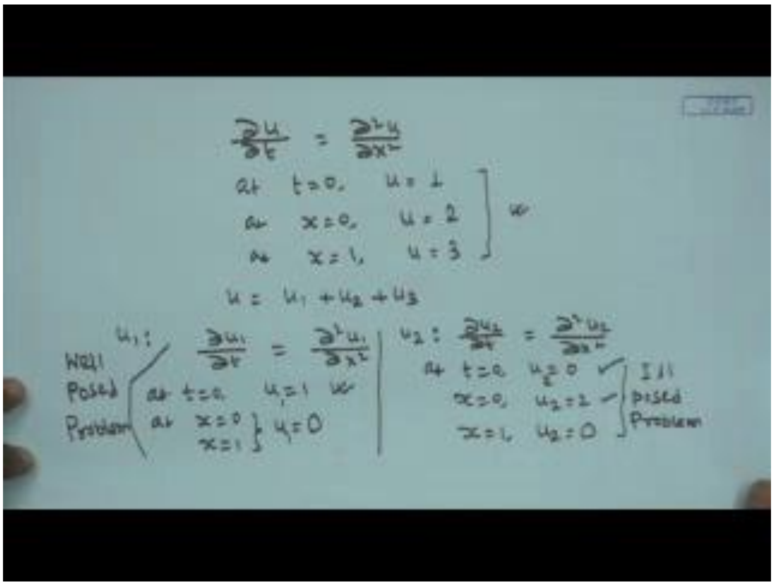

For example if I have a and the partial differential equation like this del u del t is equal to del square u del x square at t is equal to 0 u is equal to let us say 1 at for a v x at x is equal to 0 u is equal to 2 at x is equal to 1 u is equal to 3.

So, if this is the case then how will you breakdown this problem. So, there are three sources of non-homogeneities in this governing equation in the system the non-homogeneity is appearing at initial condition non-homogeneities are appearing in the boundary conditions as well. So, how this problem will be solved this problem will be divided into three sub problems considering one as non-homogeneity at a time. For example, u will be divided into u 1 plus u 2 plus u 3.

So, let us look into the definition of u 1 u 1 will be del u 1 del t is equal to del square u 1 del x square and boundary conditions are at t is equal to 0 my u is equal to 1 at x is equal to 0 and at x is equal to 1 I force the non-homogeneities to vanish. So, this is a non-zero initial condition and what is the definition of u 2 u 2 will be del u 2 del t is equal to del square u 2 del x square at t is equal to 0 put u is equal to 0 at x is equal to 0 put this is 1 this is 1 so this u 2, u 2 is equal to 2 and at x is equal to 1 u 2 is equal to 0. So, I force the other two non-homogeneous to vanish and I keep this non-homogeneity intact. So, that is the definition of u 2 and what is u 3.

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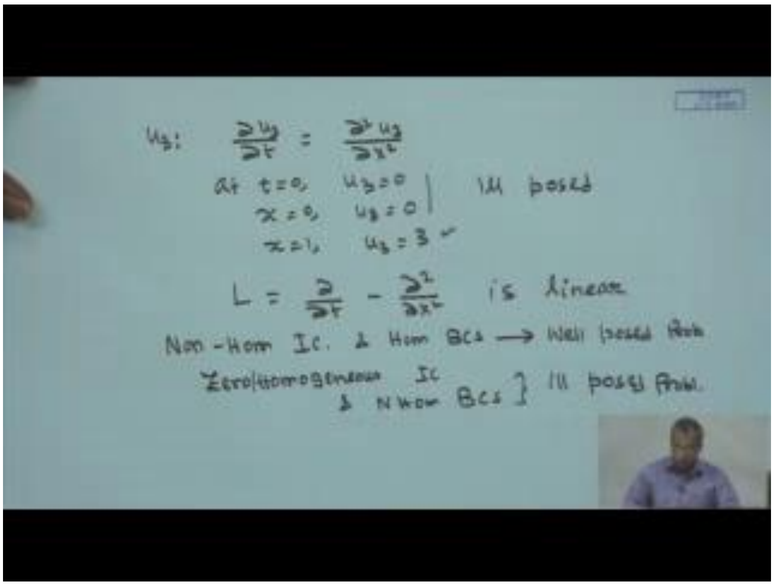

U 3 will be nothing, but del u 3 del t is equal to del square u 3 del x square and what are the boundary and initial conditions at t is equal to 0 my u 3 is equal to 0 at x is equal to 0 my u 3 is equal to 0 and at x is equal to 1 my u 3 is equal to 3. So, here I force these two non-homogeneous to vanish I keep this non-homogeneity intact.

So, what essentially you have done we are divide the problem into three sub problems and we have seen that this how this and it is easier to solve each such sub problems and I by using the principle of linear superposition we can get the complete solution by adding up all the solution of sub problem. And why this can be done because our operator what is the operator here; operator is del t minus del square del x square these known as parabolic operator and this operator is linear. Now, let us look into the nature of sub problem as well if you have a partial differential equation where the initial condition is non zero and the boundary conditions are homogeneous initial condition is non homogeneous and boundary condition are homogeneous then it is known as a well posed problem.

On the other hand if you have a partial differential equation let us say parabolic differential equation with a zero with a homogeneous initial condition and non zero boundary conditions. For example, you have the homogeneous initial condition and 1 of the boundary conditions is non-homogeneous then this is called is known as the ill posed problem. So, similarly the u 3 is also an ill posed problem because the initial condition is homogeneous. So, homogeneous non-homogeneous initial condition and homogeneous boundary conditions are generally called well posed problem and zero or homogeneous initial condition and non-homogeneous boundary conditions are known as the ill posed problem why they are called, in case of well posed problem we can have a directly separation of variable type of solution as we will be seen later on in our course.

In case of ill posed problem we cannot do that because the final constant that has to be evaluated in case of separation variable will become zero in that case. So, what is the solution, the solution is how to convert an ill posed problem into a well posed problem.

So, I will stop in this class in the next class again will be look will be coming back reverting back to this problem and we will be seeing how an ill posed problem is converted to an well posed problem and then will be going it with the separation of variable type of solution.

Thank you very much.