Partial Differential Equations (PDE) for Engineers:
Solution by Separation of Variables
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Lecture – 19
Examples of Application Oriented Problems

Good morning every one. So, in the last class what we did we were solving for a three dimensional problem which is a heat conduction problem one dimension in it is a transient problem and the temperature is varying as a function of x y and t and we have made the system non-dimensionalized and we have set up the boundary conditions also in the form in the in terms of non dimensional variables.

Now in today's class we will be going ahead with the complete solution of this problem. So, let us write down the non-dimensional version of the governing equation and the boundary conditions that we have already you know we left in the last class.

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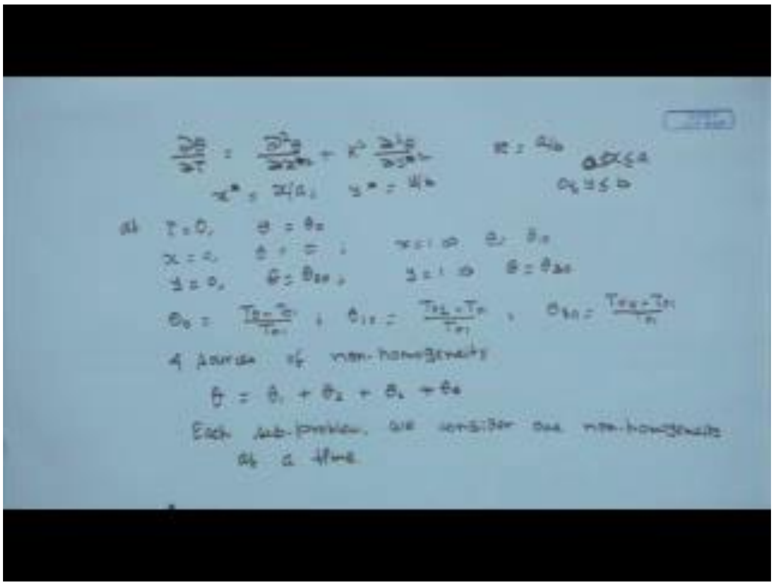

So, del theta del tau is equal to del square theta del x square plus k square del square theta del y square where k is equal to a by b and a is the length in the x direction. So, x is basically lying between a and 0 and a and y is lying between 0 and b. So, these are governing equation and all x star is basically x star square and y star square where x star is equal to x over a and y star is equal to y over b and we set up the boundary conditions as well in the non-dimensional term tau is equal to 0 theta is equal to theta naught and x is equal to 0. We have theta is equal to 0 at x is equal to 1 we had theta is equal to theta 1 0 and y is equal to 0 we have theta is equal to theta two 0 and y is equal to 1 we had theta is equal to theta three 0 where this theta 0 theta 2 0 theta 1 0 theta 3 0 they are the non-dimensional temperature and their definitions were theta need naught is equal to t naught minus t naught 1 divided by t naught 1 theta 1 0 is equal to t naught 2 minus t naught 1 divided by t naught 1 and theta 3 0 is equal to t naught 4 minus t 0 1 divided by t 0 1.

So, now if you look into the original problem in the original problem we had five sources of non-homogeneity by virtue of being non-dimensionalization. We have reduced the one non-homogeneity and now we are dealing with four non-homogeneities. So, these problems has to be divided into 4 sub problems because there are four sources of non-homogeneities in this problem four sources of non-homogeneity. Therefore, theta has to be broken down into four sub problem theta 1 plus theta 2 plus theta 3 plus theta 4 and then each such sub problem will be considering one non-homogeneity at a time.

So, each sub problem we consider one non-homogeneity at a time. So, let us define the sub problem theta 1 and going at right to the solution.

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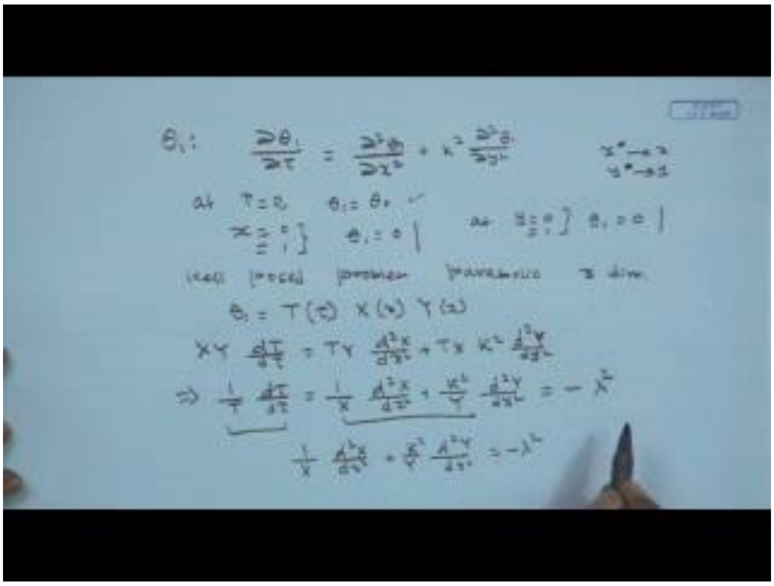

So, theta 1 will be del the governing equation of theta 1 will be del theta 1 del tau is equal to del square theta 1. So, basically we are substituting theta is equal to theta 1 theta 2 theta 3 plus theta 4 in the governing equations and collect the similar terms and then we will be getting the governing equation of each theta 1 theta 2 and theta 3 theta 4.

So, del square theta 1 del x square, now I write x star and y star as x and y they are the non-dimensional variable k square del square theta 1 del y square. So, I just substitute x star by x and y star by y understanding that now x and y will be representing the non-dimensional variables x by a and y by b. So, the boundary the initial condition will be tau is equal to 0 theta 1 is equal to theta naught at x is equal to 0 and 1 and my theta 1 is equal to 0 at y is equal to 0 and 1 my theta 1 is equal to 0. So, in this sub problem theta 1 we have kept non-homogeneity in an initial condition intact forcing the other non-homogeneous on the other boundaries to be equal to be equal to be 0. We will force them to be homogeneous. So, therefore, this is a well posed problem this is a well posed problem a parabolic one and three dimensional.

So, we go ahead with the separation of variable type of solution. So, theta 1 is equal to function of time multiplied by function of x function of y. So, therefore if I substitute in the governing equation this becomes x y d t d tau is equal to t y d square x d x square plus t x k square d square y d y square. So, I divide by t x and y. So, what will be getting is 1 over t d t d tau is equal to 1 over x d square x d x square plus k square by y d square y d y square. So, this left hand side is a function of time only the right hand is a function of space they are equal, they are equal to some constant this constant has to be a negative constant in order to have a Non-Trivial Solution.

So, now let us look into the special varying part. So, 1 over x d square x d x square plus k square by y d square y d y square is equal to minus lambda square. Now we see that the boundary conditions in x directions are homogeneous boundary conditions in the y directions are homogeneous, will be having standard eigenvalue problem independent eigenvalue problem in both x direction and y direction.

So, now let us solve the x varying part and y varying part.

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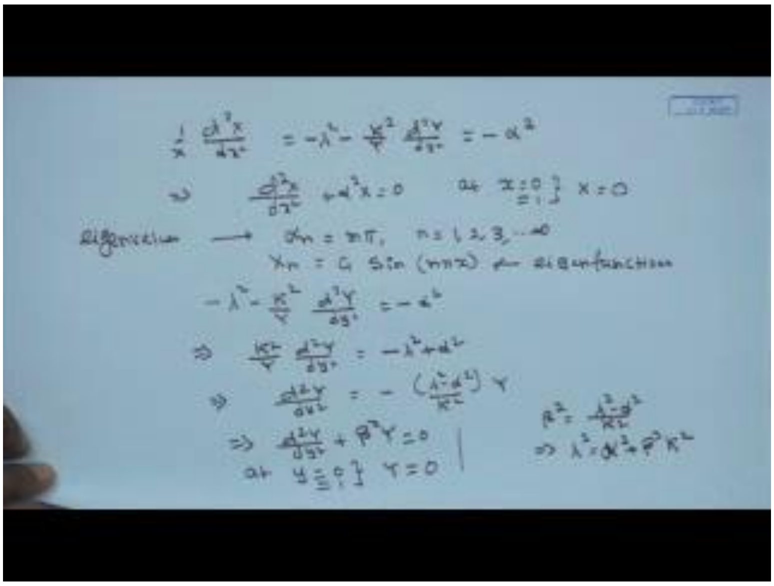

So, d square x 1 over x d x square is equal to minus lambda square minus k square by y d square y d y square and these will be again this will be the function of x this is a function of y alone, they are equal they will be equal to some constant in order to have a Non-Trivial Solution this has to be a negative constant and we formulate the standard eigenvalue problem in the x direction. And the boundary condition of the original problem in the x direction has to be satisfied by this part. So, therefore, at x is equal to 0 and 1 we have capital x which is equal to 0.

So, therefore, you know the solution of this alpha n is equal to n pi at the eigenvalues where n is equal to 1 2 3 up to infinity and x n is equal to c 1 sin n pi x are the Eigen function. So, these are the Eigen functions, these are the eigenvalues. Now let us look into the y varying part minus lambda square minus k square divided by y d square y d y square is equal to minus alpha square. So, k square over y d square y d y square will be nothing, but minus lambda square plus alpha square and we write d square y d y square is equal to minus lambda square minus alpha square over k square y. So, this we call this constant. So, is equal to beta square. So, we write it as d square y d y square plus beta square y is equal to 0 where beta square is equal to lambda square minus alpha square by (Refer Time: 09:58) square. So, I can get lambda square is equal to alpha square plus beta square (Refer Time: 10:08) square and at y is equal to 0 and 1 capital y will be equal to 0.

So, this will be the eigenvalue problem in the y direction where lambda square is equal to alpha square plus beta square (Refer Time: 10:23) square and we know the solution of this since it is an independent eigenvalue problem.

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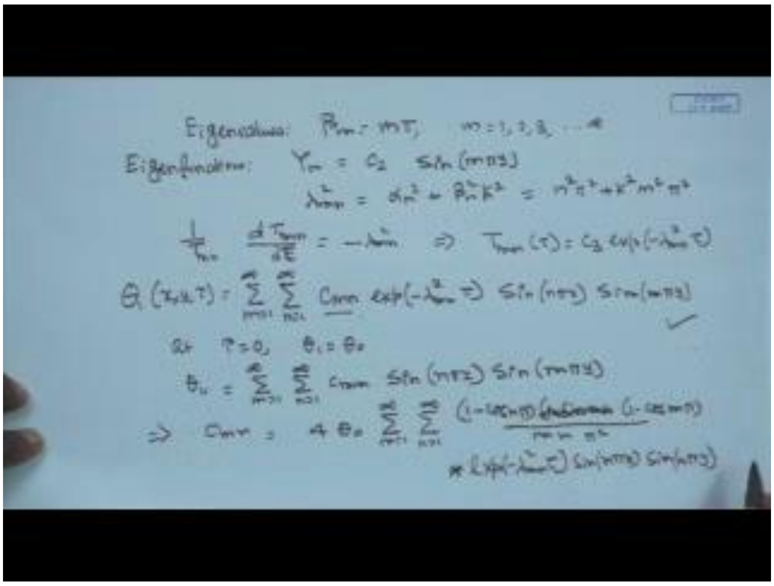

Eigenvalues are beta m is equal to n pi m pi this m index is for the independent eigenvalue problem in this particular case this will be from 1 2 3 up to infinity and Eigen functions are the sin functions y m is equal to c two sin m pi y. So, your lambda m n square will be nothing, but alpha n square plus beta m square k square. So, these will be nothing, but n square pi square plus k square m square pi square. So, that is the definition of lambda m n. Now we are position to obtain the solution of the time varying part. So, 1 over t m n d t m n d t will be nothing, but minus lambda m n square and the solution of this will be nothing, but t m n d tau as a function of tau will be a let us say another constant c 3 exponential minus lambda m n square of tau. So, this gives the solution of the time varying part.

Now, we are in a position to combine all these three independent three solutions and using the principle of linear superposition we can formulate construct the complete solution if we construct the complete solution the theta 1 which will be a function of x y and tau should have been should be m is equal to double summation m is equal to 1 to infinity n is equal to 1 to infinity. There will be two summations because we are dealing with two independent eigenvalue problems. So, c m n these will be a multiplication of the all three constants c 1 c 2 and c 3 exponential minus lambda m n square tau sin n pi x sin m pi y and we evaluate the constants c m n from the initial condition that was that tau is equal to 0 we have theta 1 is equal to theta naught and theta naught is equal to summation m is equal to 1 to infinity n is equal to 1 to infinity c m n sin n pi x sin m pi y.

Now, c m n will be evaluated as you have done several times using the Orthogonal Property of the Eigen function. So, I multiply both side by sin m pi x sin n pi y d x d y and integrate over the domain of x and y from 0 to 1 and then only in the right hand side only one term will survive when m is equal to n and when you open up the summation series. So, we will be getting c m n is equal to four we have done this exercise earlier four theta naught m is equal to 1 infinity n is equal to 1 to infinity 1 minus cosine n pi 1 minus cosine m pi 1 minus cosine m pi divided by m n pi square multiplied by exponential minus lambda m n square tau sin n pi x sin m pi y. So, that gives the complete solution. So, c m n will be substituted here and that will be giving the complete solution of theta one theta one part of the sub problem theta one out of four sub problems that we have defined earlier.

Now, we will be looking into the second sub problem theta two and see how this solution will be evolving out.

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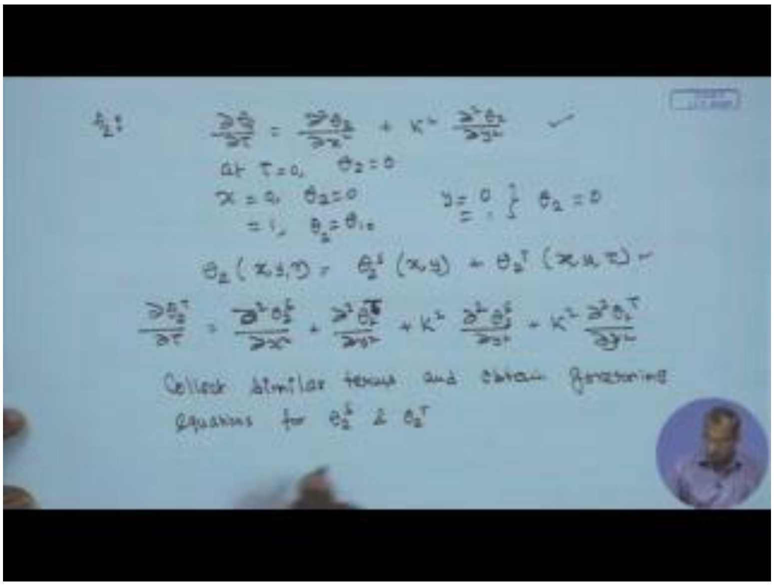

So, for theta two let us write down the governing equation. The governing equations is obtained when you substitute theta is equal to theta 1 plus theta 2 plus theta 3 plus theta 4 in the governing equation and then we collect the equal equivalent terms in terms of the all the terms collecting the terms for theta 1 collect the terms of theta 2 collect the terms for theta 3 and theta 4. So, therefore, we will be able to formulate the governing equation of all the four sub problems theta 1 to theta 4.

So, for theta 2 this will be del theta 2 del tau will be is equal to del square theta 2 del x square plus k square del square theta 2 del y square. So, at tau is equal to 0, theta 2 is equal to 0 I force this non-homogeneity to vanish in this particular problem at x is equal to 0, theta 2 is equal to 0 at x is equal to 1 my theta was equal to theta 1. So, we force all the non-homogeneities. So, y is equal to 0 and 1 theta is equal to theta 2 is equal to 0. So, we force all the non-homogeneties to vanish in this particular sub-problem and keeping the non-homogeneity at x is equal to 1 theta 2 is equal to theta 1 theta 1 naught.

So, in this sub problem we will be keeping one non-homogeneity in one of the boundary conditions at x is equal to 1. So, since this in this particular problem it will be having a zero initial condition we have to divide this sub-problem into two sub parts one will be time dependent, another will be time independent. So, let us do that. So, theta 2 will be further divided into two sub parts one will be theta 2 steady state which will be function of x and y only and another will be theta 2 tau which will be function of x y and tau all.

So, next we will be what we will be doing. So, this is the mother problem for this sub-problem. So, therefore, we are substituting these expression into a governing equation and then collect the similar terms and formulate the governing equation of theta 2 s and theta 2 t, if you do that we substitute it over there this will be function of space only. So, therefore, derivative with respect to tau will be equal to 0. So, left hand side will be getting only del theta 2 tau del tau and in the right hand side we will be getting d square del square theta 2 s del x square plus del square theta 2 s del y square plus (Refer Time: 17:40) square del square.

So, there will be del theta tau theta 2 tau and (Refer Time: 17:55) square del square theta 2 s del y square plus k square del square theta 2 tau del y square then I collect the similar terms and formulate the governing equation of you collect similar terms and obtain governing equation for theta 2 s and theta 2 tau. So, if you do that the governing equation of theta 2 s will be nothing, but del square theta 2 s del x square plus k square del square theta 2 s del y square is equal to 0 and I will associate.

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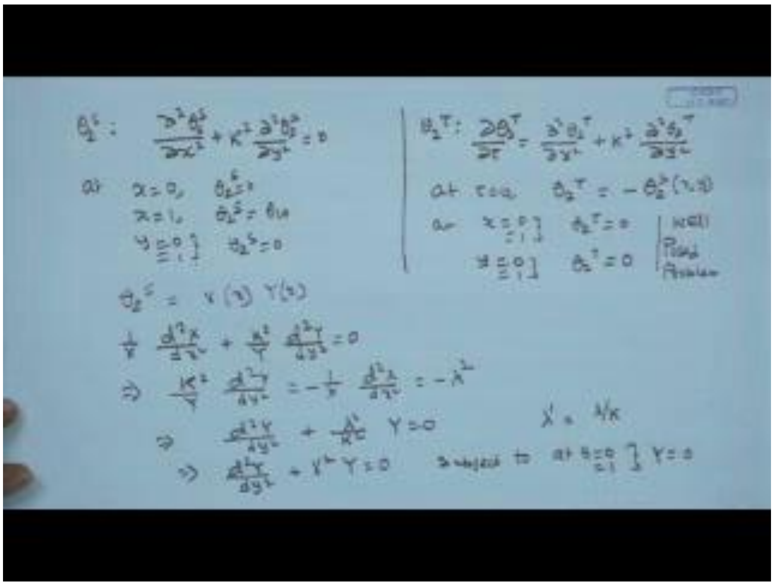

So, this is the steady state part and then we will be formulating the governing equation of theta 2 tau as well.

So, if you do that then it becomes del theta 2 tau del tau is equal to del square theta 2 tau del x square plus k square del square theta 2 tau del y square. Now, we associate the non-homogeneous boundary condition of the original problem along with the special varying parts forcing the boundary condition of the time varying part to be homogeneous, if you do that and as we have done earlier, x is equal to 0, theta 2 s is equal to 0 at x is equal to 1, we have theta 2 s is equal to theta 1 naught and at y is equal to 0 and 1 we have theta two s is equal to 0. Since, we are associating the bound the non-homogeneous boundary conditions of at x is equal to 1 with the steady state part in the transient part that will become zero.

So, therefore, at tau is equal to 0 we have done all these things in detail earlier. So, then theta 2 tau will be nothing, but the solution of the steady state part as a function of x and y and at x is equal to 0 and 1, we have theta 2 tau will be is equal to 0 at y is equal to 0 and 1, we have theta 2 tau will be equal to 0. So, therefore, we will be having a standard eigenvalue problem in the x direction as well as in the in the y direction these two are independent eigenvalue problem and we will be having a non-zero initial condition in this particular case. So, this will be a Well Posed Problem. So, this is definitely a Well Posed Problem and we will be now in the steady state part we the solution of the steady state will be the with the negative sign will become the initial condition of the transient part, but the steady state part itself is a partial differential equation and if you examine this equation these equation is nothing, but an Elliptical Partial Differential Equation.

Now, we have already seen the solution of this let us go ahead with a separation of variable type of solution. So, theta 2 s will be a product of function of x only and function of y only. So, if you put into the governing equation and separate out the variables then this becomes 1 over x d square x d x square plus k square by y d square y d y square is equal to 0. So, we will be and please note that only in the y direction in this sub-problem the boundary conditions are homogeneous, not in x direction. So, therefore, we will be having a standard eigenvalue problem in the y direction for the Elliptical Partial Differential Equation not in x direction.

So, we will be formulating a standard eigenvalue problem or Sturm Liouville problem in the y direction only. So, let us do that. So, (Refer Time: 22:16) K square y by y d square y d y square will be equal to minus 1 over x d square x d x square is equal to minus lambda square, you know that to have a Non-Trivial Solution. So, if you do that you will be formulating the standard eigenvalue problem in the y direction plus lambda square over k square times y is equal to 0 we put lambda prime is equal to lambda by k. So, this becomes d square y d y square plus lambda prime square times y is equal to 0 subject to boundary condition at y is equal to 0 and 1 capital Y is equal to 0.

So, Homogeneous Boundary Conditions at y is equal to 0 and y is equal to 1 and we know you all know about the solution of this particular problem it will be the eigenvalues will be n pi and the Eigen functions will be sin n pi. So, therefore, the eigenvalues will be lambda prime is equal to n pi, where the index n runs from 1,2,3 up to infinity and Eigen function will be C 1 sin n pi x. Now let us formulate the governing.

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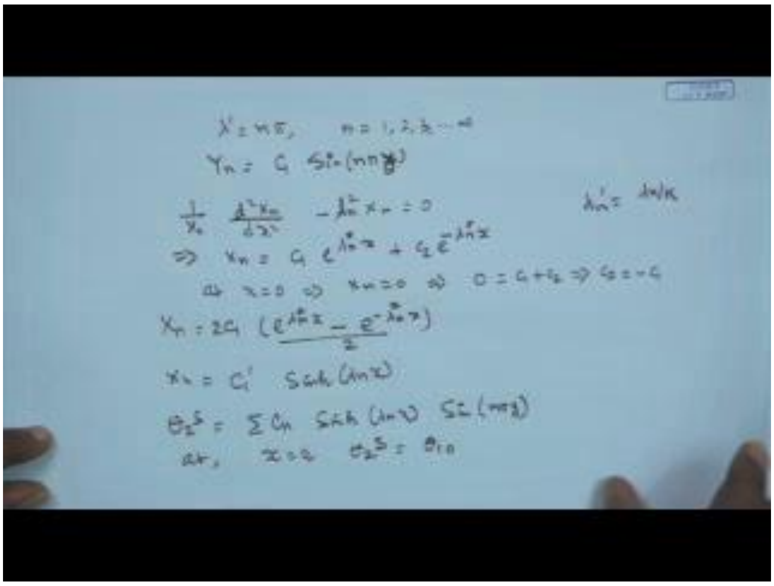

Let us look into the rest of the problem x varying part. So, 1 over x d square x n, let us put corresponding to n'th eigenvalue d x square minus lambda n square lambda n prime square x n will be equal to 0. So, x n will be nothing, but C 1 e to the power lambda n prime x plus c two e to the power minus lambda n prime x, where lambda n prime is nothing, but lambda n over k.

So, this will be the solution of that. Now we have the boundary condition at x is equal to 0 is equal to x n is equal to 0. So, we this boundary condition has to be satisfied. So, 0 is equal to c 1 plus c 2. So, your c 2 will be equal to minus c 1. So, I have the solution of x varying part as c 1 e to the power lambda n x lambda n prime x minus e to the power minus lambda n prime x. So, if you look into the governing equation this will be 1 over x d square x d x square it will be taking on the other side plus minus lambda square x. So, it will be not lambda prime it will be lambda only.

So, this will be lambda, this will be lambda, this will be lambda, this will be lambda, this will be lambda. So, now, I divide and multiply by 2. So, this x n becomes C 1 prime sin hyperbolic lambda n x. Now I get the solution theta 2 s will be summation of this C 1 prime multiplied by C 1 for the y. So, it will be a new constant, let us say C n sin hyperbolic lambda n x multiplied by sin n pi y. So, it should be n pi y, sin n pi y and then at x is equal to 0 we have theta 2 s is equal to theta 1 naught this initial condition was not utilized. So, I am going to substitute this initial condition here and evaluate C 1. So, if I do that what I will be getting is, theta 1, 0 is equal to summation of C n sin hyperbolic lambda n at x is equal to 1 sin n pi y.

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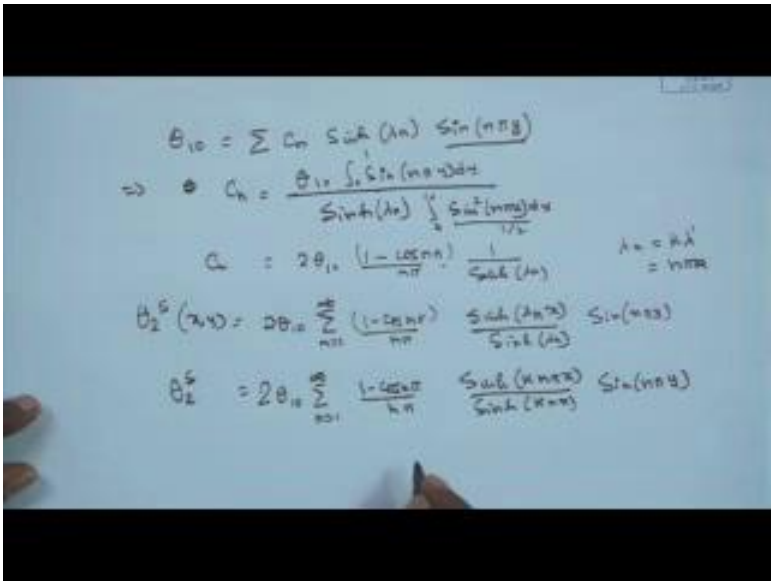

So, I multiply I use this Orthogonal Property of the sin function and evaluate, so theta 1 0. So, C n will be nothing, but sin hyperbolic lambda n integral 0 to 1 sin square n pi y d y and this will be theta 1 naught sin n pi y d y from 0 to 1 and we will be having that C n is equal to 2 theta 1 naught 1 minus cosine n pi over n pi and this will be half. So, this will be half, this half will be going that has two and we will be having 1 over sin hyperbolic lambda n.

So, we will be getting a complete solution of theta 2 s as a function of x and y as 2 theta 1 naught n is equal to 1 to infinity 1 minus cosine n pi divided by n pi sin hyperbolic lambda n x divided by sin hyperbolic lambda n and we will be having sin n pi y and we know the lambda n is nothing, but k times lambda prime and lambda prime is n pi. So, it will be n pi k. So, this will be 2 theta 1 naught summation n is equal to 1 to infinity 1 minus cosine n pi over n pi sin hyperbolic k n pi x divided by sin hyperbolic k n pi sin n pi y. So, that gives the complete solution of the steady state part and then we will be I will stop you in this class we will be looking hide with this problem and go ahead with the rest of the solution.

Thank you very much.